INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 17 NO. 2 PAGE 306–316 (2024)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1551016

Algebraic and Geometric Properties of a Family of Rational Curves

J. William Hoffman and Haohao Wang*

(Communicated by Kazım İlarslan)

ABSTRACT

This paper consists of two components - an application part and a theoretical part, where the former targets the applications of computer aided geometric designs in generating parametric curves, and the latter focuses on the algebraic analysis of rational space curves. At the application level, we construct a family of rational space curves via quaternion products of two generating curves. At the theoretical level, we use algebraic methods to extract a μ-basis for this family of curves, and describe a basis for a special submodule of the syzygy module in terms of a μ-basis for the syzygy module of this family of curves. A commutative diagram is provided to summarize these results.

Keywords: *Syzygy module, rational curve, matrix factorization.*
AMS Subject Classification (2020): *Primary: 13D02; Secondary: 14Q05.*

1. Introduction

A rational curve is a curve that is birationally equivalent to a line, and therefore can be parametrized with rational functions in one variable. Rational curves are widely used in computer graphics and geometric modeling in addition to their applications in many other areas [3], [8], [9], [11]. It is important to know how to parametrize rational curves, since rational curves are fundamental for geometric modeling, and there is a great demand for the inclusion of the full class of rational curves and surfaces in computer aided geometric design.

The application component of this paper is to create a family of rational curves for the purpose of computer aided geometric designs. We are going to invoke quaternions and quaternion multiplication to help represent shapes. We shall generalize the idea of rotation along a fixed axis by taking advantage of the fact that quaternions provide a natural way to represent rotations. We start with two rational curves represented by two generic 1-1 parametrizations: $r(t) = (r_0, r_1, r_2, r_3)$, and $d(t) = (d_0, d_1, d_2, d_3)$, where $r_i, d_i \in \mathbb{K}[t]$ where $\mathbb{K}$ is a field of characteristic 0. Let $d^*(t) = (d_0, -d_1, -d_2, -d_3)$, then the quaternion product $x(t) = (x_0, x_1, x_2, x_3) = d(t)r(t)d^*(t)$ represents a rational space curve generated by rotating the curve $r(t)$ along the direction of the curve $d(t)$ at any parameter t. With this construction, we generate a family of rational space curves using two given curves.

The theoretical component of this paper is to study some algebraic and geometric properties of this family of curves. We are particularly interested in the syzygy modules of this family of curves. There is a large literature on syzygies and their application to geometry, see e.g., [5], [6] and [7]. Recall given a finitely generated R-module M where R is a commutative ring and a set $z_1, \dots, z_n$ of generators, a syzygy of M is an element $(a_1, \dots, a_n) \in R^n$ for which $a_1 z_1 + \cdots + a_n z_n = 0$. The set of all syzygies is a submodule of R^n. Hilbert's Syzygy Theorem states that every finitely generated graded module $M = \oplus_{i=0}^{\infty} M_i$ over the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$ has a free resolution of length at most n, that is, its n-th syzygy is free.

For rational space curves we are interested in the polynomial ring $\mathbb{K}[t]$ in one variable, or in the case of graded modules over the graded ring $\mathbb{K}[t, v]$. In both cases, the Hilbert - Burch theorem yields that for a rational curve in 3-space, the syzygy module of this curve is generated by three elements called a *μ-basis*, which is a powerful tool for analyzing rational parametric curves. The μ-bases serve as a bridge between their parametric forms and

Received : *06-03-2024,* ***Accepted :*** *08-09-2024*
*** Corresponding author***

their implicit forms [4]. Geometrically, a μ-basis for a rational parametric space curve consists of three moving planes or syzygies that are orthogonal to the curve at each point, and intersect exactly at that point on the curve. We are interested in the homogeneous graded case when we study μ-bases. Let $\{\mathbf{p}_i\}_{i=1,2,3}$ and $\{\mathbf{P}_i\}_{i=1,2,3}$ be the μ-bases for the syzygy modules of the curves r and x in the homogeneous graded ring $S = \mathbb{K}[t, v]$. We attempt to understand the geometry of the basis $\{\mathbf{p}_i\}_{i=1,2,3}$ under a rotation, and the relationship between two bases $\{\mathbf{p}_i\}_{i=1,2,3}$ and $\{\mathbf{P}_i\}_{i=1,2,3}$. In particular, we show how to extract $\{\mathbf{P}_i\}_{i=1,2,3}$ using the information of $\{\mathbf{p}_i\}_{i=1,2,3}$ over S.

It is known that the quaternion product has a matrix representation, so one can express the generators for of x in terms of matrix multiplication, $x = drd^* = \mathbf{R}r^T$ as a column vector, where $\mathbf{R}$ is the rotation matrix corresponding to the curve (or quaternion) d. Hence, in our construction, the curve x is generated by a matrix product. Papers such as [12] study the properties of ideals generated by a matrix product. However, to our knowledge, there is no affirmative answer to the question on how to describe the basis of the submodule generated by $\mathbf{R}\mathbf{p}_i$ in terms of the basis $\{\mathbf{P}_i\}_{i=1,2,3}$ using the information of matrix $\mathbf{R}$. When we work over the non-homogeneous ring $\mathbb{K}[t]$, we seek a link between these modules using the Smith normal form of $\mathbf{R}$ as a bridge.

This paper is structured as the following. We begin Section 2 by a brief review of quaternions; we then construct a family of rational space curves by a quaternion product of two rational generating curves in Proposition 2.1. In addition, we prove our parametrization is generically one-one in Theorem 2.2. In Section 3, we first recall a few properties of the rotation matrix $\mathbf{R}$. Then in Theorem 3.1, we extract a μ-basis for the rational space curve $x = drd^*$ using the μ-basis for the rational curve r over the homogeneous graded ring $\mathbb{K}[t, v]$. Furthermore, over the non-homogeneous ring $\mathbb{K}[t]$. we describe a basis of the submodule generated by $\mathbf{R}\mathbf{p}_i$ in terms of the generators of $\mathrm{Syz}(x_0, \ldots, x_3)$ using the information of matrix $\mathbf{R}$ in Theorem 3.2. Finally, we summarize our results in a commutative diagram in Theorem 3.4. Examples are included to illustrate our theorems.

2. Generating Rational Curves by Quaternions

2.1. Quaternions

In this section, we review some basic facts about quaternions. Recall that an arbitrary quaternion has the form $q = s_q + \mathbf{v}_q$, where s_q is a scalar and $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is a vector in $\mathbb{R}^3$. If $p = s_p + \mathbf{v}_p$ and $q = s_q + \mathbf{v}_q$ are two quaternions, then quaternion multiplication has the simple form

$$pq = s_p s_q - \mathbf{v}_p \cdot \mathbf{v}_q + s_p\mathbf{v}_q + s_q\mathbf{v}_p + \mathbf{v}_p \times \mathbf{v}_q.$$

The *conjugate* of q is denoted by $q^* = s_q - \mathbf{v}_q = s_q - v_1\mathbf{i} - v_2\mathbf{j} - v_3\mathbf{k}$. Notice that the product $qq^* = s_q^2 + v_1^2 + v_2^2 + v_3^2 = |q|^2$ is a scalar; $|q|$ is called the *norm* of q. If $|q| = 1$, then q is called a *unit quaternion*.

