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Global Existence in a Predator-Prey Model with Nonlinear Indirect Chemotaxis Mechanism

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Highlights:

- Nonlinear parabolic equations
- PDEs in connection with biology
- Cell movement (chemotaxis, etc.)
- Alikakos-Moser iteration

Keywords:

- Predator-prey model
- Indirect chemotaxis mechanism
- Global boundedness

ABSTRACT:

One of the fundamental processes in ecology is the interaction between predator and prey. Predator-prey interactions refer to the relative changes in population density of two species as they share the same environment and one species preys on the other. There are many studies global existence or blow-up of solutions on the predator-prey model. Our this paper related to the predator-prey model with nonlinear indirect chemotaxis mechanism under homogeneous Neumann boundary conditions. We establish the global existence and boundedness of classical solutions of our problem by using parabolic regularity theory. Namely, firstly we show that u and v boundedness in L^p for some $p > 1$, then we obtain the L^∞-bound of u and v by using Alikakos-Moser iteration. Thus, it is proved that the model has a unique global classical solution under suitable conditions on the parameters in a smooth bounded domain.

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INTRODUCTION

In our this paper, we deal with the following predator-prey chemotaxis model with nonlinear indirect chemotaxis mechanism

$$\begin{cases} u_t = \Delta u + \xi\nabla\cdot(u\nabla\omega) + au(1-u^{r-1}-bv), & (x,t)\in\Omega\times(0,T), \\ v_t = \Delta v - \chi\nabla\cdot(v\nabla\omega) + cv(1-v^{k-1}+du), & (x,t)\in\Omega\times(0,T), \\ 0 = \Delta\omega - \omega + z^{\gamma}, & (x,t)\in\Omega\times(0,T), \\ 0 = \Delta z - z + u^{\alpha}, & (x,t)\in\Omega\times(0,T), \\ u_v = v_v = \omega_v = z_v = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & (x,t)\in\Omega\times(0,T), \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outward normal to $\partial\Omega$, $u(x,t)$ and $v(x,t)$ denote the respectively densities of prey and predator populations. $\omega(x,t)$ and $z(x,t)$ represent the concentration of chemical attractants, $z(x,t)$ is produced by $u(x,t)$ and $v(x,t)$, and $\omega(x,t)$ is secreted by $z(x,t)$. The initial data u_0, v_0 are nonnegative functions and the constants $a,b,c,d,\chi,\xi,r,k,\gamma,\alpha>0$. The terms $\xi\nabla\cdot(u\nabla\omega)$ and $-\chi\nabla\cdot(v\nabla\omega)$ describe that the prey moves away from the higher concentration of the chemical secreted by the predator (chemorepulsion), and the predator moves toward the higher concentration of the chemical secreted by the prey (chemoattraction) with chemotaxis sensitivity coefficients ξ and χ. The kinetic terms describe mutual effect between predator and prey, where the population of the predator has a negative effect on the density of the prey, the population of the prey has an effect positively on the density of the predator, a and c denote the growth rates of two species, b and d measure interaction between two species.

System (1) is an extended version of the Keller-Segel system which is one of the most widely used models of chemotaxis introduced by Keller and Segel (1971). Chemotaxis is the movement of an organism in response to a chemical stimulus. One of the best-known examples of chemotaxis is the movement of the bacterium Escherichia Coli (E. Coli). With the development of modern cell biology and biochemistry in the 1960s and 1970s, many new techniques were developed and the decision-making mechanism of bacteria was explained by Adler. Adler observed crawling band movement of bacteria by placing E. Coli on one side of the tube and food and oxygen on the other side (Adler, 1966). The mathematical model of chemotaxis was expressed by Keller and Segel (1971), which successfully fitted the experimental studies by Adler. In past decades, the classical Keller-Segel and some modified Keller-Segel models have been extensively studied by different researchers (see: Horstmann, 2004 for detailed information). For example, some researchers examined the global existence or blow-up of solutions for the following Keller-Segel model with a logistic source

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla v) + \mu u(1-u), & (x,t)\in\Omega\times(0,T), \\ \tau v_t = d_2\Delta v + \alpha u - \gamma v, & (x,t)\in\Omega\times(0,T), \\ u_v = v_v = 0, & (x,t)\in\partial\Omega\times(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{2}$$

