Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 27(2), 654-666, (2025)**

DOI:10.25092/baunfbed. 1549042 **J. BAUN Inst. Sci. Technol., 27(2), 654-666, (2025)**

An alternating inertial forward-backward-forward algorithm for solving monotone inclusion problem and its applications

Ebru ALTIPARMAK*

Erzurum Technical University Faculty of Science, Department of Mathematics, Erzurum.

Geliş Tarihi (Received Date): 13.09.2024
Kabul Tarihi (Accepted Date): 05.05.2025

Abstract

We present an alternating inertial forward-backward-forward algorithm designed to find the zeros of the sum of a maximally monotone operator and a single-valued monotone operator that is also Lipschitz continuous. This study aims to extend Tseng's forward-backward-forward algorithm by incorporating alternating inertial effects. We then apply our enhanced algorithm to address convex minimization problems. Key topics include the monotone inclusion problem, forward-backward-forward algorithm, the alternating inertial method, and convex minimization problems. Lastly, we explore the application of our proposed approach in image restoration, emphasizing its effectiveness and adaptability.

Keywords: *Monotone inclusion problem, forward-backward-forward algorithm, alternating inertial method, convex minimization problem, image restoration problem.*

Monoton kapsama problemini çözmek için alternatif eylemsiz ileri-geri-ileri ayırma algoritması ve uygulamaları

Öz

Maksimum monoton bir operatör ile tek değerli, aynı zamanda Lipschitz sürekli olan monoton bir operatörün toplamının sıfırlarını bulmak amacıyla tasarlanmış alternatif eylemsiz ileri-geri-ileri algoritmasını sunuyoruz. Bu çalışma, Tseng'in ileri-geri-ileri algoritmasını alternatif atalet etkilerini ekleyerek genişletmeyi amaçlamaktadır. Ardından, geliştirilmiş algoritmamızı konveks minimizasyon problemlerini ele almak için

*Ebru Altıparmak, ebru.altiparmak@erzurum.edu.tr, https://orcid.org/0000-0001-6722-0807

uyguluyoruz. Ana konular arasında monoton kapsama problemi, ileri-geri-ileri algoritması, alternatif eylemsiz yöntemi ve konveks minimizasyon problemleri yer almaktadır. Son olarak, önerdiğimiz yaklaşımın görüntü iyileştirme uygulamasını inceleyerek etkinliğini ve uygulanabilirliğini vurguluyoruz.

Anahtar kelimeler: *Monoton kapsama problemi, ileri-geri-ileri algoritması, alternatif eylemsiz metot, konveks minimizasyon problemi, görüntü iyileştirme problemi.*

1. Introduction

Let H be a real Hilbert space with the inner product $\langle .,. \rangle$, C be a nonempty closed, and convex subset of H and the induced norm $\| x \|$. This study aims to investigate the monotone inclusion problems as follows:

$$\text{find } x^* \in H \text{ such that } 0 \in (A+B)x^* \tag{1}$$

where $A: H \to H$ be a single-valued mapping and $B: H \to 2^H$ multi-valued mapping. This problem is of significant interest because it is central to various mathematical issues, including convex programming, variational inequalities, the split feasibility problem, and minimization problems, as discussed in references [1,2, 3, 4, 5]. It also has applications in machine learning, image processing, and linear inverse problems, see for detail these references [6,7,8,9,10,11,12,13,14,15,16].

Proximal point algorithm is a highly effective tool for solving monotone inclusion problems. To enhance its convergence, the literature presents an inertial proximal point algorithm. Among the most widely used methods for addressing monotone inclusion problems is the forward-backward algorithm, introduced by Lions and Mercier [17], which is formulated as:

$$x_{n+1} = J_{\lambda_k A}(I - \lambda_k B)x_k,$$

The forward-backward-forward algorithm was suggested by Tseng [18] and it generates an iterative the sequence $\{x_k\}$ via

$$\begin{cases} y_k = J_{\lambda_k A}(I - \lambda_k B)x_k, \\ x_{k+1} = y_k - \lambda_k(By_k - Bx_k), \quad \forall k \geq 0 \end{cases}$$

where $x_0 \in H$ is the starting point. The sequence $\{x_k\}$ converges weakly to a solution of if the sequence of stepsizes $\{\lambda_k\}$ is chosen in the interval $\left(0, \frac{1}{L}\right)$, where $L > 0$ is the Lipschitz constant of B. In [19], an inertial version of the forward-backward-forward primal-dual splitting algorithm was introduced.

