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INTEGRAL REPRESENTATIONS FOR MERSENNE AND HORADAM-FERMAT NUMBERS

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Abstract

In this note we first derive integral representations for Mersenne numbers M_{kn} and Horadam-Fermat numbers $\mathcal{F}_{kn}$, then we use those to provide integral representations for Mersenne numbers M_{kn+r} and Horadam-Fermat numbers $\mathcal{F}_{kn+r}$, where $n \in \mathbb{Z}_{>0} = \{1,2,3,...\}$ is a positive integer, $k \in \mathbb{Z}_{>0} = \{1,2,3,...\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

Keywords: Mersenne number, Fermat number, Horadam number, integral representation

1. Introduction

Recurrence sequences are essential in mathematics, defining sequences based on previous terms. They are fundamental in number theory, combinatorics, and computer science, aiding in the study of patterns and problem-solving. Recurrence sequences help uncover numerical relationships, provide a framework for algorithm development, and facilitate the analysis and prediction of dynamic systems, making them invaluable across scientific disciplines.

Integral representation of numbers is crucial in mathematics for studying number properties and relationships systematically. It enables the application of various analytical techniques, leading to deeper insights. This is particularly valuable in number theory for understanding prime numbers.

Numerous studies explore representing special numbers as integrals. Using calculus techniques like integration by parts and substitution, these studies simplify and evaluate integrals, enhancing our understanding of these representations.

A brief overview of notable works on integral representations of numbers like Catalan, Fibonacci, Lucas, and Motzkin is provided below.

In [10] and [17], we recall that the Catalan numbers C_n are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n},\ n = 0,1,2,....$$

The origin of Catalan numbers can be traced back to the 19th century. They are applied in probability, number, and graph theories. Numerous studies [3-7], [15], [17-19] have explored their integral representations.

From [1], the Fibonacci numbers $F_n, n = 0,1,2,...$, are defined by $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n,\ n = 0,1,2,...$$

and Lucas numbers $L_n, n = 0,1,2,...$, by $L_0 = 2, L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n,\ n = 0,1,2,....$$

Fibonacci and Lucas numbers are used in mathematics, including number theory, probability, and cryptography. In economics and computer science, they help identify data patterns. Many studies, including on integral representations, have been conducted over the past few decades (please refer to [9], [20] and the references therein).

From [2], the Motzkin numbers $M_n,\ n = 0,1,2,...$, are defined by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,\ n = 0,1,2,....$$

Motzkin numbers are a sequence related to Catalan numbers, modeling problems in mathematics, biology, and physics. Several studies have investigated their integral representations. For more details, see [14].

A Mersenne number is an integer of the form $M_n = 2^n - 1, n \in \mathbb{N}$ [8]. A Fermat number F_n is an integer of the form $F_n = 2^{2^n} + 1, n \in \mathbb{Z}, n \geq 0$ [8].

Mersenne and Fermat numbers have a rich history connected to famous mathematicians. Their primality questions are a key part of prime number history.

Horadam numbers, defined by a specific recurrence relation, generalize special numbers. By choosing different initial values and coefficients, they can represent various sequences. They are important in mathematics and science, related to concepts like Fibonacci, Pell, and Catalan numbers.

The Horadam sequence $\{W_n\} = \{W_n(a, b; p, q)\}$ is defined by

$$W_O = a, W_1 = b \text{ and } W_n = pW_{n-1} - qW_{n-2} \text{ for } n \geq 2 \tag{1}$$

where a, b, p, q are real numbers with $(a, b) \neq (0,0)$ and $pq \neq 0$ [16].

The recurrence relation (1) has characteristic equation $x^2 - px + q = 0$ with roots α and β satisfying

$$\alpha + \beta = p,\ \alpha\beta = q,\ \alpha - \beta = \sqrt{p^2 - 4q}.$$

Binet's formula for Horadam numbers is:

$$W_n = A\alpha^n + B\beta^n, \tag{2}$$

where $A = \frac{b - a\beta}{\alpha - \beta}$ and $B = \frac{a\alpha - b}{\alpha - \beta}$.