A *pure quaternion* is a quaternion q whose scalar part $s_q = 0$. A pure quaternion $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is interpreted geometrically as the vector from the origin to the point located at (v_1, v_2, v_3) in $\mathbb{R}^3$.

It will be convenient to use quaternions to represent points in affine 3-space as well. The convention is that the quaternion $q = (q_0, q_1, q_2, q_3)$ represents the point $(q_1/q_0, q_2/q_0, q_3/q_0)$ if $q_0 \neq 0$. In other words, we regard affine 3-space as the subset of projective 4-space (x_0, x_1, x_2, x_3) where $x_0 \neq 0$. We will distinguish between vectors in 3-space and 4-space (i.e., affine 3-space and projective 3-space), by the convention that given $x = (x_0, x_1, x_2, x_3) = (x_0, \mathbf{x})$, if $x_0 \neq 0$, then we get the point $\mathbf{x}/x_0$ in affine 3-space from the point x in projective 3-space.

Theorem 2.1. *(Quaternion Rotation, [10].) Let $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a pure unit quaternion and set*

$$q = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}_q.$$

Then q is a unit quaternion and the map $\mathbf{x} \to q\mathbf{x}q^*$ rotates points and vectors in $\mathbb{R}^3$ by the angle θ around the line through the origin in the direction of the vector $\mathbf{v}_q$ in $\mathbb{R}^3$.

If we write the unit quaternion $q = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}_q = (q_0, q_1, q_2, q_3)$ where $\sum_{i=0}^{3} q_i^2 = 1$, and set $x = (0, \mathbf{x})$ with $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, then the image of the rotation in $\mathbb{R}^3$ can be represented by

$$q\mathbf{x}q^* = qxq^* = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{R}_{3\times 3} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \text{ where } \mathbf{0} = [0,0,0] \text{ and}$$

$$\mathbf{R}_{3\times 3} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \tag{2.1}$$

is the rotation matrix representing the quaternion q with orthonormal columns (and rows).

We will extend this to an action of the nonzero quaternions on projective 3-space $(x_0, \mathbf{x})$.

$$qxq^* = \begin{bmatrix} q_0^2 + q_1^2 + q_2^2 + q_3^2 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{R}_{3\times 3} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}, \text{ where } \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T, \ \mathbf{R}_{3\times 3} \text{ is given in (2.1).}$$

2.2. *Generating Rational Curves by Quaternions*

A vector-valued function in a single variable $q(t) = (q_0, q_1, q_2, q_3)$ can be viewed as a quaternion function or a space curve $\mathbf{q}(t) = (q_1, q_2, q_3)/q_0$ in affine 3-space. Taking advantage of this and the fact that quaternions represent space rotations, we shall generate a large variety of rational curves by the quaternion multiplications of two rational space curves.

We start with two rational space curves typically represented by two generically one-to-one parametrizations in projective 3-space – the director curve $d(t)$, and the radius curve $r(t)$:

$$d(t) = (d_0(t), \ldots, d_3(t)) = (d_0, \mathbf{d}), \quad r(t) = (r_0(t), \ldots, r_3(t)) = (r_0, \mathbf{r}), \tag{2.2}$$

where $d_\ell, r_\ell \in \mathbb{R}[t]$, $\gcd(d_0, \ldots, d_3) = \gcd(r_0, \ldots, r_3) = 1$, and $\max\{\deg(d_\ell)\} = m$, $\max\{\deg(r_\ell)\} = n$, $\ell = 0, \ldots, 3$.

Proposition 2.1. *In affine 3-space, the rational curve*

$$x(t) = (x_0(t),\, x_1(t),\, x_2(t),\, x_3(t)) = d(t)\, r(t)\, d^*(t) \tag{2.3}$$

is generated by rotating the radius $\mathbf{r}(t)/r_0(t)$ about the director $\mathbf{d}(t)/d_0(t)$. The quaternion curve $x(t)$ has a matrix representation

$$\begin{aligned} x(t) &= \mathbf{R}r^T \\ &= \begin{bmatrix} \sum_{i=0}^{3} d_i^2 & 0 & 0 & 0 \\ 0 & d_0^2 + d_1^2 - d_2^2 - d_3^2 & 2(d_1d_2 - d_3d_0) & 2(d_1d_3 + d_2d_0) \\ 0 & 2(d_1d_2 + d_3d_0) & d_0^2 - d_1^2 + d_2^2 - d_3^2 & 2(d_2d_3 - d_1d_0) \\ 0 & 2(d_1d_3 - d_2d_0) & 2(d_2d_3 + d_1d_0) & d_0^2 - d_1^2 - d_2^2 + d_3^2 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} \end{aligned} \tag{2.4}$$

$$\det(\mathbf{R}(t)) = \left(\sum_{i=0}^{3} d_i^2\right)^4, \ \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \left(\sum_{i=0}^{3} d_i^2\right)^2 \mathbf{I}_{4\times 4}.$$

Proof. We consider the curve $d(t) = (d_0(t), d_1(t), d_2(t), d_3(t))$ as a quaternion, where

$$d(t) = (d_0(t), d_1(t), d_2(t), d_3(t)) = |d| \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) \frac{(d_1, d_2, d_3)}{\sqrt{\sum_{i=1}^{3} d_i^2}} \right) \tag{2.5}$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{d_0}{|d|}, \ \sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{\sum_{i=1}^{3} d_i^2}}{|d|}, \ \theta(t) = 2\tan^{-1}\frac{\sqrt{\sum_{i=1}^{3} d_i^2}}{d_0}, \ |d| = \sqrt{\sum_{i=0}^{3} d_i^2}.$$

Note that $\frac{d}{|d|}(t)$ is a unit quaternion. For each t value, the map $r(t) \to \left(\frac{d}{|d|}(t)\right) r(t) \left(\frac{d}{|d|}(t)\right)^*$ rotates the radius $r(t)$ by the angle $\theta(t) = 2\tan^{-1}\frac{\sqrt{d_1^2+d_2^2+d_3^2}}{d_0}(t)$, where $\cos(\theta/2) = \frac{d_0(t)}{|d(t)|}$ is the scalar part of $\frac{d}{|d|}(t)$, around the line in the direction of the vector part of $\frac{d}{|d|}(t)$, that is $\frac{\langle d_1(t), d_2(t), d_3(t)\rangle}{|d(t)|}$, which is the same as the direction of

$\langle d_1(t), d_2(t), d_3(t)\rangle$. Therefore we can define a rational curve by setting $x(t) = \left(\frac{d}{|d|}(t)\right) r(t) \left(\frac{d}{|d|}(t)\right)^*$. Since we work with homogeneous coordinates, for any real parameter t the denominators in $\frac{d}{|d|}(t)$ are scalars which can be ignored. Thus these rational curves are generated by the formula $x = d(t)r(t)d^*(t)$.