where $\Omega\subset\mathbb{R}^n$ is a smooth bounded domain and ν is the unit outward normal to $\partial\Omega$. The solutions of the problem (2) by describing in terms of spatial dimensions have been proven that if $\mu=0$, then the solutions are always globally bounded for $n=1$ (Osaki & Yagi, 2001), and may blow-up in finite or infinite time for $n\geq 2$ (Horstmann & Wang, 2001; Nagai, 2001; Winkler, 2013). Some authors proved that if $\mu>0$, then the blow-up phenomena may be prevented. For instance, when $\tau=0$, Tello and Winkler (2007) proved the existence of global bounded classical solutions under the assumption that either the space dimension does not exceed two, or that the logistic damping effect is strong enough. For the fully parabolic case when $\tau=1$, Osaki et al. (2002) showed that any blow-up phenomenon can be completely suppressed for arbitrarily small $\mu > 0$ for $n = 2$. Winkler (2010) extended this result obtained by Osaki et al. (2002) for higher dimensional bounded convex domain, and showed that if μ is sufficiently large, then the problem (2) possesses a bounded unique global classical solution in $\Omega \times (0, \infty)$. Also, many studies have considered the boundedness of the global solutions (Cao & Zheng, 2014; Yang et al., 2015; Li & Xiang, 2016; Xu & Zheng, 2018; Ayazoglu, 2022; Ayazoglu & Akkoyunlu 2022; Liu & Dai, 2022; Tian et al., 2022; Ayazoglu & Salmanova, 2024; Ayazoglu et al., 2024).

The interaction between predator and prey is one of the most fundamental processes in ecology and this interaction is critical in community dynamics for the management of renewable resources. For this reason, many mathematicians, ecologists, and biologists have researched this topic and examined the dynamic behavior that defines the interaction between predator and prey. In predator-prey models, the interaction between prey and predator populations is reviewed over time. For example, assuming that the predator population's only food source is prey, a high predator population will lead to a decrease in the prey population. A decrease in the prey population will lead to a reduction of the predator population, whose main food source is prey. The prey population will increase because the prey population will find a suitable environment for reproduction in the face of a decreasing predator population. Therefore, the increasing prey population creates a suitable feeding area for the predator population and contributes to the increase in the predator population. Thus, the interaction between the prey and predator populations cyclically continues in this way. Most previous theoretical analyses of predator-prey systems have taken as their starting point Volterra's equations (Volterra, 1926). If the prey and the predator target the same living creature as a food source, they become rivals. In contrast, if the predator chooses the prey as a food source, a hostile relationship begins between them. The competition and hostility relationship between the prey and predator populations is deal with in the Volterra's equations. Recently, studies on mathematical modeling of predator-prey systems have increased.

Tello and Winkler (2012) studied the following two-competing-species and one-stimuli chemotaxis model

$$\begin{cases} u_{1t} = d_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) + \mu_1 u_1 (1 - u_1 - e_1 u_2), (x,t) \in \Omega \times (0,T), \\ u_{2t} = d_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) + \mu_2 u_2 (1 - e_2 u_1 - u_2), (x,t) \in \Omega \times (0,T), \\ \tau v_t = d_3 \Delta v + \alpha u_1 + \beta u_2 - \gamma v, \quad (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u_1(x,0) = u_{1,0}(x), u_2(x,0) = u_{2,0}(x) \ \ v(x,0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{3}$$

when $\tau = 0$. The authors proved that the system (3) has a unique stationary solution that is globally asymptotically stable under suitable assumptions on the parameters. For $\mu_1 = \mu_2 = \tau = 0$ and γv being replaced by a positive constant ($\gamma v = 1$), Espejo et al. (2009) investigated simultaneous finite-time blow-up of (3) when Ω is a circle in $\mathbb{R}^2$. For $\mu_1 = \mu_2 = \tau = 0$, Biler et al. (2013) obtained the blow-up properties of (3) with $\Omega = \mathbb{R}^n$ ($n \geq 2$). Similar blow-up mechanisms, in particular related to the initial data size, have been studied by Conca et al. (2011) and Espejo et al. (2013) for $\Omega = \mathbb{R}^2$.

In case $\tau = 1$, for the fully parabolic chemotaxis system (3), many authors extensively studied the global existence and large time behavior of solutions (Lin et al., 2015; Bai & Winkler, 2016; Lin & Mu, 2017; Mizukami, 2017; Wang et al., 2017; Zhang & Li, 2018; Li & Wang, 2019).

Wang and Ke (2024) considered the following predator-prey system involving nonlinear indirect signal production

$$\begin{cases} u_t = \Delta u + \xi\nabla\cdot(u\nabla\omega) + a_1u(1-u^{r_1-1}-b_1v), & x\in\Omega, t>0,\\ v_t = \Delta v - \chi\nabla\cdot(v\nabla\omega) + a_2v(1-v^{r_2-1}+b_2u), & x\in\Omega, t>0,\\ \omega_t = \Delta\omega - \omega + z^{\gamma}, & x\in\Omega, t>0,\\ 0=\Delta z - z + u^{\alpha}+v^{\beta}, & x\in\Omega, t>0,\\ u_\nu = v_\nu = \omega_\nu = z_\nu = 0, & x\in\partial\Omega, \mathrm{t}>0,\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \omega(x,0)=\omega_0(x), & x\in\Omega, \mathrm{t}>0, \end{cases} \quad (4)$$

under homogeneous Neumann boundary conditions in a bounded and smooth domain $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$, where the parameters $\xi,\chi,a_1,a_2,b_1,b_2,\alpha,\beta,\gamma>0$. It has been shown that if $r_1>1$, $r_2>2$ and $\gamma(\alpha+\beta)<\frac{2}{n}$, then there exist some suitable initial data such that the system (4) has a global classical solution (u,v,ω,z), which is bounded in $\Omega\times(0,\infty)$. Wang and Ke (2024) determined the boundedness criteria only by the power exponents $r_1,r_2,\alpha,\beta,\gamma$ and spatial dimension n instead of the coefficients of the system and the sizes of initial data.