In 2020, Bot et al. [19] introduced a relaxed inertial version of the following forward-backward-forward algorithm (RIFBF), to solve monotone inclusion problems:

$$\begin{cases} z_k = x_k + \alpha_k(x_k - x_{k-1}) \\ y_k = J_{\lambda_k A}(I - \lambda_k B)z_k \\ x_{n+1} = (1 - \rho_k)z_k + \rho_k\big(y_k - \lambda_k(By_k - Bz_k)\big), \forall k \geq 1 \end{cases}$$

where $x_{0,}\ x_1 \in H$ are initial point, $\{\lambda_k\}$ and $\{\rho_k\}$ are sequences of positive numbers, and $\{\alpha_k\}$ is a sequence of nonnegative numbers, namely, inertial term. By taking $\rho_k = 1$ in RIFBF, the inertial forward-backward-forward Algorithm [20] can be obtained as follows:

$$\begin{cases} z_k = x_k + \alpha_k(x_k - x_{k-1}) \\ y_k = J_{\lambda_k A}(I - \lambda_k B)z_k \\ x_{n+1} = y_k - \lambda_k(By_k - Bz_k), \forall k \geq 1 \end{cases}$$

It is our aim in this study to propose an alternating inertial forward-backward-forward splitting algorithm in which Fejer monotonicity of $\|x_n - x\|$, namely, $\|x_{n+1} - x\| \leq \|x_n - x\|$, $x \in C \subset H$ is regained to some extent [21,22,23]. The article is structured as follows: we begin by recalling some basic definitions and lemmas, followed by the presentation and analysis of our algorithms in Section 3. In Section 4 and section 5, we discuss applications of the proposed method.

2. Preliminaries

In this part, let H be a real Hilbert space and C be a nonempty closed, and convex subset of H. Then, the weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, and the strong convergence of $\{x_n\}$ to x is denoted by $x_n \to x$ as $n \to \infty$.

Definition 1: [24] Let $A: H \to H$ be a mapping. A is said to be, for all $u, v \in H$,

1. Monotone if

$\langle Au - Av, u - v \rangle \geq 0,$

2. $L-$Lipschitz continuous on H if there exists a constant $L > 0$ such that

$\|Au - Av\| \leq L\|u - v\|.$

If $L = 1$, then A is called nonexpansive.

Definition 2: A mapping $B: H \to 2^H$ is said to be monotone, if for every $u, v \in H$, $x \in Bu$ and $y \in Bv$, $\langle x - y, u - v \rangle \geq 0$. Moreover, B is said to be maximal monotone if it is monotone and if for every $(x, u) \in H, \langle x - y, u - v \rangle \geq 0$ for every $(v, y) \in Graph(B)$, $x \in Bu$.

Lemma 1: [24] Let $A: H \to H$ be a operator and let $B: H \to 2^H$ be a maximal monotone operator. Define for $\lambda > 0$, $T_\lambda := (I + \lambda B)^{-1}(I - \lambda A)$. Then we have, for all $\lambda > 0$, $F(T_\lambda) = (A + B)^{-1}(0)$.

Lemma 2: [24] Let $A: H \to H$ be a Lipschitz continuous and monotone operator and let $B: H \to 2^H$ be a maximal monotone operator. Then, $A + B$ is a maximal monotone operator.

Lemma 3: [25] Let H be a real Hilbert space. Hence, the following properties hold:

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2,$

2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$

3. $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$

for all $x, y \in H$, and $\lambda \in [0,1]$.

Definition 3: A sequence $\{x_n\} \subset H$ is said to converges weakly to $z \in H$ if

$$\lim_{n\to\infty} \langle x_n, y\rangle = \langle z, y\rangle, \qquad \forall y \in H.$$

Lemma 4: [1] Let C be a nonempty subset of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:

1. for every $x \in C$, $\lim_{n\to\infty}\|x_n - x\|$ exists,
2. every sequential weak cluster point of $\{x_n\}$ is in C,

Then, the sequence $\{x_n\}$ converges weakly to a point in C.

Lemma 5: Let C be a nonempty subset of H, and let $\{x_n\}$ be a sequence in H. Then x_n is Fejèr monotone with respect to a set C if

$$\|x_{n+1} - x^*\| \le \|x_n - x^*\|, \forall x^* \in \mathrm{C}.$$

3. Main result

In this part, we examine an alternating inertial forward-backward-forward splitting algorithm, as well as a self-adaptive alternating inertial forward-backward-forward splitting algorithm, to address the monotone inclusion problem. The convergence of given algorithms is proven under the following conditions.

Condition 1: The solution set of monotone inclusion problem is a nonempty closed and convex subset of H, namely, $\Omega := (A + B)^{-1}(0) \neq \emptyset$.

Condition 2: $A: H \to H$ is monotone and $L-$Lipchitz continuous, and $B: H \to 2^H$ maximally monotone.

Algorithm 6 Alternating Inertial Forward-Backward-Forward Algorithm
Initialization: Choose $\mu \in (0,1)$, $0 \le \alpha_n \le \alpha < \frac{1-\mu}{1+\mu}$.
Iterative Steps:
Step 1: Compute

$$w_n = \begin{cases} x_n, & when\ n \text{ is } even \\ x_n + \alpha_n(x_n - x_{n-1}), & when\ n \text{ is } odd. \end{cases}$$

Step 2: Compute
$y_n = J_{\lambda_n, B}(I - \lambda_n A)w_n$.
If $w_n = y_n$, then stop and y_n is a solution of problem. Else, go to Step 3.
Step 3: Compute
$x_{n+1} = y_n - \lambda_n(Ay_n - Aw_n)$,
where the stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \begin{cases} min\left\{\lambda_n, \frac{\mu\|y_n - w_n\|}{\|Ay_n - Aw_n\|}\right\}, & Ay_n \neq Aw_n \\ \lambda_n, & otherwise. \end{cases} \tag{2}$$

Remark 1: Let $\mu \in (0,1)$ and $\lambda > 0$. The sequence $\{\lambda_n\}$ generated by (2) is $\lambda_{n+1} \le \lambda_n$,namely, nonicreasing and

$\lim_{n\to\infty}\lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}$.