Examples of special Horadam numbers generated from (1) based on a, b, p, and q :

- $W_n(0,1; 1, -1)$ and $W_n(2,1; 1, -1)$ are the nth Fibonacci and Lucas numbers, respectively.
- $W_n(0,1; 2, -1)$ and $W_n(2,2; 2, -1)$ are the nth Pell and Pell-Lucas numbers, respectively.
- $W_n(0,1; 1, -2)$ and $W_n(2,1; 1, -2)$ are the nth Jacobsthal and JacobsthalLucas numbers, respectively.
- $W_n(0,1; 3,2)$ and $W_n(2,3; 3,2)$ are the nth Mersenne and Horadam-Fermat numbers, respectively.

See [11-13, 16] for more details on Horadam numbers.

(2) gives us the numbers $W_n(0,1; 3,2)$ and $W_n(2,3; 3,2)$ in the following forms:

$$W_n(0,1; 3,2) = 2^n - 1 \tag{3}$$

and

$$W_n(2,3; 3,2) = 2^n + 1 \tag{4}$$

$W_n(0,1; 3,2)$ is the nth Mersenne number M_n. $W_n(2,3; 3,2)$ is similar to the nth Fermat number $F_n = 2^{2^n} + 1$. Thus, the number $W_n(2,3; 3,2)$ is called the Horadam-Fermat number. For simplicity, the number $W_n(2,3; 3,2)$ is noted as $\mathcal{F}_n$.

In this study, we obtain integral representations of the Mersenne M_{kn} and Horadam-Fermat $\mathcal{F}_{kn}$ numbers, and from these we derive integral representations of Mersenne M_{kn+r} and Horadam-Fermat $\mathcal{F}_{kn+r}$ numbers, where $n \in \mathbb{Z}_{>0} = \{1,2,3, \ldots\}$ is a positive integer, $k \in \mathbb{Z}_{>0} = \{1,2,3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

2. Preliminaries

This section presents identities related to Mersenne and Horadam-Fermat numbers for use in subsequent sections.

As deduced from (3), one can derive

$$M_n = 2^n - 1 \tag{5}$$

which is commonly referred to as Binet's formula for Mersenne numbers. Similarly, according to (4),

$$\mathcal{F}_n = 2^n + 1 \tag{6}$$

is identified as Binet's formula for Horadam-Fermat numbers.

1. Using (5) and (6), we establish for $n \in \mathbb{Z}_{\geqslant 0}$ that the relationship between Mersenne and Horadam-Fermat numbers is defined by

$$2^n = \frac{\mathcal{F}_n + M_n}{2}. \tag{7}$$

2. To elucidate the relationship between Mersenne numbers and Horadam-Fermat numbers, a direct computation derived from (5) and (6) for $n \in \mathbb{Z}_{\geqslant 0}$ leads to

$$\begin{aligned} \mathcal{F}_n^2 - M_n^2 = &\quad (2^n + 1)^2 - (2^n - 1)^2 \\ = &\quad (2^{2n} + 1 + 2^{n+1}) \\ &\quad -(2^{2n} + 1 - 2^{n+1}) \\ = &\quad 2^{n+2}. \end{aligned} \tag{8}$$

3. Direct calculation gives the Mersenne index addition formulae for $m, r \in \mathbb{Z}_{\geqslant 0}$, from (5) and (6)

$$\begin{aligned} M_r\mathcal{F}_m + \mathcal{F}_rM_m = &\quad (2^r - 1)(2^m + 1) \\ &\quad +(2^r + 1)(2^m - 1) \\ = &\quad (2^{r+m} + 2^r - 2^m - 1) \\ &\quad +(2^{r+m} - 2^r + 2^m - 1) \\ = &\quad 2(2^{r+m} - 1) \\ = &\quad 2M_{m+r}. \end{aligned} \tag{9}$$