Furthermore, in affine 3-space, the curve $x = \left(\frac{d}{|d|}\right) r \left(\frac{d}{|d|}\right)^*$ has a matrix expression $\begin{bmatrix} 1 & \mathbf{0} \\ 0 & \mathbf{R}_{3\times 3}\end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{r}/r_0 \end{bmatrix}$. By clearing the denominator, the homogeneous form of the quaternion curve $x = drd^* = \mathbf{R}\, r^T$ where the matrix $\mathbf{R}$ is the rotation matrix representing the quaternion d in Equation (2.4). □

The following example illustrates Propositions 2.1, and the generating curves and the resulting curve are shown in Figure 1.

Example 2.1. Given two space curve $d(t)$ and $r(t)$, where

$$\begin{aligned} d(t) &= (t^2+1, 0, t^2-1, 2t),\ r(t) = (1, t, t^2, t^3), \text{we generate } x(t) = d(t)r(t)d^*(t) \\ x(t) &= \mathbf{R}(t)\, r^T(t) = \begin{bmatrix} 2(t^2+1)^2 & 0 & 0 & 0 \\ 0 & 0 & -4t(t^2+1) & -2t^4+2 \\ 0 & 4t(t^2+1) & 2(t^2-1)^2 & 4t(t^2-1) \\ 0 & -2(t^4-1) & 4t(t^2-1) & 8t^2 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} \\ &= 2\left[(t^2+1)^2,\quad t^3(t^2-3)(t^2+1),\quad t^2(3t^4-2t^2+3),\quad t(5t^4-2t^2+1)\right]^T. \end{aligned}$$

In Euclidean 3-space, the curve $d(t)$ is a circle $y^2+z^2=1$ (in red), the radius $r(t)$ is the twisted cubic (t, t^2, t^3) (in green), and the curve

$$x(t) = d(t)r(t)d^*(t) = \left(\frac{t^3(t^2-3)}{t^2+1}, \frac{t^2(3t^4-2t^2+3)}{(t^2+1)^2}, \frac{t(5t^4-2t^2+1)}{(t^2+1)^2}\right)$$

(in blue) is generated by rotating the twisted cubic $r(t)$ about the director circle $d(t)$. These curves are shown in Figure 1.

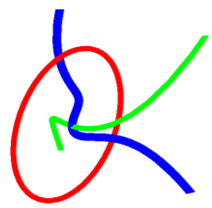

Figure 1. A rational curve (in blue) is generated by rotating a twisted cubic (in green) about a circle (in red).

2.3. *Homogeneous Parametrization*

To describe the homogeneous parametrization of this family of rational curves, we first homogenize the parametrization of the rational generating curves – the director $d(\bar{t}) = d(t, v)$, and the radius $r(\bar{t}) = r(t, v)$:

$$d(\bar{t}) = (d_0(\bar{t}), d_1(\bar{t}), d_2(\bar{t}), d_3(\bar{t})),\ r(\bar{t}) = (r_0(\bar{t}), r_1(\bar{t}), r_2(\bar{t}), r_3(\bar{t})), \tag{2.6}$$

where $\gcd(d_0, \dots, d_3) = \gcd(r_0, \dots, r_3) = 1$, $d_i, r_i \in \mathbb{K}[\bar{t}]$ with $\deg(d) = \deg(d_i) = m$, $\deg(r) = \deg(r_i) = n$ in variable $\bar{t}$ for $i = 0, 1, 2, 3$, $d_0, r_0 \not\equiv 0$, and $(d_1, \dots, d_3)$, $(r_1, \dots, r_3) \not\equiv (0, 0, 0)$. Then

$$x : \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^3_{\mathbb{K}} : \quad x(\bar{t}) = (x_0(\bar{t}),\, x_1(\bar{t}),\, x_2(\bar{t}),\, x_3(\bar{t})) = d(\bar{t})r(\bar{t})d(\bar{t})^*.$$

Theorem 2.2. *The homogeneous parametrization of the family of rational curves generated by Proposition 2.1 has the following properties:*

$$\deg(x_i) = 2m+n,\ i = 0, 1, 2, 3,\ \gcd(x) = \gcd(x_0, x_1, x_2, x_3) = 1.$$

Furthermore, for a generic choice of a pair of curves d and r, the parametrization is generically 1-1.

Proof. It is easy to observe that $\deg(x_i) = 2m + n, i = 0, 1, 2, 3$. In addition, since four homogeneous polynomials $x_0, \ldots, x_3$ in one pair of variables t, v in general do not have a common solution, we conclude that $\gcd(x) = \gcd(x_0, x_1, x_2, x_3) = 1$.

Next, we sketch a proof that for a generic choice of a pair of curves d and r, the parametrization is generically 1-1. Let

$$\{a_{00}, \ldots, a_{0m}, a_{10}, \ldots, a_{1m}, a_{20}, \ldots, a_{2m}, a_{30}, \ldots, a_{3m}\},$$
$$\{b_{00}, \ldots, b_{0n}, b_{10}, \ldots, b_{1n}, b_{20}, \ldots, b_{2n}, b_{30}, \ldots, b_{3n}\}$$

be the coefficients of an arbitrary degree m curve $d = (d_0, \ldots, d_3)$ and a degree n curve $r = (r_0, \ldots, r_3)$; and let $\mathrm{Res}(f_1, f_2, s)$ denote the resultant of polynomials f_1, f_2 with respect to s.

First, we observe that for a generic choice of the curves $d(t)$ and $r(t)$, $\gcd(x_i, x_j) = 1$, $i \neq j \in \{0, 1, 2, 3\}$. To see this, suppose if there exists a polynomial $h(t, v) \mid x_i(t, v)$ where $i = 0, 1$. Then Equation (2.4) yields

$$h \mid \sum_{i=0}^{3} d_i^2 \, r_0, \text{ and } h \mid [(d_0^2 + d_1^2 - d_2^2 - d_3^2)r_1 + 2(d_1 d_2 - d_3 d_0)r_2 + 2(d_1 d_3 + d_2 d_0)r_3].$$

Thus the coefficients of the curve d and r must satisfy certain algebraic conditions in order to vanish on the points $\{(t, v) \in \mathbb{P}^1 \mid h(t, v) = 0\}$. This contradicts the generic conditions for the choice of curves d and r. Similar argument applies for any pairs of x_i, x_j where $i \neq j$. Thus, we can assume that for a generic choice of the curves $d(t)$ and $r(t)$, $\gcd(x_i, x_j) = 1$, $i \neq j \in \{0, 1, 2, 3\}$.