Also,

$\|Ay_n - Aw_n\| \leq \frac{\mu}{\lambda_{n+1}}\|y_n - w_n\|, \forall n \geq 1.$

Lemma 7: Assume that Condition 1 and Condition 2 hold and let $\{x_n\}$ is generated by Algorithm 6. Then $\{x_{2n}\}$ is Fejer monotone with respect to Ω and $lim_{n\to\infty}\|x_{2n} - p\|$ exist, where $p \in \Omega$.

Proof: Let $p \in \Omega$. Note that,

$$\|y_{2n+1} - p\|^2 = \|w_{2n+1} - p\|^2 + 2\langle y_{2n+1} - w_{2n+1}, w_{2n+1} - p\rangle + \|w_{2n+1} - y_{2n+1}\|^2 \quad (3)$$

From the definition of $\{\lambda_n\}$ and definition of x_{n+1}, we have that

$$\begin{aligned}
\|x_{2n+2} - p\|^2 &= \|y_{2n+1} - \lambda_{2n+1}(Ay_{2n+1} - Aw_{2n+1}) - p\|^2 \\
&= \|y_{2n+1} - p\|^2 - 2\lambda_{2n+1}\langle y_{2n+1} - p, Ay_{2n+1} - Aw_{2n+1}\rangle \\
&\quad + \lambda_{2n+1}^2\|Ay_{2n+1} - Aw_{2n+1}\|^2 \\
&= \|w_{2n+1} - p\|^2 + 2\langle y_{2n+1} - w_{2n+1}, w_{2n+1} - p\rangle \\
&\quad + \|w_{2n+1} - y_{2n+1}\|^2 - 2\lambda_{2n+1}\langle y_{2n+1} - p, Ay_{2n+1} - Aw_{2n+1}\rangle \\
&= \|w_{2n+1} - p\|^2 - 2\langle y_{2n+1} - w_{2n+1}, y_{2n+1} - w_{2n+1}\rangle \\
&\quad + \lambda_{2n+1}^2\|Ay_{2n+1} - Aw_{2n+1}\|^2 \\
&\quad + 2\langle y_{2n+1} - w_{2n+1}, y_{2n+1} - p\rangle + \lambda_{2n+1}^2\|Ay_{2n+1} - Aw_{2n+1}\| \\
&\quad + \|w_{2n+1} - y_{2n+1}\|^2 - 2\lambda_{2n+1}\langle y_{2n+1} - p, Ay_{2n+1} - Aw_{2n+1}\rangle \\
&= \|w_{2n+1} - p\|^2 - \|y_{2n+1} - w_{2n+1}\|^2 + \lambda_{2n+1}^2\|Ay_{2n+1} - Aw_{2n+1}\| \\
&\quad - 2\langle y_{2n+1} - p, w_{2n+1} - y_{2n+1} + \lambda_{2n+1}(Ay_{2n+1} - Aw_{2n+1})\rangle \\
&= \|w_{2n+1} - p\|^2 - \|y_{2n+1} - w_{2n+1}\|^2 + \mu^2\frac{\lambda_{2n+1}^2}{\lambda_{2n+2}^2}\|y_{2n+1} - w_{2n+1}\|^2 \\
&\quad - 2\langle y_{2n+1} - p, w_{2n+1} - y_{2n+1} + \lambda_{2n+1}(Ay_{2n+1} - Aw_{2n+1})\rangle \\
&= \|w_{2n+1} - p\|^2 - \left(1 - \mu^2\frac{\lambda_{2n+1}^2}{\lambda_{2n+2}^2}\right)\|y_{2n+1} - w_{2n+1}\|^2 \\
&\quad - 2\langle y_{2n+1} - p, w_{2n+1} - y_{2n+1} + \lambda_{2n+1}(Ay_{2n+1} - Aw_{2n+1})\rangle. \quad (4)
\end{aligned}$$

In what follows, we are going to show that

$$\langle y_{2n+1} - p, w_{2n+1} - y_{2n+1} + \lambda_{2n+1}(Ay_{2n+1} - Aw_{2n+1})\rangle \geq 0. \quad (5)$$

Since $y_{2n+1} = (I + \lambda_{2n+1}B)^{-1}(I - \lambda_{2n+1}A)w_{2n+1}$, we get $(I - \lambda_{2n+1}A)w_{2n+1} \in (I + \lambda_{2n+1}B)y_{2n+1}$ and since B maximal monotone there exists $v_{2n+1} \in By_{2n+1}$ such that $(I - \lambda_{2n+1}A)w_{2n+1} = y_{2n+1} + \lambda_{2n+1}v_{2n+1}$.