4. The Horadam-Fermat index addition formulae derive directly from (5) and (6) for $m, r \in \mathbb{Z}_{\geqslant 0}$ and

$$
\begin{aligned}
\mathcal{F}_m\mathcal{F}_r + M_mM_r = &\ (2^m+1)(2^r+1) \\
&\ +(2^m-1)(2^r-1) \\
= &\ 2^{m+r}+2^m+2^r+1 \\
&\ +2^{m+r}-2^m-2^r+1 \\
= &\ 2(2^{m+r}+1) \\
= &\ 2\mathcal{F}_{m+r}.
\end{aligned} \tag{10}
$$

5. For $n \in \mathbb{Z}_{\geqslant 0}$, direct calculation from (7) and (8) yields

$$
\begin{aligned}
\frac{1}{2^n} &= \frac{2}{\mathcal{F}_n + M_n} \\
&= \frac{2(\mathcal{F}_n - M_n)}{(\mathcal{F}_n + M_n)(\mathcal{F}_n - M_n)} \\
&= \frac{2(\mathcal{F}_n - M_n)}{\mathcal{F}^2 - M_n^2} \\
&= \frac{2}{2^{n+2}}(\mathcal{F}_n - M_n) \\
&= \frac{1}{2^{n+1}}(\mathcal{F}_n - M_n).
\end{aligned}
$$

Consequently, it is evident that

$$
1 = \frac{\mathcal{F}_n - M_n}{2}. \tag{11}
$$

3. Integral representations for the Mersenne numbers M_{kn} and the Horadam-Fermat numbers $\mathcal{F}_{kn}$

The purpose of this section is to present integral representations of Mersenne numbers M_{kn} and for the Horadam-Fermat numbers $\mathcal{F}_{kn}$, where $n \in \mathbb{Z}_{>0} = \{1,2,3,\ldots\}$ is a positive integer and $k \in \mathbb{Z}_{>0} = \{1,2,3,\ldots\}$ is an arbitrary but fixed positive integer.

The subsequent theorem provides an integral representation of the Mersenne numbers M_{kn}. Building upon this theorem, we obtain an integral representation of the Mersenne numbers M_n, an integral representation of the Mersenne numbers M_{2n} with even integer indices, an integral representation of the Mersenne numbers M_{2n+1} with odd integer indices, and Binet's formula for M_{kn}.

Theorem 3.1 An integral representation of the Mersenne numbers M_{kn} is given by the integral

$$
M_{kn} = \frac{nM_k}{2^n}\int_{-1}^{1}(\mathcal{F}_k + M_k x)^{n-1}dx \tag{12}
$$

for $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. Let I be the desired integral. We use the substitution

$u = g(x) = \mathcal{F}_k + M_k x$

since its differential is $du = M_k dx$, which is present in the integral, aside from the factor M_k. Therefore, dx converts to $\frac{1}{M_k} du$. Before applying the substitution, we need to find the new integration limits. When $x = -1$, the new lower limit becomes $u = g(-1)$, and when $x = 1$, the new upper limit is $u = g(1)$. With these limits, we proceed with the substitution to obtain

$$\begin{aligned}
\frac{nM_k}{2^n} I &= \frac{nM_k}{2^n} \int_{-1}^{1} (\mathcal{F}_k + M_k x)^{n-1} dx \\
&= \frac{nM_k}{2^n} \frac{1}{M_k} \int_{g(-1)}^{g(1)} u^{n-1} du \\
&= \frac{n}{2^n} \frac{1}{n} [u^n]_{g(-1)}^{g(1)} \\
&= \frac{1}{2^n} [(\mathcal{F}_k + M_k x)^n]_{-1}^{1} \\
&= \left[\left(\frac{\mathcal{F}_k + M_k x}{2} \right)^n \right]_{-1}^{1} \\
&= \left[\left(\frac{\mathcal{F}_k + M_k}{2} \right)^n - \left(\frac{\mathcal{F}_k - M_k}{2} \right)^n \right]. \qquad (13)
\end{aligned}$$

Based on the results of (11) and (13), it can be deduced that

$$\begin{aligned}
\frac{nM_k}{2^n} I &= [(2^k)^n - 1] \\
&= [2^{kn} - 1] \\
&= M_{kn}.
\end{aligned}$$

Therefore, the demonstration of Theorem 3.1 is concluded.