In this study, we deal with the global boundedness of the solution of problem (1), such that the exponents are $r,k,\alpha,\gamma>0$. Also, compared to the above studies, we remove the restrictions on the coefficients of the system, and our conclusions depend only on the power exponents r,k,γ,α. The model considered in this study is more general than the models discussed so far and the conditions are optimal in some sense.

The main result of this paper can be stated as follows.

Theorem 1. Let $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ be a bounded domain with smooth boundary and the parameters satisfy $\xi,\chi,a,b,c,d,\gamma>0$. If $\alpha>\frac{N-2}{N}$, $r>1+\gamma\alpha$, and $k>\max\{1+\gamma\alpha,2\}$ for any nonnegative initial data $u_0,v_0\in C^{\delta}(\bar{\Omega})$, with $\delta\in(0,1)$ and $\omega_0\in W^{1,\infty}(\Omega)$ are nonnegative, then the system (1) has a nonnegative global classical solution

$(u,v,\omega,z)\in\left(C^0(\bar{\Omega}\times[0,\infty))\cap C^{2,1}(\bar{\Omega}\times[0,\infty))\right)^3\times C^{2,0}(\bar{\Omega}\times[0,\infty))$,

which is bounded in $\Omega\times(0,\infty)$, namely, there exists a constant $C>0$ such that

$\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty+\|\omega(\cdot,t)\|_{1,\infty}+\|z(\cdot,t)\|_{1,\infty}\leq C$ for all $t>0$.

MATERIALS AND METHODS

Preliminaries

Standard parabolic regularity theory in a suitable fixed point framework can be used to obtain the local solution of the problem (1) for sufficiently smooth initial data. In fact, one can thereby also derive a sufficient condition for extensibility of a given local-in-time solution. Details of the proof can be found in (Ladyzhenskaia et al., 1968; Tello & Winkler, 2007; Winkler, 2013; Ding & Wang, 2019).

We denote $\|u\|_{L^p(\Omega)}:=\|u\|_p$, $\|u\|_{W^{1,p}(\Omega)}:=\|u\|_{1,p}=\|u\|_p+\|\nabla u\|_p$,

$\|u\|_{W^{2,p}(\Omega)}:=\|u\|_{2,p}=\|u\|_p+\|\Delta u\|_p$ $(1\leq p\leq\infty)$.

Lemma 1. Let $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ be a bounded domain with smooth boundary and the parameters satisfy $\xi,\chi,a,b,c,d,r,k,\gamma,\alpha>0$. Assume that initial data $u_0,v_0\in C^{\delta}(\bar{\Omega})$, with $\delta\in(0,1)$ and $\omega_0\in W^{1,\infty}(\Omega)$ are nonnegative. Then there exists $T_{max}\in(0,\infty]$ such that the system (1) has a nonnegative classical solution (u,v,ω,z) satisfying

$(u,v,\omega,z)\in\left(C^0(\bar{\Omega}\times[0,T_{max}))\cap C^{2,1}(\bar{\Omega}\times[0,T_{max}))\right)^3\times C^{2,0}(\bar{\Omega}\times[0,T_{max}))$.

Furthermore, if $T_{max}<\infty$, then

$\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty+\|\omega(\cdot,t)\|_{1,\infty}+\|z(\cdot,t)\|_{1,\infty}\to\infty$ as $t\to T_{max}$.

The following lemmas are essential to prove Theorem 1. We need the well-known Gagliardo-Nirenberg interpolation inequality (Nirenberg, 1966).

Lemma 2. Let l and k be two integers satisfying $0 \leq l < k$. Suppose $1 \leq q,\ r \leq \infty$, $p > 0$, and $\frac{l}{k} \leq a \leq 1$ with

$$\frac{1}{p} - \frac{l}{k} = a\left(\frac{1}{q} - \frac{k}{N}\right) + (1-a)\frac{1}{r}. \tag{5}$$

Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two positive constants c_1 and c_2 depending only on Ω, q, k, r and N such that

$$\|D^l u\|_p \leq c_1 \|D^k u\|_q^a \|u\|_r^{1-a} + c_2 \|u\|_r$$

holds with the following exception: the condition (5) is assumed only for $a \in \left[\frac{l}{k}, 1\right)$ if $k - l - \frac{N}{q}$ is a non-negative integer with $1 < q < \infty$.

