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CONFORMABLE EIGENVALUE PROBLEMS WITH TWO PARAMETERS

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Abstract. This study used conformable derivatives to define the eigenvalue problems with two parameters and examined various associated spectral properties. Firstly, the conformable eigenvalue problems with two parameters were reduced to the simpler one parameter problems. Additionally, we focused on the orthogonality properties of eigenfunctions. Secondly, investigating the reality of eigenvalues is important to understand the physical relevance and practical usability of the considered eigenvalue problem. Finally, we examined integral relations, which explain important connections and relationships between different aspects of the problem.

1. Introduction

In many important problems in various fields such as basic sciences, natural sciences, finance, and medicine, differential equations are encountered in the mathematical modeling of these problems. The functions that satisfy these equations are also the mathematical solutions to these problems. Therefore, the first step in researching the solutions to any scientific problem is formulating the differential equation. The problems such as the heat flow in a non-uniform rod, the motion of a stretched vibrating string attached at both ends and the computation of the electrostatic field on the surface of a volume are modeled by an eigenvalue problem with an unknown parameter, known in the literature as the Sturm-Liouville differential equation [16, 17, 23]. This equation is considered along with initial or boundary conditions according to the characteristics of the models to be established. The goal here is to determine the unknown parameter and the unknown function that constitutes the problem. The most fundamental properties of Sturm-Liouville problems include the reality of the eigenvalues, the orthogonality of the eigenfunctions, and the completeness of the eigenfunctions [2,9,10]. Sturm-Liouville theory has numerous applications in fields such as physics, mathematics, and engineering [18,22].

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Depending on the nature of the models to be established, the equation may contain more than one parameter [5–8, 11].

Many mathematicians, including Liouville, Riemann, Weyl, Fourier, Laplace, Lagrange, Euler, Abel, Lacroix, Caputo, and Leibniz, have defined fractional-order derivatives in various ways [19]. Fractional differential equations are equations that contain fractional derivatives and fractional integrals [13, 21]. It is a generalization of the classical integer-order derivative concept, allowing the differentiation process to be performed at non-integer (fractional) orders [15]. This concept is widely used in many scientific fields, particularly dynamic systems, control theory, and physics. Because fractional derivatives only satisfy the linearity property of the fundamental characteristics in classical derivatives and are not applicable for all other properties, Khalil and colleagues proposed the definition of the conformable fractional derivative in 2014 to mitigate this complexity [14]. The conformable derivative is a type of fractional derivative that aims to make the concept of fractional derivatives more comprehensible and easier to compute [1]. By preserving some fundamental properties of the classical derivative operator, the conformable derivative allows for broader applicability of fractional derivatives. Therefore, it is increasingly used in scientific research and engineering applications.

Many authors [3, 4, 12, 20] consider the conformable fractional derivative and Sturm-Liouville theory together.

Now, a brief explanation of two-parameter eigenvalue problems is provided. Additionally, we will provide a general overview of some fundamental definitions and accepted results in the field of fractional calculus. This will include a brief introduction to the concepts and notations used in fractional calculus, and it also encompasses a summary of some important results and theorems widely accepted and utilized in the field.

Arscott [7] focused on a series of related eigenvalue problems common to a simple linear homogeneous differential equation dependent on two parameters and stated that the solution must satisfy three limiting conditions:

$$\frac{d^2u}{dz^2} + \{\lambda + \mu f(z) + g(z)\} u = 0, \tag{1.1}$$

$$u(a) = u(b) = u(c) = 0, \tag{1.2}$$

where, $f(z)$ and $g(z)$ are functions defined on the interval $[a, c]$ and λ and μ are spectral parameters. Here, this eigenvalue problem with two parameters has been reduced to a one-parameter problem. The spectral properties of the two-parameter eigenvalue operator such as orthogonality, the realness of eigenvalues, and the expansions theorem of eigenfunction have been investigated and some integral relations have been given. Additionally, various integration methods were examined, and results were obtained [5, 6]. In fact, the given problem is the case of a Sturm-Liouville problem with multi-parameters and significant results are obtained [8].