The subsequent corollary presents an integral expression for the Mersenne numbers M_n.

Corollary 3.1 An integral representation of the Mersenne numbers M_n is given by the integral

$$M_n = \frac{n}{2^n} \int_{-1}^{1} (3 + x)^{n-1} dx$$

for $n \in \mathbb{Z}_{>0}$.

Proof. By substituting $k = 1$ into (12), we derive the integral representations of Mersenne numbers M_n as follows:

$$\begin{aligned}
M_n &= \frac{nM_1}{2^n} \int_{-1}^{1} (\mathcal{F}_1 + xM_1)^{n-1} dx \\
&= \frac{n}{2^n} \int_{-1}^{1} (3 + x)^{n-1} dx.
\end{aligned}$$

Hence, the proof of Corollary 3.1 is concluded.

The subsequent corollary presents an integral formula for Mersenne numbers with even integer indices.

Corollary 3.2 An integral representation of the Mersenne numbers M_{2n} is given by the integral

$$M_{2n} = \frac{3n}{2^n}\int_{-1}^{1} (5+3x)^{n-1}dx \tag{14}$$

for $n \in \mathbb{Z}_{>0}$.

Proof. By substituting $k = 2$ into (12), an integral representation of the Mersenne numbers with even integer index is obtained, as shown by

$$\begin{aligned} M_{2n} &= \frac{nM_2}{2^n}\int_{-1}^{1} (\mathcal{F}_2 + xM_2)^{n-1}dx \\ &= \frac{3n}{2^n}\int_{-1}^{1} (5+3x)^{n-1}dx. \end{aligned}$$

Consequently, this completes the proof of Corollary 3.2.

The subsequent corollary provides an integral formulation for Mersenne numbers characterized by an odd integer index.

Corollary 3.3 An integral representation of the Mersenne numbers M_{2n+1} is given by the integral

$$M_{2n+1} = \frac{1}{2^{n+1}}\int_{-1}^{1} (5+9n+3(n+1)x)(5+3x)^{n-1}dx \tag{15}$$

for $n \in \mathbb{Z}_{>0}$.

Proof. For any given positive integer n, the following holds true $M_{n+1} = 2M_n + 1$, thus establishing

$$M_{n+2} = 3M_{n+1} - 2M_n. \tag{16}$$

By reindexing of $n \mapsto 2n$ in (16) and $n \mapsto n+1$ in (14) from (14) and (16) straightforward computation yields

$$
\begin{aligned}
M_{2n+1} &= \frac{1}{3}(M_{2n+2} + 2M_{2n}) \\
&= \frac{1}{3}\frac{3(n+1)}{2^{n+1}}\int_{-1}^{1}(5+3x)^n dx + \frac{2}{3}\frac{3n}{2^n}\int_{-1}^{1}(5+3x)^{n-1}dx \\
&= \frac{1}{2^{n+1}}\int_{-1}^{1}[(n+1)(5+3x)+4n](5+3x)^{n-1}dx \\
&= \frac{1}{2^{n+1}}\int_{-1}^{1}[5n+5+3(n+1)x+4n](5+3x)^{n-1}dx \\
&= \frac{1}{2^{n+1}}\int_{-1}^{1}(5+9n+3(n+1)x)(5+3x)^{n-1}dx.
\end{aligned}
$$

The demonstration of Corollary 3.3 is hereby concluded.

The following corollary gives a thinly disguised form of the Binet's formula for M_{kn}.