Lemma 3. Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded and smooth domain and the parameters satisfy $\xi, \chi, a, b, c, d, \gamma, \alpha > 0$ and $r, k > 1$. Let (u, v, ω, z) be a solution of system (3). Then there exist the constants $M_1, M_2 > 0$ such that

$$\|u(\cdot,t)\|_1 \leq M_1, \|v(\cdot,t)\|_1 \leq M_2 \text{ for all } t \in (0, T_{max}). \tag{6}$$

Lemma 4. Under the assumptions of Lemma 1 the solution of (1) satisfies

$$\int_\Omega u(\cdot,t)^l + \int_\Omega \omega(\cdot,t)^l + \int_\Omega z(\cdot,t)^l \leq \bar{C}_0 \text{ for all } t \in (0, T_{max}),$$

where $\bar{C}_0 > 0$ and $l \in \left[1, \frac{N}{(N-2)_+}\right)$.

A detailed proof of Lemma 4 is available in Tang et al. (2023).

We establish the global existence and boundedness of classical solutions of the system (1) by using well-known Alikakos-Moser iteration. Namely, if u and v boundedness in L^p for some $p > 1$, then L^∞-bound of u and v can be obtained by using Alikakos-Moser iteration (Alikakos, 1979).

RESULTS AND DISCUSSION

Global Existence and Boundedness

This section deal with the global existence and boundedness of classical solutions to the system (1). Now, we establish the L^p-boundedness of u and v for some $p > 1$. We should at first establish that for any $p > 1$, there exists $C > 0$ such that

$$\|u(\cdot,t)\|_p + \|v(\cdot,t)\|_p \leq C \text{ for all } t \in (0, T_{max}).$$

Lemma 5. Let the assumptions stated in Lemma 1 hold. Then there exist constants $\varepsilon > 0$ and $C(\varepsilon) > 0$ such that

$$\int_\Omega u^p \leq \varepsilon \int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + C(\varepsilon) \text{ for all } t \in (0, T_{max}) \text{ and } p > 1. \tag{7}$$

Proof. The proof of the inequality (7) comes from an application of a general case of the Gagliardo-Nirenberg inequality (by Lemma 2): in particular, for any $p > 1$ and for some $C_{GN} > 0$, we get

$$\int_\Omega u^p = \left\|u^{\frac{p}{2}}\right\|_2^2 \leq C_{GN}\left(\left\|\nabla u^{\frac{p}{2}}\right\|_2^{2\lambda} \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^{2(1-\lambda)} + \left\|u^{\frac{p}{2}}\right\|_{\frac{2}{p}}^2\right) \leq \tilde{C}\left(\left\|\nabla u^{\frac{p}{2}}\right\|_2^{2\lambda} + 1\right)$$

where $\lambda = \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{N}-\frac{1}{2}} = \frac{(p-1)N}{pN+2-N} \in (0,1)$ and the fact of the boundedness to $\|u\|_1$ by using Lemma 3. Due to $p>1$, we conclude that $\frac{2(p-1)N}{pN+2-N} < 2$. Using Young inequality, the inequality (7) can be derived directly. This completes the proof of Lemma 5.

Lemma 6. Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and the parameters satisfy $\xi, \chi, a, b, c, d, \gamma > 0$. If $\alpha > \frac{N-2}{N}$, $r > 1+\gamma\alpha$ and $k > \max\{1+\gamma\alpha, 2\}$, then there exists a constant $C > 0$ such that

$$\int_\Omega u^p(\cdot,t) + \int_\Omega v^p(\cdot,t) \leq C \tag{8}$$

for all $t \in (0, T_{max})$ and any $p > \max\left\{1, \gamma(1-\alpha), \alpha(1-\gamma), \frac{(2-r)(k-1)}{k-2}\right\}$.

Proof. Multiply the first equation in the system (1) by u^{p-1} for any $p > \max\left\{1, \gamma(1-\alpha), \alpha(1-\gamma), \frac{(2-r)(k-1)}{k-2}\right\}$, then integrate over Ω by parts and taking into account the inequality (6), we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \xi\int_\Omega u^{p-1}\nabla\cdot(u\nabla\omega) + a\int_\Omega u^p - a\int_\Omega u^{p+r-1} - ab\int_\Omega u^p v \\ &\leq -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 - \xi(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla\omega + a\int_\Omega u^p - a\int_\Omega u^{p+r-1} \\ &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \frac{\xi(p-1)}{p}\int_\Omega u^p\Delta\omega + a\int_\Omega u^p - a\int_\Omega u^{p+r-1} \\ &:= I_1 + I_2 + I_3\end{aligned} \tag{9}$$

for all $t \in (0, T_{max})$. We estimate the terms $I_1 + I_2 + I_3$.