In this study, consider the conformable eigenvalue problems with two parameters

$$D_t^\alpha \left(D_t^\alpha u(t)\right) + \{\lambda + \mu f(t) + g(t)\} u = 0, \tag{1.3}$$

$$u(a) = u(b) = u(c) = 0, \tag{1.4}$$

where $b \in [a, c]$, $f(t)$ and $g(t)$ are real-valued continuous functions defined on the interval $[a, c]$ and λ and μ are spectral parameters. This paper aims to reduce two-parameter conformable eigenvalue problems to one-parameter problems; to investigate the orthogonality properties of eigenfunctions and the reality of eigenvalues; and to examine for integral relations that explain important connections and relationships between different aspects of the problem.

1.1. **The conformable fractional derivative.**

In this part, we give some basic definitions and properties of the conformable fractional calculus theory [1, 14].

Definition 1.1. Consider the function $u : [0, \infty) \to \mathbb{R}$. Then, the "conformable fractional derivative ($\alpha-$ derivative)" of u order $\alpha \in (0, 1]$ is defined by:

$$D_t^{\alpha} u(t) := \lim_{h \to 0} \frac{u(t + ht^{1-\alpha}) - u(t)}{h}.$$

Here, the symbol D_t^{α} is conformable fractional derivative of $\alpha-$order with respect to t.

If u is $\alpha-$differentiable in some $(0, \alpha)$ and $\lim_{t \to 0^+} D_t^{\alpha} u(t)$ exits, then define

$$D_t^{\alpha} u(0) = \lim_{t \to 0^+} D_t^{\alpha} u(t).$$

If u is usual differentiable, then $D_t^{\alpha} u(t) = t^{1-\alpha} u'(t)$.

One can easily show that D_t^{α} satisfies all the properties in the following theorem:

Theorem 1.2. *Let $\alpha \in (0, 1]$ and u, v be $\alpha-$differentiable at a point t. Then:*

- **i.** $D_t^{\alpha} (\xi u + \eta v) = \xi D_t^{\alpha} (u) + \eta D_t^{\alpha} (v)$, *for all* $\xi, \eta \in \mathbb{R}$.
- **ii.** $D_t^{\alpha} (t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
- **iii.** $D_t^{\alpha} (uv) = D_t^{\alpha} (u) v + u D_t^{\alpha} (v)$.
- **iv.** $D_t^{\alpha} \left(\frac{u}{v}\right) = \frac{v D_t^{\alpha}(u) - u D_t^{\alpha}(v)}{v^2}$.
- **v.** $D_t^{\alpha} (c) = 0$, *c is a constant.*
- **vi.** *If u is usual differentiable, then* $D_t^{\alpha} (f)(t) = t^{1-\alpha} \frac{du}{dt}$.

Definition 1.3. Consider the function $u : [0, \infty) \to \mathbb{R}$. Then, the "conformable fractional integral ($\alpha-$ integral)" of u order $\alpha \in (0, 1]$ is defined by:

$$I_{\alpha} u(t) := \int_0^t u(\xi) d_{\alpha} \xi = \int_0^t \xi^{\alpha-1} u(\xi) d\xi$$

for $t > 0$. Integral to the right of the last equality is the usual Riemann integral.

Theorem 1.4. *Consider two $\alpha-$differentiable functions $u, v : [a, b] \to \mathbb{R}$. Then,*

$$\int_a^b u(t) D_t^{\alpha} v(t) d_{\alpha} t = uv\big|_a^b - \int_a^b v(t) D_t^{\alpha} u(t) d_{\alpha} t.$$

This formula is called $\alpha-$integration by parts.

