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Wijsman Summability Through Orlicz Function Sequences

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ABSTRACT

The Wijsman convergence is a type of convergence for sequences of closed sets in metric spaces, utilizing the distance from a point to a set. This study introduces a new sequence space by defining a summability concept for sequences of closed sets in the Wijsman sense, using sequences of Orlicz functions. Various inclusion theorems related to the space of Wijsman statistically convergent sequences of sets have been presented, considering different parameters used in the definition of this set sequence space. Additionally, in the obtained results, a concept of density has been employed using weight functions instead of asymptotic density.

Keywords: Density, Orlicz function, Statistical convergence, Wijsman convergence

Orlicz Fonksiyon Dizileri ile Wijsman Toplanabilirlik

ÖZ

Wijsman yakınsaması, metrik uzaylarda kapalı küme dizileri için bir yakınsama türüdür ve bir noktanın bir kümeye olan uzaklığını kullanır. Bu çalışmada, Orlicz fonksiyonlarının dizileri kullanılarak, Wijsman anlamında kapalı kümeler dizileri için bir toplanabilirlik kavramı tanımlanarak yeni bir dizi uzayı önerilmiştir. Bu küme dizileri uzayının tanımlanmasında kullanılan parametrelerin farklılaşması durumunda veya Wijsman istatistiksel yakınsak küme dizilerinin uzayıyla ilişkili çeşitli kapsama teoremleri sunulmuştur. Ayrıca, elde edilen sonuçlarda, asimptotik yoğunluk yerine ağırlık fonksiyonları kullanılarak elde edilen bir yoğunluk kavramı kullanılmıştır.

Anahtar Kelimeler: Yoğunluk, Orlicz fonksiyonu, İstatistiksel yakınsaklık, Wijsman yakınsaklık

INTRODUCTION

The concept of convergence is a critical tool used in various areas of mathematics. Many properties of given space can be expressed in terms of sequences that either converge or do not converge. Different types of convergence are defined based on the elements that constitute sequences or nets, or according to the underlying mathematical structure. Comparisons can also be made between different types of convergence defined within the same structure, and they can be considered together.

One such type of convergence is the Wijsman convergence, which is defined using the distance of a point from a set with respect to a metric function. This form of convergence was initially defined by R.A. Wijsman and applied to sequences of convex subsets in Euclidean spaces ([1,2]). Subsequently, various generalizations of this type of convergence, its application to different mathematical structures, and comparisons with other types of convergence have been explored (see [3]).

Nuray and Rhoades were the first to examine Wijsman convergence alongside statistical convergence [4]. Later, it has been addressed from multiple perspectives with many generalizations (see [5], [6], [7, 8], [9], [10], [11], [12])

Aral et al. [13] defined Wijsman $\rho-$statistical convergence and Wijsman strongly $\rho-$summable sequences of sets in a metric space using a density function different from the natural density function. Additionally, they defined a sequence space consisting of sequences of sets by utilizing Orlicz functions.

The primary motivation for our work has been [13]. Instead of using a single Orlicz function as defined in [13], we consider a sequence of Orlicz functions to obtain generalized similar results.

MATERIAL and METHOD

Let (E, d) be a metric space and U be non empty closed subsets of E. The distance of an element $p \in E$ to U is defined as follows:

$$d(p, U) = \inf\{d(p, a) : a \in U\}.$$

For simplicity, for a metric space (E, d), the family of all closed subsets will be denoted by $cl(P(E))$.

Definition 2.1 [1] Let (E, d) be a metric space, $\{U_k\} \subset cl(P(E)$ and $U \in cl(P(E))$. Then, $\{U_k\}$ is said to be Wijsman convergent to U provided that

$$\lim_{k\to\infty} d(p,U_k) = d(p,U)$$
holds for all $p \in E$.