$$I_1 = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2. \tag{10}$$

Using the third equation of the system (1), we get

$$I_2 = \frac{\xi(p-1)}{p}\int_\Omega u^p\Delta\omega = \frac{\xi(p-1)}{p}\int_\Omega u^p(\omega - z^\gamma) \leq \frac{\xi(p-1)}{p}\int_\Omega u^p\omega$$

For all $z \geq 0$. by using young inequality, we can obtain

$$I_2 \leq \frac{\xi(p-1)}{p}\int_\Omega u^p\omega \leq C_1\int_\Omega u^{p+\gamma\alpha} + C_2\int_\Omega \omega^{\frac{p+\gamma\alpha}{\gamma\alpha}}, \tag{11}$$

where $C_1 = \frac{\xi(p-1)}{p+\gamma\alpha} > 0$ and $C_2 = \frac{\xi\gamma\alpha(p-1)}{p(p+\gamma\alpha)} > 0$. Next, we estimate the integral $\int_\Omega \omega^{\frac{p+\gamma\alpha}{\gamma\alpha}}$ according to a procedure similar to that employed by Tao and Wang (2013). In the following, we provide a brief outline for the sake of completeness. Noting that ω solves the following linear elliptic equations

$$\begin{cases} -\Delta\omega + \omega = u^\gamma, x \in \Omega, \\ \frac{\partial\omega}{\partial\nu} = 0, \quad x \in \partial\Omega, \end{cases}$$

for $t \in (0, T_{max})$. Thus applying the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations with the homogeneous Neumann boundary condition, we conclude that there exists $B > 0$ depending only on γ and Ω such that

$$\|\omega(\cdot,t)\|_{2,p} \leq B\|u^\gamma(\cdot,t)\|_p, \;\; \forall u^\gamma(\cdot,t) \in L^p(\Omega), \;\; p > 1 \tag{12}$$

for all $t \in (0, T_{max})$ (Agmon et al., 1964). For any $p > 1$, we can find $\mu \in \left[1, \frac{N}{(N-2)_+}\right)$. Then we can use the Gagliardo-Nirenberg inequality (Lemma 2) and the inequality (12) the estimate of ω (Lemma 4) to obtain some positive constants C_3 and C_4 such that

$$C_2 \int_\Omega \omega^{\frac{p+\gamma\alpha}{\gamma\alpha}} = C_2\|\omega(\cdot,t)\|_{\frac{p+\gamma\alpha}{\gamma\alpha}}^{\frac{p+\gamma\alpha}{\gamma\alpha}} \leq \; C_3 \, \|\Delta\omega(\cdot,t)\|_{\frac{p+\gamma\alpha}{\gamma}}^{\frac{p+\gamma\alpha}{\gamma\alpha}\theta} \|\omega(\cdot,t)\|_{\mu}^{\frac{p+\gamma\alpha}{\gamma\alpha}(1-\theta)} + C_3\|\omega(\cdot,t)\|_{\mu}^{\frac{p+\gamma\alpha}{\gamma\alpha}}$$
$$\leq \; C_4\left(\|u^\gamma\|_{\frac{p+\gamma\alpha}{\gamma}}^{\frac{p+\gamma\alpha}{\gamma\alpha}\theta} + 1\right) \; = C_4\left(\|u\|_{p+\gamma\alpha}^{\frac{p+\gamma\alpha}{\alpha}\theta} + 1\right) \tag{13}$$

for all $t \in (0, T_{max})$, where $\theta = \frac{\frac{1}{\mu} - \frac{\gamma\alpha}{p+\gamma\alpha}}{\frac{1}{\mu} + \frac{2}{N} - \frac{\gamma}{p+\gamma\alpha}} \in (0,1)$. Due to $\alpha > \frac{N-2}{N}$, we conclude

$$\frac{p+\gamma\alpha}{\alpha} . \frac{\frac{N-2}{N} - \frac{\gamma\alpha}{p+\gamma\alpha}}{1 - \frac{\gamma}{p+\gamma\alpha}} < p + \gamma\alpha. \tag{14}$$

Therefore, by using (13), (14) and Young inequality, we obtain

$$C_2 \int_\Omega \omega^{\frac{p+\gamma\alpha}{\gamma\alpha}} \leq C_4 \int_\Omega u^{p+\gamma\alpha} + C_4 \tag{15}$$

for all $t \in (0, T_{max})$. Substituting (15) into (11), we derive

$$I_2 \leq C_5 \int_\Omega u^{p+\gamma\alpha} + C_4, \tag{16}$$

where $C_5 = C_1 + C_4$. Recall the following inequality

$$a_0\Gamma^i - b_0\Gamma^j \leq a_0\left(\frac{a_0}{b_0}\right)^{\frac{i}{j-i}}, \;\; \forall \Gamma > 0, \tag{17}$$

where $a_0 \geq 0$, $b_0 > 0$ and $0 \leq i < j$. We can rewrite I_3 as following

$$I_3 = \; a\int_\Omega u^p - a\int_\Omega u^{p+r-1} = \frac{a}{2}\int_\Omega u^p + \frac{a}{2}\int_\Omega u^p - \frac{a}{2}\int_\Omega u^{p+r-1} - \frac{a}{2}\int_\Omega u^{p+r-1}. \tag{18}$$