Next, to show the parametrization is generic 1-1, set $v = 1$, and $x(t) \in (\mathbb{R}[t])^4$. If $x(t_1) = x(t_2)$, then

$$x(t_1) - x(t_2) = \mathbf{R}(t_1) r^T(t_1) - \mathbf{R}(t_2) r^T(t_2) = (0, 0, 0, 0)^T,$$

which yields that $(t_1 - t_2)$ is a factor of $x_i(t_1) - x_i(t_2)$, $i = 0, \ldots, 3$, and

$$\mathrm{Res}\left(\frac{x_0(t_1) - x_0(t_2)}{t_1 - t_2}, \frac{x_1(t_1) - x_1(t_2)}{t_1 - t_2}, t_2\right) = F(t_1)$$
$$\mathrm{Res}\left(\frac{x_2(t_1) - x_2(t_2)}{t_1 - t_2}, \frac{x_3(t_1) - x_3(t_2)}{t_1 - t_2}, t_2\right) = G(t_1)$$
$$\mathrm{Res}(F(t_1), G(t_1), t_1) = f(a_{00}, \ldots, a_{4m}, b_{00}, \ldots, b_{4n}).$$

Note that polynomials F and G are not identically zero, since we have estabilished that $\gcd(x_i, x_j) = 1$, $i \neq j \in \{0, 1, 2, 3\}$. This means that $x(t_1) = x(t_2)$ only holds for degree m curves d and degree n curves r whose coefficients satisfies the condition $f(a_{00}, \ldots, a_{4m}, b_{00}, \ldots, b_{4n}) = 0$.

This is true because one can always choose a generic 1-1 parametrization for the curve $d(t)$ of degree m and the curve $r(t)$ of degree n; and for any t such that $\det(\mathbf{R}) = (d_0^2 + d_1^2 + d_2^2 + d_3^2)^4 \neq 0$, the curve $x(t) = \mathbf{R} r^T$ generated by the quaternion product is isomorphism of curves $r(t)$, hence it is generic 1-1.

Therefore, for a generic choice of a pair of curves d and r, the parametrization is generically 1-1. ☐

Example 2.2. Recall in Example 2.1, we generated a space curve $x = drd^*$ where

$$\begin{aligned} d(t) &= (t^2 + 1, 0, t^2 - 1, 2t), \; r(t) = (1, t, t^2, t^3), \\ x(t) &= 2\left((t^2 + 1)^2, \quad t^3(t^2 - 3)(t^2 + 1), \quad t^2(3t^4 - 2t^2 + 3), \quad t(5t^4 - 2t^2 + 1)\right). \end{aligned}$$

We check that this parametrization is indeed generic 1-1.

3. Bases and Commutative Diagrams

3.1. Properties of the Rotation matrix R

In this section, we investigate the rotation matrix $\mathbf{R}$ given in Equation (2.4), where $d_0, \ldots, d_3 \in S = \mathbb{K}[t, v]$ are homogeneous forms of the same degree. First, we would like to briefly summarize a few technical algebraic properties of the matrix $\mathbf{R}$.

Lemma 3.1. *([13, Lemma 3.1]) Let $\mathbf{R} = (R_{ij})_{1 \leq i,j \leq 4}$, where R_{ij} is the entry located at i-th row and j-th column of the rotation matrix* $\mathbf{R}$.

1. *There is no common factor among the entries in any of the last three rows of the matrix R, or any of the last three columns of the matrix R, that is,*

$$\gcd(R_{i1}, R_{i2}, R_{i3}, R_{i4}) = \gcd(R_{1j}, R_{2j}, R_{3j}, R_{4j}) = 1, \text{ for } i, j = 2, 3, 4.$$

2. $R_{1j}^2 + R_{2j}^2 + R_{3j}^2 + R_{4j}^2 = (\sum_{i=0}^3 d_i^2)^2,\ j = 1, 2, 3, 4,$ *that is, the sum of squares of the entries in each column of* $\mathbf{R}$ *is* $(\sum_{i=0}^3 d_i^2)^2$.
3. $R_{j1}^2 + R_{j2}^2 + R_{j3}^2 + R_{j4}^2 = (\sum_{i=0}^3 d_i^2)^2,\ j = 1, 2, 3, 4,$ *that is, the sum of squares of the entries in each row of* $\mathbf{R}$ *is* $(\sum_{i=0}^3 d_i^2)^2$.
4. *Any* 2×2 *minor of* $\mathbf{R}$ *contains a factor* $\sum_{i=0}^3 d_i^2$*; any* 3×3 *minor of* $\mathbf{R}$ *contains a factor* $(\sum_{i=0}^3 d_i^2)^2$.

3.2. Basis

Now, we will find a μ-basis for the rational curve $x(\bar{t})$, and a basis for the special submodule generated by $\mathbf{R}\mathbf{p}_i$ of the syzygy module. We begin by a brief review of a μ-basis for a syzygy module.

Let $I = (f_0, f_1, f_2, f_3) \subset S = \mathbb{K}[\bar{t}]$ be an arbitrary homogeneous graded ideal generated by four homogeneous polynomials of degree s in the graded ring R over an infinite field $\mathbb{K}$ with $\gcd(f_0, f_1, f_2, f_3) = 1$. The Hilbert-Burch theorem [5, Chapter 6] says that the minimal free resolution of the ideal I has the following form:

$$0 \longrightarrow \bigoplus_{i=1}^3 S(-s-\mu_i) \xrightarrow{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3} S^4(-s) \xrightarrow{f_0, f_1, f_2, f_3} I \longrightarrow 0,$$

where $\mu_1 \le \mu_2 \le \mu_3$ and $\mu_1 + \mu_2 + \mu_3 = s = \deg(f_i)$. Thus the syzygies module $\mathrm{Syz}(f_0, f_1, f_2, f_3)$ is a free module generated by $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ called a μ-basis. It is known that for three elements $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathrm{Syz}(f_0, f_1, f_2, f_3)$ form a μ-basis if and only if the outer product

$$\begin{aligned}
\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}_o &= \begin{Bmatrix} p_{01} & p_{02} & p_{03} \\ p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{Bmatrix}_o \\
&= \left(\begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix}, - \begin{vmatrix} p_{01} & p_{02} & p_{03} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{vmatrix}, \begin{vmatrix} p_{01} & p_{02} & p_{03} \\ p_{11} & p_{12} & p_{13} \\ p_{31} & p_{32} & p_{33} \end{vmatrix}, - \begin{vmatrix} p_{01} & p_{02} & p_{03} \\ p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix} \right) \\
&= \lambda \begin{bmatrix} f_0, & f_1, & f_2, & f_3 \end{bmatrix}, \text{ where } \lambda \in \mathbb{K} \setminus \{0\}.
\end{aligned}$$

Note that the μ-basis elements are not unique, but the degrees $\deg(\mathbf{p}_i)$ for $i = 1, 2, 3$ are unique and $\sum_{i=1}^3 \deg(\mathbf{p}_i) = s$.