This means that

$$v_{2n+1} = \frac{1}{\lambda_{2n+1}}(w_{2n+1} - y_{2n+1} - \lambda_{2n+1}Aw_{2n+1}) \quad (6)$$

On the other part, we have $0 \in (A+B)p$ and $Ay_{2n+1} + v_{2n+1} \in (A+B)y_{2n+1}$. Since $A + B$ is maximal monotone, we also have

$$\langle Ay_{2n+1} + v_{2n+1}, y_{2n+1} - p\rangle \geq 0. \quad (7)$$

By equation (6) and inequality (7), we obtain

$$\frac{1}{\lambda_{2n+1}}\langle w_{2n+1} - y_{2n+1} - \lambda_{2n+1}Aw_{2n+1} + \lambda_{2n+1}Ay_{2n+1}, y_{2n+1} - p\rangle \geq 0.$$

This shows that

$$\langle w_{2n+1} - y_{2n+1} - \lambda_{2n+1}(Aw_{2n+1} - Ay_{2n+1}), y_{2n+1} - p\rangle \geq 0.$$

By combining (4) with (5), we get

$$\|x_{2n+2}-p\|^2 \le \|w_{2n+1}-p\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n+1}-w_{2n+1}\|^2. \tag{8}$$

Now, we observe the following norm,

$$\|w_{2n+1}-p\|^2 = \|x_{2n+1}+\alpha_{2n+1}(x_{2n+1}-x_{2n})-p\|^2$$
$$= \|(1+\alpha_{2n+1})(x_{2n+1}-p)-\alpha_{2n+1}(x_{2n}-p)\|^2$$
$$= (1+\alpha_{2n+1})\|x_{2n+1}-p\|^2 - \alpha_{2n+1}\|x_{2n}-p\|^2 + \alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2$$

(9)

By using a similar line of reasoning as in the derivation of (9), one can demonstrate that

$$\|x_{2n+1}-p\|^2 \le \|w_{2n}-p\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|^2$$
$$= \|x_{2n}-p\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|^2. \tag{10}$$

By using (9) and (10), we obtain

$$\|w_{2n+1}-p\|^2 \le (1+\alpha_{2n+1})\left[\|x_{2n}-p\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|\right]$$
$$-\alpha_{2n+1}\|x_{2n}-p\|^2 + \alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2$$
$$\le \|x_{2n}-p\|^2 - (1+\alpha_{2n+1})\left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|$$
$$+\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2. \tag{11}$$

It follows from (11) that

$$\|x_{2n+2}-p\|^2 \le \|x_{2n}-p\|^2 - (1+\alpha_{2n+1})\left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|^2$$
$$+\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n+1}}{\lambda^2_{2n+2}}\right)\|y_{2n+1}-w_{2n+1}\|^2.$$

Observe that

$$\|x_{2n+1}-x_{2n}\| = \|y_{2n}-\lambda_{2n}(Ay_{2n}-Aw_{2n})-x_{2n}\|$$
$$\le \|y_{2n}-x_{2n}\| + \lambda_{2n}\|Ay_{2n}-Aw_{2n}\|$$
$$\le \left(1+\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)\|y_{2n}-x_{2n}\|. \tag{12}$$

By (12) we have

$$\|x_{2n+2}-p\|^2 \le \|x_{2n}-p\|^2 - (1+\alpha_{2n+1})\left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\|y_{2n}-w_{2n}\|^2$$
$$+\alpha_{2n+1}(1+\alpha_{2n+1})\left(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)^2\|y_{2n}-w_{2n}\|^2$$
$$-\left(1-\mu^2\frac{\lambda^2_{2n+1}}{\lambda^2_{2n+2}}\right)\|y_{2n+1}-w_{2n+1}\|^2$$
$$= \|x_{2n}-p\|^2 - (1+\alpha_{2n+1})\left[\left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)\right.$$
$$\left.-\alpha_{2n+1}\left(1-\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)^2\right]\|y_{2n}-w_{2n}\|^2 - \left(1-\mu^2\frac{\lambda^2_{2n+1}}{\lambda^2_{2n+2}}\right)\|y_{2n+1}-w_{2n+1}\|^2. \tag{13}$$

Since $\lambda_n \to \lambda > 0$, $\mu \in (0,1)$, and $0 \le \alpha_n \le \alpha < \frac{1-\mu}{1+\mu}$ we have

$$\lim_{n\to\infty}\left[\left(1-\mu^2\frac{\lambda^2_{2n}}{\lambda^2_{2n+1}}\right)-\alpha\left(1+\mu\frac{\lambda_{2n}}{\lambda_{2n+1}}\right)^2\right] = (1-\mu^2)-\alpha(1+\mu)^2 > 0, \tag{14}$$

and

$$\lim_{n\to\infty}\left[\left(1-\mu^2\frac{\lambda_{2n}^2}{\lambda_{2n+1}^2}\right)\right]=(1-\mu^2)>0. \quad (15)$$

Let ϵ be fixed such that

$0<\epsilon<(1-\mu^2)-\alpha(1+\mu)^2\le(1-\mu^2)$.

Consequently, based on equations (14) and (15) there exists $n_0\in\mathbb{N}$ such that

$$\left(1-\mu^2\frac{\lambda_{2n+1}^2}{\lambda_{2n+2}^2}\right)>\left(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)-\alpha\left(1+\mu\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)^2\ge\epsilon>0,\forall n\ge n_0.$$

From (12) we get,

$$\|x_{2n+2}-p\|^2\le\|x_{2n}-p\|^2-(1+\alpha_{2n+1})\epsilon\|y_{2n}-w_{2n}\|^2$$
$$-\epsilon\|y_{2n+1}-w_{2n+1}\|^2, \quad (16)$$

which implies

$\|x_{2n+2}-p\|\le\|x_{2n}-p\|$.