By inequality (17), we have

$$\frac{a}{2}\int_\Omega u^p - \frac{a}{2}\int_\Omega u^{p+r-1} \leq C_6, \tag{19}$$

where $C_6 = \frac{a}{2}|\Omega| > 0$. So, substituting (19) into (18), we get

$$I_3 \leq \frac{a}{2}\int_\Omega u^p - \frac{a}{2}\int_\Omega u^{p+r-1} + C_6. \tag{20}$$

Substituting (10), (16), (20) into (9), and by using (7), we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le C_5\int_\Omega u^{p+\gamma\alpha} + \frac{a}{2}\int_\Omega u^p - \frac{4(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 - \frac{a}{2}\int_\Omega u^{p+r-1} + C_7$$
$$\le C_5\int_\Omega u^{p+\gamma\alpha} + \left(\frac{a\varepsilon}{2} - \frac{4(p-1)}{p^2}\right)\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 - \frac{a}{2}\int_\Omega u^{p+r-1} + C_8,$$

where $C_7 = C_4 + C_6$ and $C_8 = \frac{a}{2}C(\varepsilon) + C_7$. Taking $\varepsilon = \frac{8(p-1)}{ap^2}$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \le C_5\int_\Omega u^{p+\gamma\alpha} - \frac{a}{2}\int_\Omega u^{p+r-1} + C_8 \tag{21}$$

for all $t \in (0, T_{max})$. Next, similarly multiplying the second equation in system (1) by v^{p-1} for any $p > \max\left\{1, \gamma(1-\alpha), \alpha(1-\gamma), \frac{(2-r)(k-1)}{k-2}\right\}$, we can deduce from integration by parts that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p = -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla v^{\frac{p}{2}}\right|^2 - \frac{\chi(p-1)}{p}\int_\Omega v^p\Delta\omega + c\int_\Omega v^p - c\int_\Omega v^{p+k-1} + cd\int_\Omega v^p u. \tag{22}$$

Using the third equation of the system (1), we can write the equation (22) as following

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p = -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla v^{\frac{p}{2}}\right|^2 - \frac{\chi(p-1)}{p}\int_\Omega v^p(\omega - z^\gamma) + c\int_\Omega v^p - c\int_\Omega v^{p+k-1} + cd\int_\Omega v^p u$$
$$\le -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla v^{\frac{p}{2}}\right|^2 + \frac{\chi(p-1)}{p}\int_\Omega v^p z^\gamma + c\int_\Omega v^p - c\int_\Omega v^{p+k-1} + cd\int_\Omega v^p u \tag{23}$$

for all $\omega \ge 0$. Further, from inequality (17), we see

$$c\int_\Omega v^p - \frac{c}{3}\int_\Omega v^{p+k-1} \le C_9, \tag{24}$$

where $C_9 = 3^{\frac{p}{k-1}}c|\Omega| > 0$. Since $k > 2$, from Young inequality, we conclude

$$cd\int_\Omega v^p u \le \frac{c}{3}\int_\Omega v^{p+k-1} + C_{10}\int_\Omega u^{\frac{p+k-1}{k-1}}, \tag{25}$$

for some $C_{10} > 0$. Similarly by using Young inequality, one may obtain

$$\frac{\chi(p-1)}{p}\int_\Omega v^p z^\gamma \le C_{11}\int_\Omega v^{p+\gamma\alpha} + C_{12}\int_\Omega z^{\frac{p+\gamma\alpha}{\alpha}}, \tag{26}$$

where $C_{11} = \frac{\chi(p-1)}{p+\gamma\alpha}$, $C_{12} = \frac{\chi\gamma\alpha(p-1)}{p(p+\gamma\alpha)}$. Substituting the inequalities (24), (25) and (26) into (23), we get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega v^p = C_{11}\int_\Omega v^{p+\gamma\alpha} + C_{12}\int_\Omega z^{\frac{p+\gamma\alpha}{\alpha}} - \frac{c}{3}\int_\Omega v^{p+k-1} + C_{10}\int_\Omega u^{\frac{p+k-1}{k-1}} + C_9. \tag{27}$$