Now, we consider the rational curve $x(\bar{t}) = (x_0, x_1, x_2, x_3) = d(\bar{t})\, r(\bar{t})\, d^*(\bar{t})$ generated by two rational space curves $d(\bar{t}) = (d_0, d_1, d_2, d_3)$ and $r(\bar{t}) = (r_0, r_1, r_2, r_3)$ of degree m and n given by the projective parametrization $\mathbb{P}^1 \to \mathbb{P}^3$ as in Proposition 2.1. We will investigate the syzygies of the curve $x(\bar{t})$.

Lemma 3.2. *Let* $\beta = \left(\sum_{i=0}^3 d_i^2\right)^2$*, and* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *be a* μ*-basis of the curve* $r(\bar{t})$*. Then* $\mathbf{R}\mathbf{p}_i$*,* $i = 1, 2, 3$*, are three linearly independent syzygies of curve* $x(\bar{t})$ *over the homogeneous graded ring* $S = \mathbb{K}[\bar{t}]$*, and the outer product*

$$\{\mathbf{R}\mathbf{p}_1, \mathbf{R}\mathbf{p}_2, \mathbf{R}\mathbf{p}_3\}_o = \beta(x_0, x_1, x_2, x_3) = \beta\, x(\bar{t}).$$

Proof. It is easy to see that $\mathbf{R}\mathbf{p}_i \in \mathrm{Syz}(x(\bar{t}))$, $i = 1, 2, 3$, since

$$\mathbf{R}\mathbf{p}_i \cdot x = \mathbf{R}\mathbf{p}_i \cdot \mathbf{R}r^T = (\mathbf{p}_i)^T \mathbf{R}^T \mathbf{R} r^T = \beta^2 \mathbf{p}_i \cdot r = 0.$$

Furthermore, these three syzygies are linearly independent over the ring S, since the condition $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ form a μ-basis implies that

$$\begin{bmatrix} \mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{R} \sum_{i=1}^3 a_i \mathbf{p}_i = \mathbf{0} \iff \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, $\mathbf{R}\mathbf{p}_i \in \mathrm{Syz}(x(\bar{t}))$, $i = 1, 2, 3$, are three linearly independent over the ring S. Moreover, a computation via Mathematica shows the outer product

$$\{\mathbf{R}\mathbf{p}_1, \mathbf{R}\mathbf{p}_2, \mathbf{R}\mathbf{p}_3\}_o = \beta \mathbf{R} r^T = \beta\, x(\bar{t}), \text{ where } \beta = \left(\sum_{i=0}^{3} d_i^2\right)^2.$$

□

Geometrically, three syzygies $\mathbf{R}\mathbf{p}_i$, $i = 1, 2, 3$, are obtained by rotating the μ-basis of the curve $r(\bar{t})$ about the axis in the direction of the curve $d(\bar{t})$. Furthermore, Lemma 3.2 shows that the outer product of the syzygies $\mathbf{R}\mathbf{p}_1, \mathbf{R}\mathbf{p}_2, \mathbf{R}\mathbf{p}_3$ is a polynomial multiple of the rational curve $x(\bar{t})$, whereas the outer product of a μ-basis yields the constant multiple of the rational curve. This suggests that syzygies $\mathbf{R}\mathbf{p}_i$ are closely related to a μ-basis of the curve $x(\bar{t})$.

Theorem 3.1. *Let $\beta = \left(\sum_{i=0}^{3} d_i^2\right)^2$, and $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a μ-basis of the curve $r(\bar{t})$. Then*

$$\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = L \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T,$$

where L is a 3×3 matrix with entries in $S = \mathbb{K}[\bar{t}]$, $\det(L)$ is a non-zero constant multiple of β, and $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ form a μ-basis for $x(\bar{t})$.

Proof. First, we recall the Primitive Factorization Theorem [2] which states: Suppose that A is an $a \times b$ matrix $(a \leq b)$ with entries in $E[X]$, where E is a Euclidean domain. Let $d(X) \in E[X]$ denote the greatest common divisor of the a-th order minors of A. Then A can be factored as the product $A = LB$ where L is an $a \times a$ matrix, B is an $a \times b$ matrix, the entries of L and B are in $E[X]$, and $\det(L) = d(X)$.

The Primitive Factorization Theorem with $a = 3$ implies that

$$\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = L \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T,$$

where L is a 3×3 matrix with entries in S, and $\det(L)$ is a non-zero constant multiple of β.

Second, to see that the vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ form a μ-basis for $x(\bar{t})$, we recall that the Cauchy-Binet formula [1, Theorem 2.34, Page 210]: Suppose that A is an $a \times b$ matrix, B is an $b \times c$ matrix, I is a subset of $\{1, \ldots, a\}$ with k elements, and J is a subset of $\{1, \ldots, c\}$ with k elements. Let $[A]_{I,J}$ be the minor of A associated to the ordered sequences of indexes I and J. Then the Cauchy-Binet formula is

$$[AB]_{I,J} = \sum_K [A]_{I,K} [B]_{K,J}, \tag{3.1}$$

where the sum extends over all subsets K of $\{1, \ldots, b\}$ with k elements.

To apply the Cauchy-Binet formula to our situation, let $A = L$, $B = \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T$, J a subset of $\{1, 2, 3, 4\}$ with three elements, and $I = K = \{1, 2, 3\}$. Then by Lemma 3.2,

$$\beta x(\bar{t}) = \{\mathbf{R}\mathbf{p}_1, \mathbf{R}\mathbf{p}_2, \mathbf{R}\mathbf{p}_3\}_o = \left\{L \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T\right\}_o = \det(L) \begin{Bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{Bmatrix}_o.$$

Since $\det(L)$ is a non-zero constant multiple of β which is not zero, it follows $x(\bar{t}) = \{\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\}_o$. Thus, $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ form a μ-basis for the curve $x(\bar{t})$. □

The following example shows how to compute a μ-basis for the syzygy module of the curve $x(\bar{t})$ using Theorem 3.1.