Therefore, $\lim_{n\to\infty}\|x_{2n}-p\|$ exists and this implies that $\{\|x_{2n}-p\|\}$ and $\{x_{2n}\}$ are bounded. Consequently, we have from (16),

$$\lim_{n\to\infty}\|y_{2n}-x_{2n}\|=0 \quad (17)$$

and

$$\lim_{n\to\infty}\|y_{2n+1}-w_{2n+1}\|=0. \quad (18)$$

Also, from (11) we conclude that

$$\lim_{n\to\infty}\|x_{2n+1}-x_{2n}\|=0. \quad (19)$$

Lemma 8: Suppose that Condition 1 and Condition 2 hold, and $\{x_n\}$ is generated by Algorithm 6. Let $x^*\in H$ denote the weak limit of the subsequence $\{x_{2n_j}\}$ of $\{x_{2n}\}$. Then $x^*\in\Omega$.

Proof: Given that $\{x_{2n}\}$ is bounded, there exists a subsequence $\{x_{2n_j}\}$ of $\{x_{2n}\}$ such that $x_{2n_j}\rightharpoonup x^*$. In view of (16) we can choose a subequence $\{y_{2n_j}\}$ of $\{y_{2n}\}$ such that $y_{2n_j}\rightharpoonup x^*$. Let $(u,v)\in Grap(A+B)$ that is, $u-Av\in Bv$ and we have

$$y_{2n_j}=\left(I+\lambda_{2n_j}B\right)^{-1}\left(I-\lambda_{2n_j}A\right)x_{2n_j}$$

also we have

$$\left(I-\lambda_{2n_j}A\right)x_{2n_j}\in\left(I+\lambda_{2n_j}B\right)$$

which implies that

$$\frac{1}{\lambda_{2n_j}}\left(x_{2n_j}-y_{2n_j}-\lambda_{2n_j}Ax_{2n_j}\right)\in By_{2n_j}.$$

By the maximal monotonicity of B we have

$$\left\langle v-y_{2n_j},u-Av-\frac{1}{\lambda_{2n_j}}\left(x_{2n_j}-y_{2n_j}-\lambda_{2n_j}Ax_{2n_j}\right)\right\rangle\ge 0$$

thus

$$\left\langle v-y_{2n_j},u\right\rangle\ge\left\langle v-y_{2n_j},Av-\frac{1}{\lambda_{2n_j}}\left(x_{2n_j}-y_{2n_j}-\lambda_{2n_j}Ax_{2n_j}\right)\right\rangle$$
$$=\left\langle v-y_{2n_j},Av-Ax_{2n_j}\right\rangle+\left\langle v-y_{2n_j},\frac{1}{\lambda_{2n_j}}\left(x_{2n_j}-y_{2n_j}\right)\right\rangle$$
$$=\left\langle v-y_{2n_j},Av-Ay_{2n_j}\right\rangle+\left\langle v-y_{2n_j},Ay_{2n_j}-Ax_{2n_j}\right\rangle$$

$$+\left\langle v - y_{2n_j}, \frac{1}{\lambda_{2n_j}}\left(x_{2n_j} - y_{2n_j}\right)\right\rangle$$
$$\geq \left\langle v - y_{2n_j}, Ay_{2n_j} - Ax_{2n_j}\right\rangle + \left\langle v - y_{2n_j}, \frac{1}{\lambda_{2n_j}}\left(x_{2n_j} - y_{2n_j}\right)\right\rangle.$$

Since $\lim_{n\to\infty}\|y_{2n} - x_{2n}\| = 0$ and A is Lipchitz continuous, we obtain

$$\lim_{j\to\infty}\left\|Ay_{2n_j} - Ax_{2nj}\right\| = 0$$

which together with $\lim_{n\to\infty}\lambda_n = \lambda$ we have

$$\langle v - p, u\rangle = \lim_{j\to\infty}\left\langle v - y_{2n_j}, u\right\rangle \geq 0.$$

Since the maximal monotonicity of $A + B$, we obtain $x^* \in (A + B)^{-1}(0)$.

Theorem 9: Suppose that Conditions 1 and 2 hold and $\{x_n\}$ is generated by Algorithm 6. Then, $\{x_n\}$ converges weakly to a element in Ω.