Corollary 3.4 The Mersenne numbers M_{kn} can be expressed by

$$
M_{kn} = n\int_{1}^{2^k} t^{n-1}dt
$$

for $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. Using the substitution $t = \frac{1}{2}(\mathcal{F}_k + M_k x)$ in (12), we have $dt = \frac{M_k}{2}dx$ and $dx = \frac{2}{M_k}dt$. Adjusting the limits of integration (12), when $x = -1$,

$$
t = \frac{1}{2}(\mathcal{F}_k - M_k) = 1
$$

and when $x = 1$,

$$
t = \frac{1}{2}(\mathcal{F}_k + M_k) = 2^k.
$$

Thus, from (12), we get

$$
\begin{aligned}
M_{kn} &= \frac{nM_k}{2^n}\int_{-1}^{1}(\mathcal{F}_k + M_k x)^{n-1}dx \\
&= \frac{nM_k}{2^n}\int_{1}^{2^k}(2t)^{n-1}\frac{2}{M_k}dt \\
&= \frac{nM_k}{2^n}2^{n-1}\frac{2}{M_k}\int_{1}^{2^k}t^{n-1}dt \\
&= n\int_{1}^{2^k}t^{n-1}dt.
\end{aligned}
$$

This concludes the proof of Corollary 3.4.

The subsequent theorem provides an integral representation for the Horadam-Fermat numbers.

Theorem 3.2 An integral representation of the Horadam-Fermat numbers $\mathcal{F}_{kn}$ is given by the following integral:

$$\mathcal{F}_{kn} = \frac{1}{2^n}\int_{-1}^{1} (\mathcal{F}_k + M_k(n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx \tag{17}$$

for $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. Let

$$I_1 = \int (\mathcal{F}_k + M_k(n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx.$$

To evaluate this integral, we use the method of integration by parts. Let

$$u = \mathcal{F}_k + M_k(n+1)x$$

and

$$dv = (\mathcal{F}_k + M_k x)^{n-1} dx.$$

Then,

$$du = M_k(n+1)dx$$

and

$$v = \int (\mathcal{F}_k + M_k x)^{n-1} dx.$$

To evaluate the integral $v = \int (\mathcal{F}_k + M_k x)^{n-1} dx$, use the substitution $t = \mathcal{F}_k + M_k x$. This gives $dt = M_k dx$, resulting in $dx = \frac{1}{M_k} dt$. Therefore,

$$\begin{aligned} v &= \int \frac{1}{M_k} t^{n-1} dt \\ &= \frac{1}{nM_k} t^n \\ &= \frac{1}{nM_k} (\mathcal{F}_k + M_k x)^n. \end{aligned}$$

If we let

$$I_2 = \frac{1}{2^n}\int_{-1}^{1} (\mathcal{F}_k + M_k(n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx,$$

then we obtain

$$
\begin{aligned}
I_2 = \quad & \frac{1}{2^n}\left\{[uv]_{-1}^{1} - \int_{-1}^{1} vdu\right\} \\
= \quad & \frac{1}{2^n}\left\{\frac{1}{nM_k}[(\mathcal{F}_k + M_k(n+1)x)(\mathcal{F}_k + M_k x)^n]_{-1}^{1}\right. \\
& \left. -M_k(n+1)\frac{1}{nM_k}\int_{-1}^{1}(\mathcal{F}_k + M_k x)^n dx\right\} \\
= \quad & \frac{1}{nM_k}\left(\frac{\mathcal{F}_k + M_k}{2}\right)^n \left(\mathcal{F}_k + M_k(n+1)\right) \\
& -\frac{1}{nM_k}\left(\frac{\mathcal{F}_k - M_k}{2}\right)^n (\mathcal{F}_k - M_k(n+1)) \\
& -\frac{n+1}{n2^n}\int_{-1}^{1}(\mathcal{F}_k + M_k x)^n dx.
\end{aligned}
\tag{18}
$$

From (12), we have that

$$
M_{k(n+1)}\frac{2^{n+1}}{(n+1)M_k} = \int_{-1}^{1}(\mathcal{F}_k + M_k x)^n dx. \tag{19}
$$

Hence, from (18) and (19) we get

$$
\begin{aligned}
I_2 = & \frac{1}{nM_k}\left(\frac{\mathcal{F}_k + M_k}{2}\right)^n (\mathcal{F}_k + M_k(n+1)) \\
& -\frac{1}{nM_k}\left(\frac{\mathcal{F}_k - M_k}{2}\right)^n (\mathcal{F}_k - M_k(n+1)) \\
& -\frac{n+1}{n2^n}\frac{2^{n+1}}{(n+1)M_k}M_{kn+k}.
\end{aligned}
$$