2. Some Spectral Properties

2.1. **Reduction to one parameter problems.** Let $u(x)$ and $u(y)$ be the solution of (1.3) and

$$U(x,y) = u(x)\,u(y)\,. \tag{2.1}$$

By $\alpha-$ differentiating twice equality (2.1) with respect to x and y, we have

$$\begin{aligned} D_x^\alpha\left(D_x^\alpha U(x,y)\right) &= D_x^\alpha\left(D_x^\alpha u(x)\right)u(y)\,,\\ D_y^\alpha\left(D_y^\alpha U(x,y)\right) &= u(x)\,D_y^\alpha\left(D_y^\alpha u(y)\right). \end{aligned} \tag{2.2}$$

Additionally, since $u(x)$ and $u(y)$ satisfy (1.3), the equations

$$\begin{aligned} &D_x^\alpha\left(D_x^\alpha u(x)\right) + \{\lambda + \mu f(x) + g(x)\}\,u(x) = 0,\\ &D_y^\alpha\left(D_y^\alpha u(y)\right) + \{\lambda + \mu f(y) + g(y)\}\,u(y) = 0 \end{aligned}$$

can be written.

If the equations are multiplied by $u(y)$ and $u(x)$, respectively and after subtracted side by side, from (2.2) we get

$$\begin{aligned} D_x^\alpha\left(D_x^\alpha U(x,y)\right) &- D_y^\alpha\left(D_y^\alpha U(x,y)\right)\\ &+ \{\mu\left(f(x) - f(y)\right) + g(x) - g(y)\}\,U(x,y) = 0. \end{aligned} \tag{2.3}$$

As a result, the problem (1.3)-(1.4) is reduced to the one parameter eigenvalue problem. Also, when the values x and y are any of the values a, b, c, using equality (1.4), it is obtained that the boundary conditions of (2.3) are of the form

$$U(x,y) = 0. \tag{2.4}$$

2.2. **Orthogonality properties.** In this section, the orthogonality property are examined separately for the one parameter and two parameter cases.

Let μ be a constant in (1.3); $u_1(t)$ and $u_2(t)$ be solutions to the problem (1.3)-(1.4) for different values of λ_1 and λ_2, respectively. Then, it can be written as follows

$$D_t^\alpha\left(D_t^\alpha u_1(t)\right) + \{\lambda_1 + \mu f(t) + g(t)\}\,u_1(t) = 0,$$

$$D_t^\alpha\left(D_t^\alpha u_2(t)\right) + \{\lambda_2 + \mu f(t) + g(t)\}\,u_2(t) = 0.$$

If above equations are multiplied by $u_2(t)$ and $u_1(t)$, respectively and after subtracted side by side, we obtain

$$D_t^\alpha\left(D_t^\alpha u_1(t)\,u_2(t) - u_1(t)\,D_t^\alpha u_2(t)\right) = \{\lambda_2 - \lambda_1\}\,u_1(t)\,u_2(t)\,.$$

Integrating for $(t_1, t_2) = (a, b)$, $(t_1, t_2) = (a, c)$ or $(t_1, t_2) = (b, c)$ and from boundary conditions (1.4), we get

$$\{\lambda_2 - \lambda_1\}\int_{t_1}^{t_2} u_1(t)\,u_2(t)\,d_\alpha t = 0.$$

Considering $\lambda_1 \neq \lambda_2$ yields

$$\int_{t_1}^{t_2} u_1(t)\, u_2(t)\, d_\alpha t = 0. \tag{2.5}$$

The solutions of equation (1.3) corresponding to different values of λ and μ provide a broader orthogonality relation. This concept of orthogonality is called double orthogonality in the classical sense [7]. In conformable fractional calculus, we express the concept of double orthogonality as follows.

Theorem 2.1. *Suppose that $v_1(t)$ and $v_2(t)$ are the solutions of the problem (1.3)-(1.4) for different values of (λ_1, μ_1) and (λ_2, μ_2), respectively. Then,*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} v_1(x)\, v_1(y)\, v_2(x)\, v_2(y)\,[f(x) - f(y)]\, d_\alpha y d_\alpha x = 0, \tag{2.6}$$

where $\lambda_1 \neq \lambda_2$, $\mu_1 \neq \mu_2$ or both; (x_1, x_2) and (y_1, y_2) represent different values of the values-pairs $(a, b), (a, c), (b, c)$.