Proof: From Lemma 7, since $\{x_{2n}\}$ is bounded, hence $\{x_{2n}\}$ has weakly convergent subsequences. Assume $x^* \in H$ denotes the weak limit of such a subsequence $\left\{x_{2n_j}\right\}$ of $\{x_{2n}\}$. By Lemma 8, we have $x^* \in \Omega$. Also, by Lemma 7, we get $\lim_{n\to\infty}\|x_{2n} - p\|$. This indicates that, based on Lemma 4, the weak convergence of the sequence $\{x_{2n}\}$ to some element in the set Ω has been established. Moreover, from (17) we have for all $\overline{x} \in H$,

$$|\langle x_{2n+1} - \overline{x}, y\rangle| = |\langle x_{2n+1} - p + x_{2n} - x_{2n}, \overline{x}\rangle|$$
$$\leq |\langle x_{2n} - p, \overline{x}\rangle| + |\langle x_{2n+1} - p, \overline{x}\rangle|$$
$$\leq |\langle x_{2n} - p, \overline{x}\rangle| + \|x_{2n+1} - x_{2n}\|\|\overline{x}\| \to 0, n \to \infty.$$

Therefore, $\{x_{2n+1}\}$ converges weakly to x^* in Ω. Hence, the sequence $\{x_n\}$ converges weakly to a element $x^* \in (A + B)^{-1}(0)$.

4. Application to convex minimization problem

Let us consider the convex minimization problem as follows:

$$f(x^*) + g(x^*) = \min_{x\in H}\{f(x) + g(x)\}$$

Let $g: H \to R$ be a proper, convex, and lower semi-continuous function, $f: H \to R$ be convex and differentiable with a gradient ∇f that has an L-Lipschitz constant. According to the Baillon-Haddad theorem, ∇f is cocoercive with respect to L^{-1}, and the subdifferential of g, ∂g is maximally monotone. A point x^* is a solution of the convex minimization problem if and only if $0 \in \nabla f(x^*) + \partial g(x^*)$. By setting $A = \nabla f$ and $B = \partial g$ in Algorithm 6, we can derive the following algorithm and corresponding theorem.

Algorithm 10

Initialization: Choose $\mu \in (0,1)$, $0 \leq \alpha_n \leq \alpha < \frac{1-\mu}{1+\mu}$.

Iterative Steps:

Step 1: Compute

$$w_n = \begin{cases} x_n, & when\ n \text{ is } even \\ x_n + \alpha_n(x_n - x_{n-1}), & when\ n \text{ is } odd. \end{cases}$$

Step 2: Compute
$y_n = J_{\lambda_n,\partial g}(w_n - \lambda_n \nabla f(w_n)).$
If $w_n = y_n$, then stop and y_n is a solution of problem. Else, go to Step 3.

Step 3: Compute
$x_{n+1} = y_n - \lambda_n(\nabla f(y_n) - \nabla f(w_n)),$
where the stepsize sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \begin{cases} min\left\{\lambda_n, \dfrac{\mu\|y_n - w_n\|}{\|\nabla f(y_n) - \nabla f(w_n)\|}\right\}, & \nabla f(y_n) \neq \nabla f(w_n) \\ \lambda_n, & otherwise. \end{cases}$$

Theorem 11: Let $f: H \to R$ be a convex, differentiable with L −Lipschitz constant of ∇f and let $g: H \to R$ be a proper convex and lower semi-continuous function. Suppose that the solution set of convex minimization problem is nonempty. The parameters are subject to the same conditions as those stated in Theorem 9. Let $\{x_n\}$ be a sequence generated by Algorithm 10. Hence, $\{x_n\}$ converges weakly to x^*, which is a solution of the convex minimization problem.

5. Application to image restoration problem

This section demonstrates the application of the alternating inertial forward-backward-forward algorithm (Alternating Inertial FBF Algorithm) to the image restoration problem. Moreover, a comparative analysis is carried out between Alternating Inertial FBF Algorithm and Tseng's Algorithm.
The inverse problem in the following form serves to define a image restoration problem:

$$z = Ax + \kappa \tag{20}$$

where $x \in \mathbb{R}^d$ is original image, $A: \mathbb{R}^d \to \mathbb{R}^m$ is a linear operator, $z \in \mathbb{R}^m$ is observed image and κ is the additive noise. It is widely recognized that problem (20) is approximately equivalent to several different optimization problems. Moreover, the l_1-norm is frequently employed as a regularization technique to address such problems. Consequently, the image restoration problem (20) can be reformulated as an l_1-regularization problem, expressed as follows:

$$\min_{x\in\mathbb{R}^d} \left\{\frac{1}{2}\|Ax - z\|^2 + \sigma\|x\|_1\right\}. \tag{21}$$

where $\sigma > 0$ is a regularization parameter. On the other hand, For $f(x) = \frac{1}{2}\|Ax - z\|^2$ and $g(x) = \sigma\|x\|_1$, the convex minimization problem can be reduced to l_1- regularization problem. Based on this selection, the gradient of f, which is Lipschitz continuous, takes the form $\nabla f(x) = A^T(Ax - b)$, where A^T denotes the transpose of A.