By (7), (9) and (11), it follows that

$$
\begin{aligned}
I_2 = \quad & \frac{1}{nM_k}2^{kn}(\mathcal{F}_k + M_k(n+1)) \\
& -\frac{1}{nM_k}(1)^n(\mathcal{F}_k - M_k(n+1)) \\
& -\frac{2}{nM_k}M_{kn+k} \\
= \quad & \frac{1}{nM_k}[(2^{kn}-1)\mathcal{F}_k \\
& +(2^{kn}+1)(n+1)M_k \\
& -2M_{kn+k}] \\
= \quad & \frac{1}{nM_k}[\mathcal{F}_k M_{kn} + (n+1)M_k\mathcal{F}_{kn} - 2M_{kn+k}] \\
= \quad & \frac{1}{nM_k}[nM_k\mathcal{F}_{kn} + \mathcal{F}_k M_{kn} + M_k\mathcal{F}_{kn} - 2M_{kn+k}].
\end{aligned}
\tag{20}
$$

The subsequent equation can be substantiated by referencing the formula provided in (9), followed by substituting k for r and kn for m :

$$\mathcal{F}_k M_{kn} + M_k \mathcal{F}_{kn} - 2M_{kn+k} = 0.$$

Thus, (20) yields $I_2 = \mathcal{F}_{kn}$. Therefore, the theorem is proved by (17), which completes the proof.

Corollary 3.5 The Horadam-Fermat numbers $\mathcal{F}_n$ are given by the integral representation

$$\mathcal{F}_n = \frac{1}{2^n}\int_{-1}^{1} (3 + (n+1)x)(3 + x)^{n-1} dx$$

for $n \in \mathbb{Z}_{>0}$.

Proof. From $k = 1$ at (17), we get integral representations for Horadam-Fermat numbers $\mathcal{F}_n$ as follows:

$$\begin{aligned}\mathcal{F}_n &= \frac{1}{2^n}\int_{-1}^{1} (\mathcal{F}_1 + M_1(n+1)x)(\mathcal{F}_1 + M_1 x)^{n-1} dx \\ &= \frac{1}{2^n}\int_{-1}^{1} (3 + (n+1)x)(3 + x)^{n-1} dx.\end{aligned}$$

Corollary 3.5 is proved.

The subsequent corollary presents an integral representation for Horadam-Fermat numbers indexed by even integers.

Corollary 3.6 An integral representation of Horadam-Fermat numbers $\mathcal{F}_{2n}$ is given by the integral

$$\mathcal{F}_{2n} = \frac{1}{2^n}\int_{-1}^{1} (5 + 3(n+1)x)(5 + 3x)^{n-1} dx \tag{21}$$

for $n \in \mathbb{Z}_{>0}$.

Proof. Substituting $k = 2$ into (17) yields an integral representation of the Horadam-Fermat numbers with even indices as demonstrated by

$$\begin{aligned}\mathcal{F}_{2n} &= \frac{1}{2^n}\int_{-1}^{1} (\mathcal{F}_2 + M_2(n+1)x)(\mathcal{F}_2 + M_2 x)^{n-1} dx \\ &= \frac{1}{2^n}\int_{-1}^{1} (5 + 3(n+1)x)(5 + 3x)^{n-1} dx.\end{aligned}$$

This completes the proof of Corollary 3.6.

The subsequent corollary provides an integral representation for the Horadam-Fermat numbers with an odd integer index.

Corollary 3.7 An integral representation of the Horadam-Fermat numbers $\mathcal{F}_{2n+1}$ is given by

$$\mathcal{F}_{2n+1} = \frac{1}{2^{n+1}}\int_{-1}^{1} (15 + 3n + 9(n + 1)x)(5 + 3x)^{n-1}dx \tag{22}$$

for $n \in \mathbb{Z}_{>0}$.