Proof. Suppose that $v_1(t)$ and $v_2(t)$ are the solutions of the problem (1.3) and

$$V_i := V_i(x, y) = v_i(x)\, v_i(y),\, i = 1, 2. \tag{2.7}$$

Now, $\alpha-$differentiating twice of the equality (2.7) with respect to x and y, we have

$$D_x^\alpha (D_x^\alpha V_i(x, y)) = D_x^\alpha (D_x^\alpha v_i(x))\, v_i(y),$$
$$D_y^\alpha \left(D_y^\alpha V_i(x, y)\right) = v_i(x)\, D_y^\alpha \left(D_y^\alpha v_i(y)\right)$$

for $i = 1, 2$.

Let us first consider the equality $V_1(x, y) = v_1(x)\, v_1(y)$.

Since $v_1(x)$ and $v_1(y)$ satisfy (1.3), the following equations

$$D_x^\alpha (D_x^\alpha v_1(x)) + \{\lambda_1 + \mu_1 f(x) + g(x)\}\, v_1(x) = 0,$$
$$D_y^\alpha \left(D_y^\alpha v_1(y)\right) + \{\lambda_1 + \mu_1 f(y) + g(y)\}\, v_1(y) = 0$$

are provided. Multiplying these equations by $v_1(y)$, $v_1(x)$, respectively and subtracting gives

$$\begin{aligned} D_x^\alpha (D_x^\alpha V_1(x, y)) &- D_y^\alpha \left(D_y^\alpha V_1(x, y)\right) \\ &+ \{\mu_1 (f(x) - f(y)) + g(x) - g(y)\}\, V_1(x, y) = 0. \end{aligned} \tag{2.8}$$

Similarly, we get

$$\begin{aligned} D_x^\alpha (D_x^\alpha V_2(x, y)) &- D_y^\alpha \left(D_y^\alpha V_2(x, y)\right) \\ &+ \{\mu_2 (f(x) - f(y)) + g(x) - g(y)\}\, V_2(x, y) = 0 \end{aligned} \tag{2.9}$$

for $V_2(x, y) = v_2(x)\, v_2(y)$.

After (2.8) is multiplied by $V_2(x,y)$ and (2.9) is multiplied by $V_1(x,y)$ hence, by subtraction,

$$\left[V_2 D_x^{\alpha}\left(D_x^{\alpha} V_1\right)-V_1 D_x^{\alpha}\left(D_x^{\alpha} V_2\right)\right]+\left[V_1 D_y^{\alpha}\left(D_y^{\alpha} V_2\right)-V_2 D_y^{\alpha}\left(D_y^{\alpha} V_1\right)\right] \\ +\left(\mu_2-\mu_1\right)(f(x)-f(y)) V_1 V_2=0$$

is obtained. $\alpha-$integrating both sides over the interval (x_1,x_2) and (y_1,y_2) on the last equation gives the following

$$\begin{aligned}&\left(\mu_2-\mu_1\right)\int_{x_1}^{x_2}\int_{y_1}^{y_2}(f(x)-f(y)) V_1 V_2 d_\alpha y d_\alpha x\\ &=\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left[V_2 D_x^{\alpha}\left(D_x^{\alpha} V_1\right)-V_1 D_x^{\alpha}\left(D_x^{\alpha} V_2\right)\right] d_\alpha x d_\alpha y+\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[V_1 D_y^{\alpha}\left(D_y^{\alpha} V_2\right)-V_2 D_y^{\alpha}\left(D_y^{\alpha} V_1\right)\right] d_\alpha y d_\alpha x\\ &=\int_{y_1}^{y_2}\int_{x_1}^{x_2} D_x^{\alpha}\left[V_2\left(D_x^{\alpha} V_1\right)-V_1\left(D_x^{\alpha} V_2\right)\right] d_\alpha x d_\alpha y+\int_{x_1}^{x_2}\int_{y_1}^{y_2} D_y^{\alpha}\left[V_1\left(D_y^{\alpha} V_2\right)-V_2\left(D_y^{\alpha} V_1\right)\right] d_\alpha y d_\alpha x\\ &=\int_{y_1}^{y_2}\left[V_2\left(D_x^{\alpha} V_1\right)-V_1\left(D_x^{\alpha} V_2\right)\right]_{x_1}^{x_2} d_\alpha y+\int_{x_1}^{x_2}\left[V_1\left(D_y^{\alpha} V_2\right)-V_2\left(D_y^{\alpha} V_1\right)\right]_{y_1}^{y_2} d_\alpha x.\end{aligned}$$