In recent times, one of the most intensively studied types of convergence is statistical convergence, first defined by Fast [14] and Steinhaus [15]. Preliminary works in this area include those by Schoenberg [16], Salat [17], Fridy [18], and Connor [19]. Over time, the concept of statistical convergence has been generalized in various perspectives and mathematical structures (see [20], [21] [22], [23]).

Statistical convergence is based on the density of sets, and the natural density of a set $G \subseteq \mathbb{N}$ is defined by the limit (if exists)
$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n: k \in G\}|$$
where $|\{k \le n: k \in G\}|$ denotes the number of elements of G that are less than or equal to n (see [18]).

The concept of Wijsman convergence was subsequently generalized in the context of statistical convergence by Nuray and Rhoades [4]. The natural (asymptotic) density function, which forms the basis of statistical convergence, is defined as follows:

Definition 2.2 [4] Given $(U_k) \subset cl(P(E)$ and $U \in cl(P(E))$, the sequence (U_k) is called Wijsman statistical convergent to U if, for $\varepsilon > 0$ and for every $p \in E$,
$$\lim_{n\to\infty} \frac{1}{n} |k \le n: |d(p,U_k) - d(p,U)| \ge \varepsilon| = 0.$$
Subsequently, it has been generalized using various mathematical tools. One such generalization is provided by Aral et al. [13], who employed a different density function. For this density function, a non-decreasing sequence $\rho = (\rho_n)$ of positive reals numbers is used such that
$$\rho_n \to \infty, \limsup_{n\to\infty} \frac{\rho_n}{n} < \infty, \quad \Delta\rho_n = \rho_{n+1} - \rho_n, \ \Delta\rho_n = O(1), \qquad (2.1)$$
for every $n \in \mathbb{Z}^+$.

Definition 2.3 [13] Let (E,d) be a metric space and $\alpha \in (0,1]$. It is called that the sequence $(U_k) \subset cl(P(E))$ is Wijsman ρ-statistical convergent to $U \in cl(P(E))$ of order α (or WS_ρ^α-convergent to U) if for every $\epsilon > 0$ and $p \in E$,
$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |k \le n: |d(p,U_k) - d(p,U)| \ge \varepsilon| = 0.$$
where $\rho = (\rho_n)$ is defined as in (2.1).

Furthermore, in [13], a new class of sequences of sets is defined as follows, using an Orlicz function $\Phi: [0,\infty) \to [0,\infty)$ such that Φ is non-decreasing, convex and continuous, $\Phi(0) = 0$ and $\Phi(p) \to \infty$ as $p \to \infty$.

Definition 2.4 [13] Let (E,d) be a metric space, Φ be an Orlicz function, $q = (q_k) \subset \mathbb{R}^+$, $\alpha \in (0,1]$ and $\rho = (\rho_n)$ be a sequence such as in (2.1). Then, for $\lambda > 0$,
$$\left(\mathcal{W}_\rho^\alpha, [\Phi,(q)]\right) = \{\{U_k\} \subset cl\big(P(E)\big):$$
$$\frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[\Phi\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k} \to 0,$$
for some U and for $p \in E\}$

A sequence $\mathcal{M} = (\Phi_k)$ of Orlicz functions is known as a Musielak-Orlicz function, and we represent the collection of all Orlicz functions by $\mathcal{O}$.

RESULTS

In the literature, numerous generalizations of the natural density concept exist. One such modification, proposed by Balcerzak et al. in [24], introduces the notion of a weight function. A function $g: \mathbb{N} \to [0,\infty)$ is termed a weight function if it meets the criteria $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$. The notation $\mathcal{G}$ stands for the set of all such kind functions.

Let $g \in \mathcal{G}$ and consider a subset $G \subseteq \mathbb{N}^+$. In [24], the density of G with respect to the weight function g, referred to as the g-weighted density, is given by the limit
$$\lim_{n\to\infty} \frac{1}{g(n)} |\{k \le n: k \in G\}|,$$
whenever it exists.