Proof. By recalling that the identity presented in (10) takes the form of

$$2\mathcal{F}_{m+r} = \mathcal{F}_m\mathcal{F}_r + M_mM_r,$$

and by substituting $2n$ in place of m and 1 in place of r, we derive the following identity:

$$\begin{aligned} 2\mathcal{F}_{2n+1} &= \mathcal{F}_{2n}\mathcal{F}_1 + M_{2n}M_1 \\ &= 3\mathcal{F}_{2n} + M_{2n}. \end{aligned} \tag{23}$$

By substituting the integral representations derived for $\mathcal{F}_{2n}$ and M_{2n} into (23), we obtain the following integral representation for $\mathcal{F}_{2n+1}$:

$$\begin{aligned} \mathcal{F}_{2n+1} &= \frac{3}{2}\mathcal{F}_{2n} + \frac{1}{2}M_{2n} \\ &= \frac{3}{2}\frac{1}{2^n}\int_{-1}^{1} (5 + 3(n + 1)x)(5 + 3x)^{n-1}dx \\ &\quad + \frac{1}{2}\frac{3n}{2^n}\int_{-1}^{1} (5 + 3x)^{n-1}dx \\ &= \frac{1}{2^{n+1}}\int_{-1}^{1} (15 + 3n + 9(n + 1)x)(5 + 3x)^{n-1}dx. \end{aligned}$$

Consequently, the proof of Corollary 3.7 is thus completed.

The following corollary presents a version of Binet's formula for $\mathcal{F}_{kn}$.

Corollary 3.8 The Horadam-Fermat numbers $\mathcal{F}_{kn}$ can be represented by

$$\mathcal{F}_{kn} = n\int_{1}^{2^k} t^{n-1}dt + 2$$

for $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$.

Proof. The proof is similar to that of Corollary 3.4.

In the subsequent section, we derive the integral representations for Mersenne Numbers M_{kn+r} and Horadam-Fermat Numbers $\mathcal{F}_{kn+r}$.

4. Integral representations for the Mersenne numbers M_{kn+r} and the Horadam-Fermat numbers $\mathcal{F}_{kn+r}$

This section presents integral representations of Mersenne numbers M_{kn+r} and Horadam-Fermat numbers $\mathcal{F}_{kn+r}$, derived from those of Mersenne numbers M_{kn} and Horadam-Fermat numbers $\mathcal{F}_{kn}$, where $n \in \mathbb{Z}_{>0} = \{1,2,3,\dots\}$ is a positive integer, $k \in \mathbb{Z}_{>0} = \{1,2,3,\dots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer. Theorem 4.1 presents integral representations of Mersenne numbers M_{kn+r}. Thus, Theorem 4.1 shows that substituting different integer pairs (k,r) yields representations for different Mersenne numbers M_{kn+r}.

Theorem 4.1 For $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Mersenne numbers M_{kn+r} can be represented by the integral

$$M_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (nM_k\mathcal{F}_r + M_r\mathcal{F}_k + M_kM_r(n+1)x)(\mathcal{F}_k + M_kx)^{n-1}dx. \tag{24}$$

Proof. The Mersenne index addition formula

$$M_r\mathcal{F}_m + \mathcal{F}_rM_m = 2M_{m+r}$$

from (9) with m replaced by kn gives

$$2M_{kn+r} = M_{kn}\mathcal{F}_r + M_r\mathcal{F}_{kn}.$$

Using formulas (12) and (17), we can express M_{kn} and $\mathcal{F}_{kn}$ in terms of $M_k, \mathcal{F}_r, M_r$ and $\mathcal{F}_k$. This provides an integral representation for M_{kn+r}. By substituting the integral forms of M_{kn} and $\mathcal{F}_{kn}$ from (12) and (17), we obtain the following result:

$$\begin{aligned}
2M_{kn+r} &= \mathcal{F}_rM_{kn} + M_r\mathcal{F}_{kn} \\
&= \mathcal{F}_r \frac{nM_k}{2^n} \int_{-1}^{1} (\mathcal{F}_k + M_kx)^{n-1}dx \\
&\quad + M_r \frac{1}{2^n} \int_{-1}^{1} (\mathcal{F}_k + M_k(n+1)x)(\mathcal{F}_k + M_kx)^{n-1}dx \\
&= \frac{1}{2^n} \int_{-1}^{1} (nM_k\mathcal{F}_r + M_r\mathcal{F}_k + M_kM_r(n+1)x)(\mathcal{F}_k + M_kx)^{n-1}dx
\end{aligned}$$

and that completes the proof.