By virtue of boundary conditions (1.4), we obtain

$$\left(\mu_2-\mu_1\right)\int_{x_1}^{x_2}\int_{y_1}^{y_2}(f(x)-f(y)) V_1 V_2 d_\alpha y d_\alpha x=0. \tag{2.10}$$

Since $\mu_1 \neq \mu_2$, we reach the result (2.6) from (2.7).

On the other hand, let $\mu_1=\mu_2$ and $\lambda_1 \neq \lambda_2$. Then, the left side of the equality (2.6) can be arranged as follows:

$$\begin{aligned}&\int_{x_1}^{x_2} f(x) v_1(x) v_2(x) d_\alpha x \int_{y_1}^{y_2} v_1(y) v_2(y) d_\alpha y\\ &-\int_{y_1}^{y_2} f(y) v_1(y) v_2(y) d_\alpha y \int_{x_1}^{x_2} v_1(x) v_2(x) d_\alpha x.\end{aligned} \tag{2.11}$$

From (2.5), the proof is completed. □

2.3. **The Reality of Eigenvalues.**

Theorem 2.2. *All eigenvalues of problem (1.3)-(1.4) are real.*

Proof. The function $u_0(t)$ is an eigenfunction associated with the complex conjugate pair (λ_0,μ_0) and the function $\overline{u_0(t)}$ is an eigenfunction associated with another complex conjugate pair $\left(\overline{\lambda_0},\overline{\mu_0}\right)$. Thus, if $\mu_1=\mu_0$, $\mu_2=\overline{\mu_0}$, $v_1=u_0, v_2=\overline{u_0}$ is taken into account on the equality (2.10), the double orthogonality results, we have

$$\left(\overline{\mu_0}-\mu_0\right)\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left(f\left(x\right)-f\left(y\right)\right)u_0\left(x\right)u_0\left(y\right)\overline{v_0}\left(x\right)\overline{v_0}\left(y\right)d_\alpha y d_\alpha x=0.$$

Here, the integrand is not zero; since $\overline{\mu_0}-\mu_0=0$, μ_0 is real.

On the other hand, let λ_0 and $\overline{\lambda_0}$ be eigenvalues of the same values of μ_0. Similar operations are performed for the $u_0\left(x\right), \overline{u_0}\left(x\right)$ eigenfunctions corresponding to these values. From (2.11), it is obtained that

$$\int_{x_1}^{x_2}\left|u_0\right|^2 d_\alpha x\neq 0$$

and since $\overline{\lambda_0}=\lambda_0$, λ_0 is real.

Consequently, since λ_0 and μ_0 are arbitrary, all eigenvalues of the problem (1.3)-(1.4) are real. □

3. Some Integral Relations

In this section, two integral relationships are given. These are derived from integral equations satisfied by the solutions of the problem (1.3)-(1.4).

Theorem 3.1. *The function*

$$U\left(t\right)=\int_{x_1}^{x_2}G\left(t,x\right)u\left(x\right)d_\alpha x \tag{3.1}$$

is a solution of the equation (1.3) the following conditions are satisfied

i. *The function $u\left(x\right)$ is a solution of the equation*

$$D_x^\alpha\left(D_x^\alpha u\left(x\right)\right)+\left\{\lambda+\mu f\left(x\right)+g\left(x\right)\right\}u\left(x\right)=0.$$

ii. *The function $G\left(t,x\right)$ is a solution of the conformable partial equation*

$$D_t^\alpha\left(D_t^\alpha G\left(t,x\right)\right)-D_x^\alpha\left(D_x^\alpha G\left(t,x\right)\right)+\left\{\mu\left(f\left(t\right)-f\left(x\right)\right)+g\left(t\right)-g\left(x\right)\right\}G=0.$$

iii. *The function*

$$G\left(t,x\right)D_x^\alpha\left(u\left(x\right)\right)-u\left(x\right)D_x^\alpha\left(G\left(t,x\right)\right)$$

has the same value at the endpoints of the intervals $\left(a,b\right),\left(a,c\right),\left(b,c\right)$

iv. *The integral*

$$\int_{x_1}^{x_2}G\left(t,x\right)u\left(x\right)d_\alpha x$$

exists.