For sets $G, F \subset \mathbb{N}$, the following properties are observed: the density $\delta_g(G)$ is zero for finite sets G; if $\delta_g(G)$ exists for $g \in \mathcal{G}$, then $\delta_g(\mathbb{N}\backslash G) = \delta_g(\mathbb{N}) - \delta_g(G)$; and if $G \subseteq F$, then $\delta_g(G) \le \delta_g(F)$ holds.

In the subsequent results, $\ell_\infty^+(\mathbb{R})$ will be used as the set of bounded sequences consisting of strictly positive real numbers. Suppose that $q = (q_k) \in \ell_\infty^+(\mathbb{R})$ such that $0 < q_k \le \sup_k\{q_k\} = H < \infty$, and $K = \max\{1, 2^{H-1}\}$. For the real sequences (c_k) and (d_k), the following inequality will be utilized in proofs:
$$|c_k + d_k|^{q_k} \le K(|c_k|^{q_k} + |d_k|^{q_k}) \qquad (3.1)$$
and
$$|c_k|^{q_k} \le \max\{1, |a|^H\}$$
for all $a \in \mathbb{R}$.

Definition 3.1 Let (E,d) be a metric space and $g \in \mathcal{G}$. The sequence $(U_k) \subset cl(P(E))$ is Wijsman g-statistical convergent (shortly $\mathcal{WS}_g$-convergent) to a closed set $U \in cl(P(E))$ if, for every $p \in E$ and for every $\epsilon > 0$,
$$\lim_{n\to\infty} \frac{1}{g(n)} |k \le n: |d(p,U_k) - d(p,U)| \ge \varepsilon| = 0.$$
We use $U_k \xrightarrow{\mathcal{WS}_g} U$ to indicate this convergence, and we denote the set of all sequences of closed sets that are Wijsman g-statistically convergent by $\mathcal{WS}_g$.

Remark 3.2 Let $\alpha \in (0,1]$ and let $\rho = (\rho_n)$ be a sequence that satisfies the conditions in (2.1). Specifically, for $g(n) = \rho_n^\alpha$, Definition 3.1 coincides with Definition 1 in [13], and $\mathcal{WS}_g$ corresponds to $\mathcal{WS}_\rho^\alpha$.Thus, Definition 3.1 generalizes the definitions in [12] and [4] (see [13]).

Consequently, given Definition 3.1 with the g-density function, results analogous to those in [13] can be easily obtained. However, this study focuses on the Wijsman summability of sequences of closed sets using g-density functions and sequences of Orlicz functions.

Definition 3.3 Let (E, d) denote a metric space. Consider $\mathcal{M} = (\Phi_k)$ as a sequence of Orlicz functions, $q = (q_k) \subset \mathbb{R}^+$, and $g \in \mathcal{G}$. We say that a sequence $\{U_k\}$ of closed subsets of E is Wijsman $(\mathcal{M}, g, q)$-summable to a closed set $U \subset E$ if there exists a $\lambda \in \mathbb{R}^+$ such that for every $p \in E$, following equality holds:

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k} = 0.$$

We denote this by $U_k \xrightarrow{\mathcal{W}_{\mathcal{M}}^{g,q}} U$. The collection of all sequences $\{U_k\}$ of closed subsets of E that are Wijsman $(\mathcal{M}, g, q)$-summable to some closed subset U is denoted by $\mathcal{W}_g[\mathcal{M}, q]$. Formally,

$$\mathcal{W}_g[\mathcal{M}, q] = \left\{\{U_k\} \subset P(E) | U_k \xrightarrow{\mathcal{W}_{\mathcal{M}}^{g,q}} U \text{ for some } U \in cl(P(E))\right\}.$$

Remark 3.4 Let $\alpha \in (0,1]$ and let $\rho = (\rho_n)$ be a sequence that satisfies the conditions in (2.1). Specifically, for $g(n) = \rho_n^\alpha$ and $\mathcal{M} = (\Phi_k)$ being a constant sequence, Definition 3.3 coincides with Definition 3 introduced in [13], and $\mathcal{W}_g[\mathcal{M}, q]$ corresponds to $\mathcal{W}_\rho^\alpha[\Phi, (q)]$.