Now, we show that Alternating Inertial FBF Algorithm is used to solve the image restoration problem (20) and also that this algorithm is compared to Tseng's Algorithm. In all comparison, we consider the motion blur functions and add random noise to the test Lena image. To evaluate the quality of the restored images, we use the signal-to-noise ratio (SNR), which is defined as follows:

$$SNR = 20\log\frac{\|x\|_2}{\|x-x_n\|_2},$$

where x and x_n are the original image and the estimated image at iteration n, respectively. All algorithms were implemented in MATLAB R2024b on an Asus computer equipped with an Intel(R) Core(TM) i9-14900HX 2.20 GHz processor and 32.0 GB of RAM. We compare Alternating Inertial FBF Algorithm with Tseng's Algorithm. We set $\alpha_n = 0.01$, $\lambda_n = 0.1$, maximum iteration=25000 and the regularization parameter $\sigma = 0.0001$. Figure 1 and Table 1 display the visual and numerical results corresponding to these selections

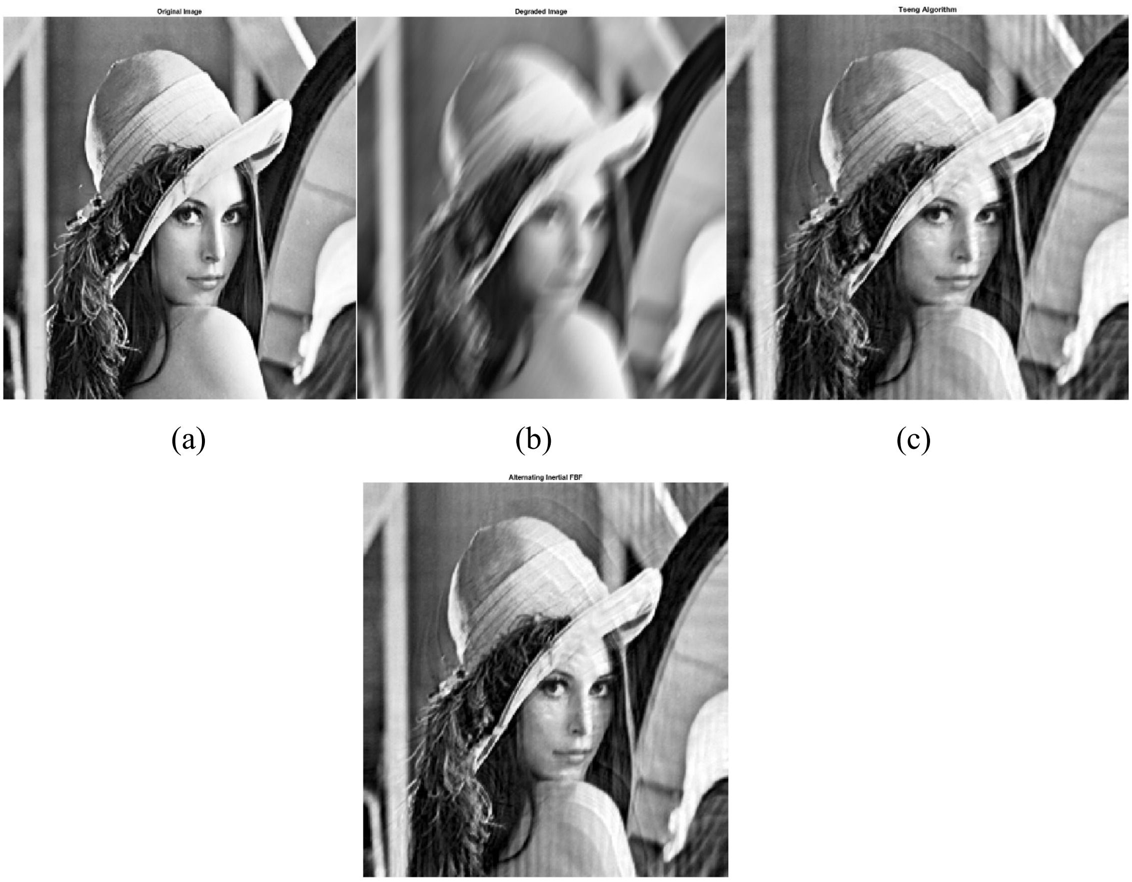

Figure 1. (a) Lena Image (b) Degraded image (c) Restorated Image by the Tseng's Algorithm (d) Restorated Image by the Alternating Inertial FBF Algorithm.

Table 1. Table of SNR values for the Lena image

	Alternating Inertial FBF Algorithm	Tseng’s Algorithm
1	15.827333792615972	15.830448675610187
100	17.938677723877210	17.936446662849352
250	19.295244901185410	19.276080054466462
500	20.560027898189340	20.544342728972160
750	21.311493957050523	21.301216323622512
1000	21.825211913004175	21.817680738151750
2000	22.911456629047414	22.906426728660062
2500	23.203808851323588	23.183338688201420

As seen from the above table, the SNR value of Alternating Inertial FBF Algorithm is better than the Tseng’s Algorithm, meaning it demonstrates better performance in image restoration.

6. Conclusion

In this study, we establish weak convergence results for the alternating forward-backward-forward splitting algorithm in Hilbert spaces, demonstrating its effectiveness in solving monotone inclusion problems. Furthermore, we apply our proposed algorithm to address convex minimization problems, showing that it can handle composite optimization tasks where the objective function is a sum of a differentiable function with a Lipschitz continuous gradient and a proper, convex, lower semi-continuous function. The numerical results demonstrate that the Alternating Inertial FBF Algorithm restores images with a higher SNR value than Tseng’s Algorithm, suggesting it outperforms Tseng's Algorithm in image restoration. Also, our findings can be extended algorithm’s applicability to a wide range of practical optimization problems in various fields, such as machine learning, and signal recovery.