Proof. To complete the proof, we need to show that

$$D_t^\alpha \left(D_t^\alpha U\left(t\right)\right) + \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} U\left(t\right) = 0. \tag{3.2}$$

By the existence of the integral in the condition (iv), the $\alpha-$differential of (3.1) under the integral sign can be taken. Thus,

$$\begin{aligned} &D_t^\alpha \left(D_t^\alpha U\left(t\right)\right) + \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} U\left(t\right) \\ &= \int_{x_1}^{x_2} \left[D_t^\alpha \left(D_t^\alpha G\left(t,x\right)\right) + \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} G\left(t,x\right)\right] u\left(x\right) d_\alpha x \end{aligned} \tag{3.3}$$

is obtained. Besides, from the condition (ii), it can be written as

$$\begin{aligned} D_t^\alpha \left(D_t^\alpha G\left(t,x\right)\right) &+ \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} G\left(t,x\right) \\ &= D_x^\alpha \left(D_x^\alpha G\left(t,x\right)\right) + \left\{\lambda + \mu f\left(x\right) + g\left(x\right)\right\} G\left(t,x\right). \end{aligned} \tag{3.4}$$

The equality (3.4) is also taken into account on the equation (3.3), we have

$$\begin{aligned} &D_t^\alpha \left(D_t^\alpha U\left(t\right)\right) + \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} U\left(t\right) \\ &= \int_{x_1}^{x_2} D_x^\alpha \left(D_x^\alpha G\left(t,x\right)\right) u\left(x\right) d_\alpha x + \int_{x_1}^{x_2} \left\{\lambda + \mu f\left(x\right) + g\left(x\right)\right\} G\left(t,x\right) u\left(x\right) d_\alpha x \end{aligned}$$

When $\alpha-$partial integration is applied to the first integral on the last equality, it is seen to be

$$\begin{aligned} D_t^\alpha \left(D_t^\alpha U\left(t\right)\right) &+ \left\{\lambda + \mu f\left(t\right) + g\left(t\right)\right\} U\left(t\right) \\ &= \left[D_x^\alpha \left(G\left(t,x\right)\right) u\left(x\right) - G\left(t,x\right) D_x^\alpha \left(u\left(x\right)\right)\right]_{x_1}^{x_2} \\ &+ \int_{x_1}^{x_2} \left[D_x^\alpha \left(D_x^\alpha u\left(x\right)\right) + \left\{\lambda + \mu f\left(x\right) + g\left(x\right)\right\} u\left(x\right)\right] G\left(t,x\right) d_\alpha x. \end{aligned}$$

The first term on the last equation and the expression in square brackets in the second term vanishes from conditions (iii) and (i), respectively.

As a result, the proof is completed by obtaining (3.2). □

Theorem 3.2. *The function*

$$U\left(z\right) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[f\left(x\right) - f\left(y\right)\right] P\left(x,y,z\right) u\left(x\right) u\left(y\right) d_\alpha y d_\alpha x \tag{3.5}$$

is a solution of the equation (1.3) the following conditions are satisfied

i. *The functions $u\left(x\right)$and $u\left(y\right)$ are the solutions of the equations*

$$\begin{aligned} D_x^\alpha \left(D_x^\alpha u\left(x\right)\right) + \left\{\lambda + \mu f\left(x\right) + g\left(x\right)\right\} u\left(x\right) = 0, \\ D_y^\alpha \left(D_y^\alpha u\left(y\right)\right) + \left\{\lambda + \mu f\left(y\right) + g\left(y\right)\right\} u\left(y\right) = 0, \end{aligned}$$

respectively.