Theorem 3.5 Let $\mathcal{M} = (\Phi_k) \subset \mathcal{O}$ and $q = (q_k)$ be a sequence in $\mathbb{R}^+$ such that $\inf_{k\in\mathbb{N}}\{q_k\} > 0$. If $\{U_k\} \in \mathcal{W}_g[\mathcal{M}, q]$ and $U_k \xrightarrow{\mathcal{W}_{\mathcal{M}}^{g,q}} U$, then $U \in cl(P(E))$ is unique.

Proof. Let us assume that $U_k \xrightarrow{\mathcal{W}_{\mathcal{M}}^{g,q}} U_1$ and $U_k \xrightarrow{\mathcal{W}_{\mathcal{M}}^{g,q}} U_2$, where $U_1, U_2 \in cl(P(X))$. Hence, there exist positive real numbers λ_1 and λ_2 such that

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_1)|}{\lambda}\right)\right]^{q_k} = 0 \qquad (3.2)$$

and

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_2)|}{\lambda}\right)\right]^{q_k} = 0. \qquad (3.3)$$

Let $\lambda = \max\{\lambda_1, \lambda_2\}$, $0 < h = \inf\{q_k\} \leq q_k \leq \sup_k\{q_k\} = H$, and $K = \max\{1, 2^{H-1}\}$. For every $k \in \mathbb{N}$, Φ_k is a non-decreasing and convex, by (3.1), we obtain the following inequalities:

$$0 \leq \left[\Phi_1\left(\frac{|d(p,U_1) - d(p,U_2)|}{\lambda}\right)\right]^{h}$$

$$\leq \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_1) - d(p,U_2)|}{\lambda}\right)\right]^{q_k}$$

$$\leq K \sum_{k=1}^{n} \frac{1}{2^{q_k}}\left(\left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_1)|}{\lambda_1}\right)\right]^{q_k} + \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_2)|}{\lambda_2}\right)\right]^{q_k}\right)$$

$$\leq K \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_1)|}{\lambda_1}\right)\right]^{q_k} + K \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_2)|}{\lambda_2}\right)\right]^{q_k}.$$

Thus, using the inequalities in (3.2) and (3.3), and taking the limit as $n \to \infty$, we have

$$\left[\Phi_1\left(\frac{|d(p,U_1) - d(p,U_2)|}{\lambda}\right)\right]^{h} = 0.$$

for every $p \in E$. This implies that $U_1 = U_2$.

Theorem 3.6 Let (E, d) be a metric space, $q = (q_k) \in \ell_\infty^+(\mathbb{R})$ such that $0 < h = \inf\{q_k\} \leq q_k \leq \sup_k\{q_k\} = H < \infty$, $g \in \mathcal{G}$ and $\mathcal{M} = (\Phi_k)$ be a sequence of Orlicz functions such that $\inf_{k\in\mathbb{N}}(\Phi_k(t)) > 0$ for every $t \in (0, \infty)$. Then,

$$\mathcal{W}_g[\mathcal{M}, q] \subset \mathcal{WS}_g.$$

Proof. Suppose that $\{U_k\} \in \mathcal{W}_g[\mathcal{M}, q]$ and $\varepsilon > 0$. For every $n \in \mathbb{N}$, it holds that

$$\frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k}$$

$$= \frac{1}{g(n)} \sum_{\substack{k=1 \\ |d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k}$$

$$+ \frac{1}{g(n)} \sum_{\substack{k=1 \\ |d(p,U_k)-d(p,U)|<\varepsilon}}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k}$$

$$\geq \frac{1}{g(n)} \sum_{\substack{k=1 \\ |d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k}$$

$$\geq \frac{1}{g(n)} \sum_{\substack{k=1 \\ |d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n} \left[\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right]^{q_k}$$

$$\geq \frac{1}{g(n)} \sum_{\substack{k=1 \\ |d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n} \min\left\{\left(\inf\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^{h}, \left(\inf\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^{H}\right\}$$

$$\geq \frac{1}{g(n)} |\{k \leq n : |d(p,U_k) - d(p,U)| \geq \varepsilon\}| \,.\, A$$

where $A = \min\left\{\left(\inf\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^{h}, \left(\inf\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^{H}\right\}$. Thus, it can be seen that