Example 3.1. Recall in Example 2.1, a rational space curve $x(\bar{t}) = d(\bar{t}) r(\bar{t}) d^*(\bar{t})$ is generated by two rational space curves $d(\bar{t})$ and $r(\bar{t})$, where

$$\begin{aligned} d(\bar{t}) &= (t^2 + v^2, 0, t^2 - v^2, 2tv), \; r(\bar{t}) = (v^3, tv^2, t^2 v, t^3), \\ x(\bar{t}) &= 2(v^3(t^2+v^2)^2, t^3(t^2 - 3v^3)(t^2+v^2), t^2 v(3t^4 - 2t^2 v + 3v^3), tv^2(5t^4 - 2t^2v^2 + v^4)). \end{aligned}$$

We check that the columns of the matrix $[\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] = \begin{bmatrix} -t & 0 & 0 \\ v & -t & 0 \\ 0 & v & -t \\ 0 & 0 & v \end{bmatrix}$ form a μ-basis for the curve $r(\bar{t})$.

Over the homogeneous graded ring $S = \mathbb{K}[\bar{t}]$, we compute $\mathbf{Rp}_1, \mathbf{Rp}_2, \mathbf{Rp}_3$, and write as the row vectors in the following matrix

$$\begin{bmatrix} -2t(t^2+v^2)^2 & 0 & 4tv^2(t^2+v^2) & 2v(-2t^4+v^4) \\ 0 & -4tv^2(t^2+v^2) & 2v(-t^4-4t^2v^2+v^4) & 2(t^5+2t^3v^2-3tv^4) \\ 0 & 6t^4v+4t^2v^3-2v^5 & -2(t^5-4t^3v^2+3tv^4) & -4t^2v(t^2-3v^2) \end{bmatrix}$$

and the outer product

$$\left\{\mathbf{Rp}_1 \quad \mathbf{Rp}_2 \quad \mathbf{Rp}_3\right\}_o = \beta x(\bar{t}), \text{ where } \beta = 4(t^2+v^2)^4, \det(\mathbf{R}) = \beta^2, \ \mathbf{R}\mathbf{R}^T = \beta \mathbf{I}_{4\times 4}.$$

Furthermore,

$$\begin{bmatrix} \mathbf{Rp}_1{}^T \\ \mathbf{Rp}_2{}^T \\ \mathbf{Rp}_3{}^T \end{bmatrix} = \begin{bmatrix} 0 & -v(t^2+v^2) & -(t^2+v^2) \\ v(t^2+v^2) & t(t^2+3v^2) & 2tv \\ -2t(t^2+v^2) & -v(3t^2-v^2) & -5t^2-v^2 \end{bmatrix} \begin{bmatrix} -5t^2 & tv & 2t^3+tv^2 \\ -3tv & -v^2 & v^3 \\ t^2+2v^2 & -3tv & -tv^2 \\ -tv & 2t^2-v^2 & -v^3 \end{bmatrix}^T = L \begin{bmatrix} \mathbf{P}_1^T \\ \mathbf{P}_2^T \\ \mathbf{P}_3^T \end{bmatrix},$$

where $$L = \begin{bmatrix} 0 & -v(t^2+v^2) & -(t^2+v^2) \\ v(t^2+v^2) & t(t^2+3v^2) & 2tv \\ -2t(t^2+v^2) & -v(3t^2-v^2) & -5t^2-v^2 \end{bmatrix}, \ \det(L) = -2(t^2+v^2)^4 = -\frac{\beta}{2}.$$

We check the outer product $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}_o = -2x(\bar{t})$, hence $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ form a μ-basis for the homogeneous graded syzygy module of the curve $x(\bar{t})$.

We note that Theorem 3.1 computes a μ-basis $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ for the syzygy module of the curve $x(\bar{t})$, $\mathrm{Syz}(x_0, x_1, x_2, x_3)$, from the syzygies $\mathbf{Rp}_1, \mathbf{Rp}_2, \mathbf{Rp}_3$ over the homogeneous graded ring $S = \mathbb{K}[\bar{t}]$. In the next theorem, we express a basis for the module generated by $\mathbf{Rp}_1, \mathbf{Rp}_2, \mathbf{Rp}_3$ in terms of a basis of $\mathrm{Syz}(x_0, x_1, x_2, x_3)$ over the non-homogeneous ring $\mathbb{K}[t]$ by setting $v = 1$.

Theorem 3.2. *Let N denote the syzygy submodule generated by $\mathbf{Rp}_1, \mathbf{Rp}_2, \mathbf{Rp}_3$, and let $M = \mathrm{Syz}(x_0, x_1, x_2, x_3)$ over the non-homogeneous ring $\mathbb{K}[t]$. Then there exists a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subset \mathbb{K}^4[t]$ for M, and a nonzero element $\alpha = \sum_{i=0}^4 d_i^2(t)$ such that $\{\mathbf{a}_1, \alpha\mathbf{a}_2, \alpha\mathbf{a}_3\}$ form a basis for the module N.*

Proof. Set $v = 1$, by Theorem 3.1,

$$\left[\mathbf{Rp}_1 \quad \mathbf{Rp}_2 \quad \mathbf{Rp}_3\right]^T = L\left[\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\right]^T,$$

where L is a 3×3 matrix with entries in $\mathbb{K}[t]$, $\det(L)$ is a non-zero constant multiple of α^2, and $\mathbf{P}_i$, $i = 1, 2, 3$, form a μ-basis for the curve $x(t)$. Without loss of generality, we assume that $\det(L) = \alpha^2$ for simplicity.

We claim that $L = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} V$, where $U, V \in \mathrm{GL}(3, \mathbb{K}[t])$.

To prove our claim, we note that the ring $\mathbb{K}[t]$ is a PID, the Smith normal form [1, Theorem 3.1, Page 307] yields that

$$L = U \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} V, \text{ where } \alpha_j = \frac{D_j}{D_{j-1}}, \ D_j \text{ is the GCD of the } j \times j \text{ minors of } L, D_0 := 1,$$

and $U, V \in \mathrm{GL}(3, \mathbb{K}[t])$ are the invertible matrices over the ring $\mathbb{K}[t]$. Since $D_3 = \det(L) = \alpha^2$, to prove that $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \alpha$, we only need to show that $D_1 = 1$ and $D_2 = \alpha$.

To compute the GCD of the $j \times j$ minors, we again invoke the Cauchy-Binet formula [1, Theorem 2.34, Page 210]. First, we apply the Cauchy-Binet formula to the matrix $L\left[\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\right]^T$ by setting $A = L$ and $B = \left[\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\right]^T$. We note that for $j = 1, 2, 3$, the GCD of the $j \times j$ minors of the matrix $L\left[\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\right]^T$ is exactly the GCD of the $j \times j$ minors of L. This is true because the GCD of the $j \times j$ minors of the matrix $\left[\mathbf{P}_1 \quad \mathbf{P}_2 \quad \mathbf{P}_3\right]^T$ is 1; otherwise, the outer product of the $\mathbf{P}_i$ would be a polynomial multiple of the curve $x(t)$ contradicting the fact that $\mathbf{P}_i$, $i = 1, 2, 3$, form a μ-basis.