ii. *$P := P(x,y,z)$ is a solution of following partial equation*

$$\{f(y)-f(z)\}\, D_x^\alpha\left(D_x^\alpha P\right)+\{f(z)-f(x)\}\, D_y^\alpha\left(D_y^\alpha P\right)+\{f(x)-f(y)\}\, D_z^\alpha\left(D_z^\alpha P\right)$$
$$=-\left[g(x)\{f(y)-f(z)\}+g(y)\{f(z)-f(x)\}+g(z)\{f(x)-f(y)\}\right] P.$$

iii. *The equalities*

$$\left[\left(D_x^\alpha P\right) u(x)-P(x,y,z)\, D_x^\alpha u(x)\right]_{x_1}^{x_2}=0,$$
$$\left[\left(D_y^\alpha P\right) u(y)-P(x,y,z)\, D_y^\alpha u(y)\right]_{y_1}^{y_2}=0$$

are satisfied on the intervals $(a,b),(a,c),(b,c)$.

iv. *The integral*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f(x)-f(y)\}\, P(x,y,z) u(x)\, u(y)\, d_\alpha y d_\alpha x$$

is also exists and convergent.

Proof. To complete the proof, we need to show that

$$D_z^\alpha\left(D_z^\alpha U(z)\right)+\{\lambda+\mu f(z)+g(z)\}\, U(z)=0. \tag{3.6}$$

By the existence of the integral in the condition (iv), the $\alpha-$differential of (3.5) under the integral sign can be taken. Therefore,

$$\begin{aligned} & D_z^\alpha\left(D_z^\alpha U(z)\right)+\{\lambda+\mu f(z)+g(z)\}\, U(z) \\ & \quad=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\{f(x)-f(y)\}\left[D_z^\alpha\left(D_z^\alpha P\right)\right. \\ & \quad\quad\left.+\{\lambda+\mu f(z)+g(z)\}\, P(x,y,z)\right] u(x)\, u(y)\, d_\alpha y d_\alpha x \end{aligned} \tag{3.7}$$

is obtained. Besides, with the help of the equality on the condition (ii), the integrand on the equation (3.7) is rearranged as follows

$$\begin{aligned} & \left[\{f(z)-f(y)\}\, D_x^\alpha\left(D_x^\alpha P\right)+\{f(x)-f(z)\}\, D_y^\alpha\left(D_y^\alpha P\right)-F(x,y,z)\right. \\ & \left.+\{\lambda+\mu f(z)+g(z)\}\{f(x)-f(y)\}\, P\right] u(x)\, u(y), \end{aligned} \tag{3.8}$$

where

$$F(x,y,z)=\left[g(x)\{f(y)-f(z)\}+g(y)\{f(z)-f(x)\}+g(z)\{f(x)-f(y)\}\right] P(x,y,z).$$

Then, the function (3.8) is taken into account on the equation (3.7), we have

$$\begin{aligned} & D_z^\alpha \left(D_z^\alpha U(z)\right) + \{\lambda + \mu f(z) + g(z)\} U(z) \\ & = \int_{y_1}^{y_2} \{f(z) - f(y)\} u(y) \int_{x_1}^{x_2} D_x^\alpha \left(D_x^\alpha P\right) u(x)\, d_\alpha x d_\alpha y \\ & + \int_{x_1}^{x_2} \{f(x) - f(z)\} u(x) \int_{y_1}^{y_2} D_y^\alpha \left(D_y^\alpha P\right) u(y)\, d_\alpha y d_\alpha x \\ & - \int_{x_1}^{x_2}\int_{y_1}^{y_2} [F(x,y,z) \\ & - \{\lambda + \mu f(z) + g(z)\} \{f(x) - f(y)\} P(x,y,z)]\, u(x)\, u(y)\, d_\alpha y d_\alpha x. \end{aligned} \tag{3.9}$$