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k} = 0$$

implies that

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$$\lim_{n\to\infty}\frac{1}{g(n)}\left|\{k\leq n: |d(p,U_k)-d(p,U)|\geq\varepsilon\}\right|=0.$$

In other words, $\{U_k\}\in\mathcal{WS}_g$.

Theorem 3.7 Let (E,d) be a metric space, $q=(q_k)\in\ell_\infty^+(\mathbb{R})$ such that $0<h=\inf\{q_k\}\leq q_k\leq\sup_k\{q_k\}=H<\infty$, $\mathcal{M}=(\Phi_k)$ be a bounded sequence of Orlicz functions and $g\in\mathcal{G}$. Then,

$$\mathcal{WS}_g\subset\mathcal{W}_g[\mathcal{M},q].$$

Proof. Suppose that $\{U_k\}\in\mathcal{WS}_g$ and $\varepsilon>0$. Since $\{U_k\}$ is bounded, there exists $C\in\mathbb{R}^+$ such that $|d(p,U_k)-d(p,U)|<C$ for every $n\in\mathbb{N}$. In that case, for any $\epsilon>0$, the following inequalities hold:

$$\frac{1}{g(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$=\frac{1}{g(n)}\sum_{\substack{k=1\\|d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$+\frac{1}{g(n)}\sum_{\substack{k=1\\|d(p,U_k)-d(p,U)|<\varepsilon}}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$\leq\frac{1}{g(n)}\sum_{\substack{k=1\\|d(p,U_k)-d(p,U)|\geq\varepsilon}}^{n}B$$
$$+\frac{1}{g(n)}\sum_{\substack{k=1\\|d(p,U_k)-d(p,U)|<\varepsilon}}^{n}\left[\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right]^{q_k}$$
$$\leq B\frac{1}{g(n)}\left|\{k\leq n: |d(p,U_k)-d(p,U)|\geq\varepsilon\}\right|$$
$$+\frac{n}{g(n)}\max\left\{\left(\sup\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^h,\left(\sup\Phi_k\left(\frac{\varepsilon}{\lambda}\right)\right)^H\right\}$$

where $0<h=\inf\{q_k\}\leq q_k\leq\sup\{q_k\}=H<\infty$ and

$$B=\max\left\{\left(\sup\Phi_k\left(\frac{C}{\lambda}\right)\right)^h,\left(\sup\Phi_k\left(\frac{C}{\lambda}\right)\right)^H\right\}.$$

It follows that $\{U_k\}\in\mathcal{W}_g[\mathcal{M},q]$.

Theorem 3.8 Let (E,d) be a metric space, $q=(q_k)\in\ell_\infty^+(\mathbb{R})$, $\mathcal{M}=(\Phi_k)$ be a bounded sequence of Orlicz functions and $g,h\in\mathcal{G}$. The following assertions are valid:

1. Assume there is a positive constant L and $k_0\in\mathbb{N}$ such that $\frac{g(n)}{h(n)}\leq L$ holds for all $n\geq k_0$. In that case,
$$\mathcal{W}_g[\mathcal{M},q]\subset\mathcal{W}_h[\mathcal{M},q].$$
2. Assume there is a positive constant l and $k_0\in\mathbb{N}$ such that $l\leq\frac{g(n)}{h(n)}$ holds for all $n\geq k_0$. In that case,
$$\mathcal{W}_h[\mathcal{M},q]\subset\mathcal{W}_g[\mathcal{M},q].$$