We estimate the integral $\int_\Omega z^{\frac{p+\gamma\alpha}{\alpha}}$. Noting that z solves the following linear elliptic equations

$$\begin{cases} -\Delta z + z = u^\alpha, x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, \quad x \in \partial\Omega, \end{cases}$$

for $t \in (0, T_{max})$. Similarly to (12), we get

$$\|z(\cdot,t)\|_{2,p} \leq \tilde{B}\|u^{\alpha}(\cdot,t)\|_{p}, \ \ \forall u^{\alpha}(\cdot,t) \in L^{p}(\Omega), \ \ p > 1 \tag{28}$$

for all $t \in (0, T_{max})$ with $\tilde{B} > 0$. For any $p > 1$, we can find $\mu_0 \in \left[1, \frac{N}{(N-2)_+}\right)$. Then we can use the Gagliardo-Nirenberg inequality (Lemma 2) and (28) the estimate of z (Lemma 4) to obtain some positive constants $\bar{C}_3$, $\bar{C}_4$ such that

$$C_{12}\int_{\Omega} z^{\frac{p+\gamma\alpha}{\alpha}} = C_{12}\|z\|_{\frac{p+\gamma\alpha}{\alpha}}^{\frac{p+\gamma\alpha}{\alpha}} \leq \bar{C}_3 \|\Delta z(\cdot,t)\|_{\frac{p+\gamma\alpha}{\alpha}}^{\frac{p+\gamma\alpha}{\alpha}\theta_1} \|z(\cdot,t)\|_{\mu_0}^{\frac{p+\gamma\alpha}{\alpha}(1-\theta_1)} + \bar{C}_3\|z(\cdot,t)\|_{\mu_0}^{\frac{p+\gamma\alpha}{\alpha}}$$

$$\leq \bar{C}_4\left(\|u^{\alpha}\|_{\frac{p+\gamma\alpha}{\alpha}}^{\frac{p+\gamma\alpha}{\alpha}\theta_1} + 1\right) \leq \bar{C}_4\left(\|u\|_{p+\gamma\alpha}^{(p+\gamma\alpha)\theta_1} + 1\right) \tag{29}$$

for all $t \in (0, T_{max})$, where $\theta_1 = \frac{\frac{1}{\mu_0} - \frac{\alpha}{p+\gamma\alpha}}{\frac{1}{\mu_0} + \frac{2}{N} - \frac{\alpha}{p+\gamma\alpha}} \in (0,1)$. Then we conclude

$$\frac{p+\gamma\alpha}{\alpha} \cdot \frac{\frac{N-2}{N} - \frac{\alpha}{p+\gamma\alpha}}{1 - \frac{\alpha}{p+\gamma\alpha}} < p + \gamma\alpha. \tag{30}$$

Therefore, by combining the inequality (29) with (30) and applying Young inequality, we obtain

$$C_{12}\int_{\Omega} z^{\frac{p+\gamma\alpha}{\alpha}} \leq \bar{C}_4\int_{\Omega} u^{p+\gamma\alpha} + \bar{C}_4. \tag{31}$$

Substituting inequality (31) into (27), it is easy to see

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega} v^{p} \leq C_{11}\int_{\Omega} v^{p+\gamma\alpha} + \bar{C}_4\int_{\Omega} u^{p+\gamma\alpha} - \frac{c}{3}\int_{\Omega} v^{p+k-1} + C_{10}\int_{\Omega} u^{\frac{p+k-1}{k-1}} + C_{14} \tag{32}$$

with $C_{14} = \bar{C}_4 + C_9$. By virtue of (21) and (32), we conclude

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega} (u^p + v^p) \leq C_{15}\int_{\Omega} u^{p+\gamma\alpha} + C_{11}\int_{\Omega} v^{p+\gamma\alpha} + C_{10}\int_{\Omega} u^{\frac{p+k-1}{k-1}}$$

$$-\frac{a}{2}\int_{\Omega} u^{p+r-1} - \frac{c}{3}\int_{\Omega} v^{p+k-1} + C_{16}, \tag{33}$$

where $C_{15} = \bar{C}_4 + C_5$, $C_{16} = C_{14} + C_8$. Due to $k > 2$, we see

$$p + r - 1 - \frac{p+k-1}{k-1} = \frac{p(k-2) + (r-2)(k-1)}{k-1} = \frac{p(k-2)}{k-1} + r - 2 > 0$$

for all $p > \max\left\{1, \gamma(1-\alpha), \alpha(1-\gamma), \frac{(2-r)(k-1)}{k-2}\right\}$. On the one hand, since $p + r - 1 > \frac{p+k-1}{k-1}$, $r > 1 + \gamma\alpha$ and $k > \max\{1+\gamma\alpha, 2\}$, from inequality (17), we can deduce

$$C_{10}\int_{\Omega} u^{\frac{p+k-1}{k-1}} - \frac{a}{6}\int_{\Omega} u^{p+r-1} \leq C_{17} \text{ with } C_{17} = C_{10}\left(\frac{6C_{10}}{a}\right)^{\frac{\frac{p+k-1}{k-1}}{p+r-1-\frac{p+k-1}{k-1}}} |\Omega| > 0, \tag{34}$$