By applying $\alpha-$partial integration twice to the integrals

$$\int_{x_1}^{x_2} D_x^\alpha \left(D_x^\alpha P\right) u(x)\, d_\alpha x, \qquad \int_{y_1}^{y_2} D_y^\alpha \left(D_y^\alpha P\right) u(y)\, d_\alpha y$$

on the equality (3.9) respectively,

$$\int_{x_1}^{x_2} D_x^\alpha \left(D_x^\alpha P\right) u(x)\, d_\alpha x = \left[\left(D_x^\alpha P\right) u(x) - P D_x^\alpha u(x)\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} P D_x^\alpha \left(D_x^\alpha u\right)(x)\, d_\alpha x,$$

$$\int_{y_1}^{y_2} D_y^\alpha \left(D_y^\alpha P\right) u(y)\, d_\alpha y = \left[\left(D_y^\alpha P\right) u(y) - P D_y^\alpha u(y)\right]_{y_1}^{y_2} + \int_{y_1}^{y_2} P D_y^\alpha \left(D_y^\alpha u\right)(y)\, d_\alpha y$$

are obtained. Here, the first terms on the right-hand side of these equalities vanish on the intervals $(a,b), (a,c), (b,c)$. Then, the equality (3.9) is rearranged that

$$\begin{aligned} & D_z^\alpha \left(D_z^\alpha U(z)\right) + \{\lambda + \mu f(z) + g(z)\} U(z) \\ & = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f(z) - f(y)\} u(y) \left\{D_x^\alpha \left(D_x^\alpha u\right)(x) + g(x) u(x)\right\} P d_\alpha y d_\alpha x \\ & + \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{f(x) - f(z)\} u(x) \left\{D_y^\alpha \left(D_y^\alpha u\right)(y) + g(y) u(y)\right\} P d_\alpha y d_\alpha x \\ & + \int_{x_1}^{x_2}\int_{y_1}^{y_2} \{\lambda + \mu f(z)\} \{f(x) - f(y)\} u(x)\, u(y)\, P d_\alpha y d_\alpha x. \end{aligned} \tag{3.10}$$

On the other hand, the condition (i) can be written as follows

$$\begin{aligned} D_x^\alpha \left(D_x^\alpha u\right)(x) + g(x) u(x) &= -\{\lambda + \mu f(x)\} u(x), \\ D_y^\alpha \left(D_y^\alpha u\right)(y) + g(y) u(y) &= -\{\lambda + \mu f(y)\} u(y). \end{aligned}$$

If these last representations are taken into consideration in the equation (3.10), we reach

$$\begin{aligned} &D_z^{\alpha}\left(D_z^{\alpha}U\left(z\right)\right)+\left\{\lambda+\mu f\left(z\right)+g\left(z\right)\right\}U\left(z\right) \\ &=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\left\{f\left(y\right)-f\left(z\right)\right\}\left\{\lambda+\mu f\left(x\right)\right\}+\left\{f\left(z\right)-f\left(x\right)\right\}\left\{\lambda+\mu f\left(y\right)\right\}\right. \\ &\left.+\left\{f\left(x\right)-f\left(y\right)\right\}\left\{\lambda+\mu f\left(z\right)\right\}\right]u\left(x\right)u\left(y\right)Pd_{\alpha}yd_{\alpha}x=0. \end{aligned} \tag{3.11}$$

As a result, the proof is completed by obtaining (3.6). □

4. Conclusion

In this study, the focus was initially on classical two-parameter eigenvalue problems. These problems with two parameters have been transformed into one-parameter eigenvalue problems using specific methodologies. Throughout this transformation process, the orthogonal properties of the eigenvalue problems have been emphasized, and certain integral relationships have been established. The recalculation of transitions in the relevant theorems was necessary to obtain the main results. Consequently, the two-parameter eigenvalue problems were formulated using conformable fractional derivatives, and their related properties were examined. Subsequently, these problems were transformed from a two-parameter to a one-parameter format with the help to the properties of conformable fractional derivatives. The research particularly emphasized the reality of the eigenvalues and presented specific integral relationships. And, it was seen to coincide with Arscott's work [7] when the case $\alpha = 1$.

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