References

[1] Opial, Z., Weak convergence of the sequence of successive approximations for nonexpansive mappings. **Bulletin of the American Mathematical Society,** 73(4), 591-597, (1967).

[2] Combettes, P.L., andWajs, V.R., Signal recovery by proximal forward-backward splitting. **Multiscale modeling and simulation** 4(4), 1168-1200, (2005).

[3] Daubechies, I., Defrise, M., and De Mol, C., An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. **Communications on Pure and Applied Mathematics: Journal Issued by the Courant Institute of Mathematical Sciences** 57(11), 1413-1457, (2004).

[4] Duchi, J., and Singer, Y., Efficient online and batch learning using forward backward splitting. **The Journal of Machine Learning Research,** 10, 2899-2934, (2009).

[5] Padcharoen, A., Kitkuan, D., Kumam, and W., Kumam, P., Tseng methods with inertial for solving inclusion problems and application to image deblurring and image recovery problems. **Computational and Mathematical Methods** 3(3), 1088, (2021).

[6] Altiparmak, E., and Karahan, I., A new preconditioning algorithm for finding a zero of the sum of two monotone operators and its application to image restoration problems. **International Journal of Computer Mathematics**, 99(12), 2482-2498, (2022).

[7] Altiparmak, E., and Karahan, I., A modified preconditioning algorithm for solving monotone inclusion problem and application to image restoration problem**. Scientific Bulletin-University Politehnica of Bucharest A**, 84, 81-92, (2022).

[8] Ungchittrakool, K., Cho, Y. J., Plubtieng, S., and Thammasiri, P., Accelerated Mann-type algorithm via two-step inertial points for solving a fixed point problem of a nonexpansive mapping and application to image restoration problems. **Numerical Computations: Theory and Algorithms NUMTA** 2023, 208, (2023).

[9] Jolaoso, L. O., Sunthrayuth, P., Cholamjiak, P., and Cho, Y. J., Inertial projection and contraction methods for solving variational inequalities with applications to image restoration problems. **Carpathian Journal of Mathematics**, 39(3), 683-704, (2023).

[10] Altıparmak, E., Jolaoso, L. O., Karahan, I., and Rehman, H. U., Pre-conditioning CQ algorithm for solving the split feasibility problem and its application to image restoration problem. **Optimization**, 1-19, (2024).

[11] Altiparmak, E., and Karahan, I., A modified inertial viscosity algorithm for an infinite family of nonexpansive mappings and its application to image restoration. **Journal of Industrial and Management Optimization**, 20(2), 453-477, (2024).

[12] Jolaoso, L. O., Bai, J., and Shehu, Y., New fast proximal point algorithms for monotone inclusion problems with applications to image recovery. **Optimization,** 1-26, (2024).

[13] Mungkala, C., Padcharoen, A., and Akkasriworn, N., Convergence of proximal gradient method with alternated inertial step for minimization problem. **Advances Fixed Point Theory,** 14, Article-ID 35, (2024).

[14] Suantai, S., Cholamjiak, P., Inkrong, P., and Kesornprom, S., A fast contraction algorithm using two inertial extrapolations for variational inclusion problem and data classification. **Carpathian Journal of Mathematics**, 40(3), 737-752, (2024).

[15] Suantai, S., Cholamjiak, P., Inkrong, P., and Kesornprom, S., Modified iterative schemes with two inertia and linesearch rule for split variational inclusion and applications to image deblurring and diabetes prediction. **Carpathian Journal of Mathematics,** 40(2), 459-476, (2024).

[16] Jolaoso, L. O., Shehu, Y., and Xu, H. K., New accelerated splitting algorithm for monotone inclusion problems. **Optimization**, 74(3), 781-810, (2025).

[17] Lions, P.L., and Mercier, B. Splitting algorithms for the sum of two nonlinear operators. **SIAM Journal on Numerical Analysis,** 16, 964-979, (1979).

[18] Tseng, P. A., Modified forward-backward splitting method for maximal monotone mappings. **SIAM Journal Control Optimization,** 38(2), 431-446, (2000).

[19] Bot, R. I., Sedlmayer, M., and Vuong, P. T. A relaxed inertial forward-backward-forward algorithm for solving monotone inclusions with application to GANs. **Journal of Machine Learning Research**, 24(8), 1-37, (2023).

[20] Boţ, R. I., and Csetnek, E. R., An inertial forward-backward-forward primal-dual splitting algorithm for solving monotone inclusion problems. **Numerical Algorithms**, 71, 519-540, (2016).
[21] Mu, Z., and Peng, Y., A note on the inertial proximal point method. **Statistics, Optimization and Information Computing,** 3(3), 241-248, (2015).
[22] Iutzeler, F., and Hendrickx, J.M., A generic online acceleration scheme for optimization algorithms via relaxation and inertia. **Optimization Methods and Software** 34(2), 383-405, (2019).
[23] Iutzeler, F., and Malick, J., On the proximal gradient algorithm with alternated inertia. **Journal of Optimization Theory and Applications,** 176(3), 688-710 (2018).
[24] Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. **Springer, Berlin,** (2011).
[25] Takahashi, W., Introduction to Nonlinear and Convex Analysis, **Yokohama Publishers,** 2009.