Second, we apply the Cauchy-Binet formula [1, Theorem 2.34, Page 210] to the matrix $\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T$ by setting $A = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}^T$ and $B = \mathbf{R}^T$. With a similar argument, we note that for $j = 1, 2, 3$, the GCD of the $j \times j$ minors of the matrix $\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T$ is exactly the GCD of the $j \times j$ minors of $\mathbf{R}^T$, hence the GCD of the $j \times j$ minors of $\mathbf{R}$.

Third, since $\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = L\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T$, the GCD of the $j \times j$ minors of L is the same as the GCD of the $j \times j$ minors of the matrix $\mathbf{R}$ for $j = 1, 2, 3$. It follows directly from Lemma 3.1 that GCD of 1×1 minor of L is 1, GCD of 2×2 minor of L is α, and GCD of 3×3 minor of L is α^2. Thus, we have shown that $D_1 = 1$, $D_2 = \alpha$, and $D_3 = \alpha^2$. Therefore, we proved our claim that

$$\alpha_1 = 1,\ \alpha_2 = \frac{D_2}{D_1} = \alpha,\ \alpha_3 = \frac{D_3}{D_2} = \alpha, \quad L = U\begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{bmatrix}V,\ U, V \in \mathrm{GL}(3, \mathbb{K}[t]).$$

Finally, since $U, V \in \mathrm{GL}(3, \mathbb{K}[t])$, the equality

$$\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = L\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T = U\begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{bmatrix}V\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T,$$

$$\iff \quad U^{-1}\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{bmatrix}V\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{bmatrix}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix}.$$

Since $U, V \in \mathrm{GL}(3, \mathbb{K}[t])$, the three rows of the matrix $U^{-1}\begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T$ form a basis for the module N; and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ which are the three rows of the matrix $V\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T$ form a basis for the module M. Therefore, we have shown that there exists a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subset \mathbb{K}^4[t]$ for M, and a nonzero element $\alpha = \sum_{i=0}^4 d_i^2(t)$ such that $\{\mathbf{a}_1, \alpha\mathbf{a}_2, \alpha\mathbf{a}_3\}$ form a basis for the module N. □

Remark 3.1. Theorem 3.2 provides the means to compute $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{K}[t]$ which form a basis for the module M over the non-homogeneous ring $\mathbb{K}[t]$. In general, the homogenization of $\mathbf{a}_i$ do not form a μ-basis for the syzygy module $\mathrm{Syz}(x(\bar{t}))$ over the homogeneous graded ring $S = \mathbb{K}[\bar{t}]$. Over the non-homogeneous ring $\mathbb{K}[t]$, Theorem 3.2 implies that $\mathrm{Ann}(M/N) = \langle\alpha\rangle \subset \mathbb{K}[t]$.

The following example is a continuation of Example 3.1, where we show how to compute a basis for the module N over the non-homogeneous ring $\mathbb{K}[t]$ using Theorem 3.2.

Example 3.2. Recall in Example 3.1, a rational space curve $x(\bar{t}) = d(\bar{t})r(\bar{t})d^*(\bar{t})$ is generated by two rational space curves $d(\bar{t})$ and $r(\bar{t})$, where

$$\begin{aligned} d(\bar{t}) &= (t^2 + v^2, 0, t^2 - v^2, 2tv),\ r(\bar{t}) = (v^3, tv^2, t^2v, t^3),\\ x(\bar{t}) &= 2(v^3(t^2 + v^2)^2, t^3(t^2 - 3v^3)(t^2 + v^2), t^2v(3t^4 - 2t^2v + 3v^3), tv^2(5t^4 - 2t^2v^2 + v^4)). \end{aligned}$$

Over the homogeneous graded ring $\mathbb{K}[\bar{t}]$, a matrix L and a μ-basis $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ for the curve $x(\bar{t})$ are obtained in Example 3.1, where

$$L = \begin{bmatrix}0 & -v(t^2 + v^2) & -(t^2 + v^2)\\ v(t^2 + v^2) & t(t^2 + 3v^2) & 2tv\\ -2t(t^2 + v^2) & -v(3t^2 - v^2) & -5t^2 - v^2\end{bmatrix}, \quad \begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix} = \begin{bmatrix}-5t^2 & tv & 2t^3 + tv^2\\ -3tv & -v^2 & v^3\\ t^2 + 2v^2 & -3tv & -tv^2\\ -tv & 2t^2 - v^2 & -v^3\end{bmatrix}.$$

Now, by Theorem 3.2, over the non-homogeneous ring $\mathbb{K}[t]$ by setting $v = 1$,

$$L = \begin{bmatrix}-1 & -t/2 & 0\\ 2t & 1 + t^2 & 0\\ -1/2 & t & 1/2\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & (1 + t^2)^2 & 0\\ 0 & 0 & (1 + t^2)^2\end{bmatrix}\begin{bmatrix}0 & 1 & -t\\ 0 & 0 & 1\\ 1 & (t + t^3)/2 & -1\end{bmatrix}^{-1} = U\Lambda V.$$

We compute a basis, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for $\mathrm{Syz}(x_0, x_1, x_2, x_3)$:

$$\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{bmatrix} = V\begin{bmatrix}\mathbf{P}_1^T\\ \mathbf{P}_2^T\\ \mathbf{P}_3^T\end{bmatrix} = \begin{bmatrix}2t(1 + t^2)^2 & 2t^2(1 + t^2) & t(-5 + t^4) & -2 + 3t^2 - t^6\\ -4t^2 & -4t & 2(1 - t^2) & 2t(t^2 - 1)\\ t & -1 & -3t & 2t^2 - 1\end{bmatrix}.$$

We verify that

$$\begin{bmatrix} \mathbf{a}_1 \\ (1+t^2)^2\mathbf{a}_2 \\ (1+t^2)^2\mathbf{a}_3 \end{bmatrix} = \Lambda \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = U^{-1} \begin{bmatrix} \mathbf{R}\mathbf{p}_1{}^T \\ \mathbf{R}\mathbf{p}_2{}^T \\ \mathbf{R}\mathbf{p}_3{}^T \end{bmatrix},$$

and hence, $\{\mathbf{a}_1, (1+t^2)^2\mathbf{a}_2, (1+t^2)^2\mathbf{a}_3\}$, which are the rows of the following matrix

$$\begin{bmatrix} 2t(1+t^2)^2 & 2t^2(1+t^2) & t(-5+t^4) & -2+3t^2-t^6 \\ -4t^2(1+t^2)^2 & -4t(1+t^2)^2 & 2(1-t^2)(1+t^2)^2 & 2t(t^2-1)(1+t^2)^2 \\ t(1+t^2)^2 & -(1+t^2)^2 & -3t(1+t^2)^2 & (2t^2-1)(1+t^2)^2 \end{bmatrix}$$

form a basis for the module generated by $\mathbf{R}\mathbf{p}_1, \mathbf{R}\mathbf{p}_2, \mathbf{R}\mathbf{p}_3$ over the non-homogeneous ring $\mathbb{K}[t]$.