3. Assume there are positive constants $l,L>0$ and $k_0\in\mathbb{N}$ such that $l\leq\frac{g(n)}{h(n)}\leq L$ holds for all $n\geq k_0$. In that case,
$$\mathcal{W}_g[\mathcal{M},q]=\mathcal{W}_h[\mathcal{M},q].$$

Proof. Suppose that $\{U_k\}\in\mathcal{W}_g[\mathcal{M},q]$. Then, there exists a closed set $U\subset E$ and $\lambda>0$ such that, for $p\in E$,

$$\lim_{n\to\infty}\frac{1}{g(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}=0.$$

By hypothesis, there exists a number $L>0$ and $k_0\in\mathbb{N}$ such that $\frac{g(n)}{h(n)}\leq L$ holds for all $n\geq k_0$. Hence, we derive the following inequlity:

$$\frac{1}{h(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$\leq L\frac{1}{g(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k} \tag{3.4}$$

Thus, by taking limit as $n\to\infty$, the equality (3.4) implies that $\{U_k\}\in\mathcal{W}_h[\mathcal{M},q]$. The proof of (ii) is similar and (iii) is a consequence of (i) and (ii).

Theorem 3.9 Let $\mathcal{M}=(\Phi_k)\in\mathcal{O}$. If $q=(q_k)$ and $r=(r_k)$ are two sequences in $\mathbb{R}^+$ such that $0<q_k\leq r_k<\infty$ for each $k\in\mathbb{N}$, then

$$\mathcal{W}_g[\mathcal{M},q]\subset\mathcal{W}_g[\mathcal{M},r].$$

Proof. For $\{U_k\}\in\mathcal{W}_g[\mathcal{M},q]$, there exists a $\lambda\in\mathbb{R}^+$ such that the following equality

$$\frac{1}{g(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$\leq\frac{1}{g(n)}\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{r_k}$$

holds for every $n\in\mathbb{N}$. It then follows that $\{U_k\}\in\mathcal{W}_g[\mathcal{M},r]$.

If we take the sequence $q=(q_k)$ as a constant one sequence, then $\mathcal{W}_g[\mathcal{M},q]$ will be denoted by $\mathcal{W}_g[\mathcal{M}]$.

Corollary 3.10 Let $\mathcal{M}=(\Phi_k)\subset\mathcal{O}$, $q=(q_k)$ be a sequence in $\mathbb{R}^+$, and $g\in\mathcal{G}$. Then,

1. If, for each $k\in\mathbb{N}$, $0<q_k<1$ then
$$\mathcal{W}_g[\mathcal{M},q]\subset\mathcal{W}_g[\mathcal{M}].$$
2. If, for each $k\in\mathbb{N}$, $q_k\geq1$ then
$$\mathcal{W}_g[\mathcal{M}]\subset\mathcal{W}_g[\mathcal{M},q].$$

Theorem 3.11 Let $\mathcal{M}=(\Phi_k)\subset\mathcal{O}$, $\mathcal{T}=(\Omega_k)\subset\mathcal{O}$, $g\in\mathcal{G}$ and $q=(q_k)\subset\mathbb{R}^+$. If the inequality $\Phi_k\leq\Omega_k$ holds for every $k\in\mathbb{N}$, then

$$\mathcal{W}_g[\mathcal{T},q]\subseteq\mathcal{W}_g[\mathcal{M},q]$$

holds.

Proof. Suppose that $\{U_k\}\in\mathcal{W}_g[\mathcal{T},q]$. The desired inclusion is readily apparent from the following inequality:

$$\sum_{k=1}^{n}\left[\Phi_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}$$
$$\leq\sum_{k=1}^{n}\left[\Omega_k\left(\frac{|d(p,U_k)-d(p,U)|}{\lambda}\right)\right]^{q_k}.$$

Remark 3.12 Any two Orlicz functions Φ_1 and Φ_2 are said to be equivalent if there are positive constants α and β, and p_0 such that $\Phi_1(\alpha p) \leq \Phi_2(p) \leq \Phi_1(\beta p)$ for all p with $0 \leq p \leq p_0$. In this instance, we derive the following result by utilizing Theorem 3.11.