$$C_{15}\int_{\Omega} u^{p+\gamma\alpha} - \frac{a}{6}\int_{\Omega} u^{p+r-1} \leq C_{18} \text{ with } C_{18} = C_{15}\left(\frac{6C_{15}}{a}\right)^{\frac{p+\gamma\alpha}{r-\gamma\alpha-1}} |\Omega| > 0, \tag{35}$$

$$C_{11}\int_{\Omega} v^{p+\gamma\alpha} - \frac{c}{6}\int_{\Omega} v^{p+k-1} \leq C_{19} \text{ with } C_{19} = C_{11}\left(\frac{6C_{11}}{c}\right)^{\frac{p+\gamma\alpha}{k-\gamma\alpha-1}} |\Omega| > 0. \tag{36}$$

From (33)-(36), we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u^p+v^p)+\frac{a}{6}\int_\Omega u^{p+r-1}+\frac{c}{6}\int_\Omega v^{p+k-1}\le C_{20} \tag{37}$$

for all $t\in(0,T_{max})$, where $C_{20}=C_{16}+C_{17}+C_{18}+C_{19}>0$. Next, adding $\int_\Omega (u^p+v^p)$ to both sides of the inequality (37) let's write the inequality as following

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u^p+v^p)+\int_\Omega (u^p+v^p)\le\int_\Omega u^p+\int_\Omega v^p-\frac{a}{6}\int_\Omega u^{p+r-1}-\frac{c}{6}\int_\Omega v^{p+k-1}+C_{20} \tag{38}$$

From the inequality (17), we obtain

$$\int_\Omega u^p-\frac{a}{6}\int_\Omega u^{p+r-1}\le C_{21} \text{ with } C_{21}=\left(\frac{6}{a}\right)^{\frac{p}{r-1}}|\Omega|>0, \tag{39}$$

$$\int_\Omega v^p-\frac{c}{6}\int_\Omega v^{p+k-1}\le C_{22} \text{ with } C_{22}\left(\frac{6}{c}\right)^{\frac{p}{k-1}}|\Omega|>0. \tag{40}$$

Finally, by combining (38)-(40), we infer

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u^p+v^p)+\int_\Omega (u^p+v^p)\le C_{23}$$

with $C_{23}=C_{20}+C_{21}+C_{22}$ for all $t\in(0,T_{max})$. By taking $z(t):=\int_\Omega (u^p+v^p)$, the absorptive differential inequality $\frac{dz(t)}{dt}\le C_{23}-z(t)$ concludes the proof through comparison reasoning for ordinary differential inequalities. Thus, one derives the inequality (8) directly.

Proof of Theorem 1. By Lemma 6, we know that $\|u(\cdot,t)\|_p+\|v(\cdot,t)\|_p\le C$ for all $t\in(0,T_{max})$ and $p>\max\left\{1,\gamma(1-\alpha),\alpha(1-\gamma),\frac{(2-r)(k-1)}{k-2}\right\}$. We deal with the fourth equation in system (1) by elliptic L^p-estimate, thus there exists $\hat{C}_1>0$ such that $\|z(\cdot,t)\|_{2,\frac{p}{\alpha}}\le\hat{C}_1$ for all $t\in(0,T_{max})$. From Sobolev imbedding theorem, we get $\|z(\cdot,t)\|_{1,\infty}\le\hat{C}_2$ for all $t\in(0,T_{max})$, for some $\hat{C}_2>0$. By using parabolic regularity for the third equation in system (1), we conclude $\|\omega(\cdot,t)\|_{1,\infty}\le\hat{C}_3$ for all $t\in(0,T_{max})$, for some $\hat{C}_3>0$. By the standard Alikakos-Moser iteration (see: Lemma A.1 in Alikakos, 1979), it is entailed from the inequality (8) of Lemma 6 that there exists $\hat{C}_4$, $\hat{C}_5>0$ such that $\|u(\cdot,t)\|_\infty\le\hat{C}_4$ and $\|v(\cdot,t)\|_\infty\le\hat{C}_5$
for all $t\in(0,T)$, where $\hat{C}_4$, $\hat{C}_5>0$ is independent of $T\in(0,T_{max})$. Thus, from Lemma 1, we obtain that $T_{max}=\infty$. Thanks to the Neumann heat semigroup estimate the solution (u,v,ω,z) is global in time and bounded (Winkler, 2013). The proof of Theorem 1 is complete.

CONCLUSION

We considered a predator-prey model with nonlinear indirect chemotaxis mechanism under homogeneous Neumann boundary conditions. In the conclusion, we proved that for all appropriate regular nonnegative initial data the system (1) has a nonnegative global classical solution under suitable conditions on the parameters.

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Conflict of Interest

The article authors declare that there is no conflict of interest between them.

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