3.3. *Commutative Diagram*

Finally, we will summarize our result in a commutative diagram.

Theorem 3.3. *Let $\mathbf{L}$ be an $n \times n$ matrix with entries homogeneous forms in $S = \mathbb{K}[\bar{t}]$ and $\det(\mathbf{L}) \not\equiv 0$. Then*

$$0 \longrightarrow S^n \xrightarrow{\mathbf{L}} S^n \longrightarrow S^n/\mathbf{L}S^n \longrightarrow 0.$$

Proof. Let A be the fraction field of S. If $\det(\mathbf{L}) \not\equiv \mathbf{0}$, then $\mathbf{L} : S^n \to S^n$ is injective. Let $\mathbf{v}$ be a vector in S^n such that $\mathbf{L}\mathbf{v} = \mathbf{0}$. Then $\mathbf{L}\mathbf{v} = \mathbf{0}$ in A^n. By linear algebra $\mathbf{v} = \mathbf{0}$ since $S^n \to A^n$ is injective. Thus we get an exact sequence

$$0 \longrightarrow S^n \xrightarrow{\mathbf{L}} S^n \longrightarrow S^n/\mathbf{L}S^n \longrightarrow 0.$$

□

Corollary 3.1. *Let $\mathbf{R}$ be the rotation matrix given in Equation (2.4), where $d_0, \ldots, d_3 \in S = \mathbb{K}[\bar{t}]$ are homogeneous forms of the same degree. Then*

$$0 \longrightarrow S^4 \xrightarrow{\mathbf{R}} S^4 \longrightarrow S^4/\mathbf{R}S^4 \longrightarrow 0.$$

Proof. This is a direct consequence of Theorem 3.3 since $\det(\mathbf{R}) \not\equiv 0$. □

Theorem 3.4. *Let $\beta = \left(\sum_{i=0}^{3} d_i^2\right)^2$, and $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a μ-basis of the curve $r(\bar{t})$. Set $\mathbf{L} = (\mathrm{Adj}(L))^T$, the transpose of the adjoint matrix of L in Theorem 3.1. Since $\det(L)$ is a non-zero constant multiple of β, for simplicity, we assume that $\det(L) = \beta$. Then the following diagram has three exact rows (columns), and the diagram commutes:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^3 & \xrightarrow{\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3} & S^4 & \xrightarrow{x=[r_0,r_1,r_2,r_3]\mathbf{R}^T} & \langle x \rangle & \longrightarrow & 0 \\
 & & \mathbf{L}\downarrow & & \mathbf{R}^T\downarrow & & \downarrow & & \\
0 & \longrightarrow & S^3 & \xrightarrow{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} & S^4 & \xrightarrow{[r_0,r_1,r_2,r_3]} & \langle r \rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^3/\mathbf{L}S^3 & \longrightarrow & S^4/\mathbf{R}^T S^4 & \longrightarrow & \langle r \rangle/\langle x \rangle & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. The first two rows of the diagram are the minimal free resolutions of the ideals generated by the space curves $x(\bar{t})$ and $r(\bar{t})$ given by the respective μ-bases, so these are exact. Furthermore, the columns are exact, where the first two columns are exact by Theorem 3.1 and Corollary 3.3. We observe that the diagram commutes, in particular, by Theorem 3.1,

$$\begin{aligned}
& L\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T = \begin{bmatrix}\mathbf{R}\mathbf{p}_1 & \mathbf{R}\mathbf{p}_2 & \mathbf{R}\mathbf{p}_3\end{bmatrix}^T = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}^T \mathbf{R}^T \\
\iff \quad & \mathrm{Adj}(L)L\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T \mathbf{R} = \mathrm{Adj}(L)\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}^T \mathbf{R}^T\mathbf{R} \\
\iff \quad & \beta\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}^T \mathbf{R} = \beta\mathrm{Adj}(L)\begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}^T \\
& \text{since } \mathrm{Adj}(L)L = \beta\mathbf{I}_{3\times 3},\ \mathbf{R}^T\mathbf{R} = \beta\mathbf{I}_{4\times 4} \\
\iff \quad & \mathbf{R}^T\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix} = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}(\mathrm{Adj}(L))^T = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3\end{bmatrix}\mathbf{L}.
\end{aligned}$$

Finally the last row is exact by the snake lemma. □

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

References

[1] Adkins, W., Weintraub, S.: Algebra: An approach via module theory. Graduate Texts in Mathematics 136. Springer. New York. (1992).
[2] Bose, N. K.: Multidimensional systems theory and applications. Springer. New York. (1995).
[3] Chaikin, G.: *An algorithm for high speed curve generation*. Computer Graphics and Image Processing. **3**, 346-349 (1974).
[4] Cox, D.: *Equations of parametric curves and surfaces via syzygies*. Contemporary Mathematics. **286**, 1-20 (2001).
[5] Cox, D., Little, J., O'Shea, D.: Using algebraic geometry. Graduate Texts in Mathematics 185. Springer. New York. (1998).
[6] Eisenbud, D.: The geometry of syzygies. Graduate Texts in Mathematics 229. Springer. New York. (2005).
[7] Evans, E. G., Griffith, P.: Syzygies. London Mathematics Society Lecture Notes Series 106. Cambridge University Press. Cambridge. (1985).
[8] Farin, G.: *Algorithms for rational Bézier curves*. Computer Aided Design. **15**, 73-77 (1983).
[9] Farin, G.: Curves and surfaces for computer aided geometric design. Morgan-Kaufmann. Massachusetts. (2001).
[10] Goldman, R.: Rethinking quaternions: theory and computation. Synthesis Lectures on Computer Graphics and Animation, ed. Brian A. Barsky, No. 13. Morgan & Claypool Publishers. San Rafael. (2010).
[11] Hagen, H.:*Geometric spline curves*. Computer Aided Geometric Design. **2**, 223-228 (1985).
[12] Northcott, D. G.: *A homological investigation of a certain residual ideal*. Math. Annalen. **150**, 99-110 (1963).
[13] Wang, H., Goldman, R.:*Surfaces of revolution with moving axes and angles*. Graphical Models. **106**, (2019). https://doi.org/10.1016/j.gmod.2019.101047.

Affiliations

J. WILLIAM HOFFMAN
ADDRESS: Lousiana State University, Baton Rouge, LA 70803, USA.
E-MAIL: hoffman@math.lsu.edu
ORCID ID: 0000-0002-3590-3029

HAOHAO WANG
ADDRESS: Southeast Missouri State University, Dept. of Mathematics, MS6700, Cape Girarduea, MO, USA.
E-MAIL: hwang@semo.edu
ORCID ID: 0000-0001-7942-5623