Corollary 3.13 Let $\mathcal{M} = (\Phi_k) \subset \mathcal{O}, \mathcal{T} = (\Omega_k) \subset \mathcal{O}$, $q \in \mathcal{G}$ and $q = (q_k) \subset \mathbb{R}^+$. If Φ_k and Ω_k are equivalent for every $k \in \mathbb{N}$, then

$$\mathcal{W}_g[\mathcal{T}, q] = \mathcal{W}_g[\mathcal{M}, q].$$

Theorem 3.14 Let $\mathcal{M} = (\Phi_k) \subset \mathcal{O}, \mathcal{T} = (\Omega_k) \subset \mathcal{O}$, $q \in \mathcal{G}$ and $q = (q_k) \subset \mathbb{R}^+$. Then, the following statements are hold:

1. $\mathcal{W}_g[\mathcal{T}, q] \cap \mathcal{W}_g[\mathcal{M}, q] \subseteq \mathcal{W}_g[\mathcal{M} + \mathcal{T}, q]$.
2. $\mathcal{W}_g[\mathcal{M} + \mathcal{T}, q] \subseteq \mathcal{W}_g[\mathcal{T}, q] \cup \mathcal{W}_g[\mathcal{M}, q]$.

Proof. (i) Let us assume that $\{U_k\} \in \mathcal{W}_g[\mathcal{T}, q] \cap \mathcal{W}_g[\mathcal{M}, q]$. Then, there exist $U \in cl(P(E))$ and positive real numbers λ_1 and λ_2 such that

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Omega_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k} = 0 \qquad (3.5)$$

and

$$\lim_{n\to\infty} \frac{1}{g(n)} \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U)|}{\lambda}\right)\right]^{q_k} = 0. \qquad (3.6)$$

If we choose $\lambda = \max\{\lambda_1, \lambda_2\}$, $0 < q_k \leq \sup_k q_k = H$, and $K = \max\{1, 2^{H-1}\}$, by using (3.1), we obtain the following inequality:

$$\sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_1) - d(p,U_2)|}{\lambda}\right)\right]^{q_k}$$
$$\leq K \sum_{k=1}^{n} \frac{1}{2^{q_k}} \left(\left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_1)|}{\lambda_1}\right)\right]^{q_k}\right.$$
$$\left. + \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_2)|}{\lambda_2}\right)\right]^{q_k}\right)$$
$$\leq K \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_1)|}{\lambda_1}\right)\right]^{q_k}$$
$$+ K \sum_{k=1}^{n} \left[\Phi_k\left(\frac{|d(p,U_k) - d(p,U_2)|}{\lambda_2}\right)\right]^{q_k}.$$

Consequently, using the inequalities in (3.5) and (3.6), it follows that $\{U_k\} \in \mathcal{W}_g[\mathcal{T} + \mathcal{M}, q]$.

On the other hand, for each $t \in \mathbb{R}^+$, since $\Phi_k(t), \Omega_k(t) \leq (\Phi_k + \Omega_k)(t)$, it follows from Theorem 3.11 that the inclusion stated in (ii) is satisfied.

CONCLUSION

In the presented study, Wijsman statistical convergence and Wijsman summability of sequences of closed sets in metric spaces are defined using Orlicz function sequences. The definitions involve the use of a weighted density function, which is a generalized form of the natural density function. Inclusion theorems related to the sequence spaces obtained from these definitions have been established. Future research on this topic could explore similar results by differentiating density functions or using lacunary sequences. Additionally, the structural properties of these sequence spaces could be examined.

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