As noted in the subsequent remark, the findings presented in Corollary 3.1, Corollary 3.2, and Corollary 3.3 can likewise be derived by employing Theorem 4.1.

Remark 4.1 In the integral representation presented in (24) within the framework of Theorem 4.1, the substitution of $(1,0), (2,0)$, and $(2,1)$ for (k,r) results in integral representations for M_n, M_{2n}, and M_{2n+1}.

Theorem 4.2 provides integral representations of Horadam-Fermat numbers $\mathcal{F}_{kn+r}$. Thus, Theorem 4.2 shows that substituting different integer pairs (k, r) results in various Horadam-Fermat numbers $\mathcal{F}_{kn+r}$.

Theorem 4.2 For $n \in \mathbb{Z}_{>0}$ and arbitrary but fixed $k \in \mathbb{Z}_{>0}$ and $r \in \mathbb{Z}_{\geqslant 0}$, the Horadam-Fermat numbers $\mathcal{F}_{kn+r}$ can be represented by the integral

$$\mathcal{F}_{kn+r} = \frac{1}{2^{n+1}} \int_{-1}^{1} (nM_k M_r + \mathcal{F}_k \mathcal{F}_r + M_k \mathcal{F}_r (n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx. \tag{25}$$

Proof. The Horadam-Fermat index addition formula

$$\mathcal{F}_m \mathcal{F}_r + M_m M_r = 2\mathcal{F}_{m+r}$$

from (10) with m replaced by kn yields

$$\mathcal{F}_{kn} \mathcal{F}_r + M_{kn} M_r = 2\mathcal{F}_{kn+r}.$$

Formulas (12) and (17) enable us to express M_{kn} and $\mathcal{F}_{kn}$ in terms of $M_k, \mathcal{F}_r, M_r$, and $\mathcal{F}_k$. Thus, an integral representation for M_{kn+r} can be derived. By substituting the integral representations of M_{kn} and $\mathcal{F}_{kn}$ provided in (12) and (17), respectively, into the specified index addition formula, the result is obtained directly:

$$\begin{aligned} 2\mathcal{F}_{kn+r} &= M_r M_{kn} + \mathcal{F}_r \mathcal{F}_{kn} \\ &= M_r \frac{nM_k}{2^n} \int_{-1}^{1} (\mathcal{F}_k + M_k x)^{n-1} dx \\ &\quad + \mathcal{F}_r \frac{1}{2^n} \int_{-1}^{1} (\mathcal{F}_k + M_k (n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx \\ &= \frac{1}{2^n} \int_{-1}^{1} (nM_k M_r + \mathcal{F}_k \mathcal{F}_r + M_k \mathcal{F}_r (n+1)x)(\mathcal{F}_k + M_k x)^{n-1} dx \end{aligned}$$

and that completes the proof.

The following remark states that the results in Corollaries 3.5, 3.6, and 3.7 can be derived using Theorem 4.2.

Remark 4.2 In the integral representation presented in (25) via Theorem 4.2, the substitution of $(1,0), (2,0)$, and $(2,1)$ in place of (k, r) results in integral representations for $\mathcal{F}_n, \mathcal{F}_{2n}$, and $\mathcal{F}_{2n+1}$.

3. Conclusion

This note first derives integral representations for Mersenne numbers M_{kn} and Horadam-Fermat numbers $\mathcal{F}_{kn}$, then uses those to provide integral representations for Mersenne numbers M_{kn+r} and Horadam-Fermat numbers $\mathcal{F}_{kn+r}$, where $n \in \mathbb{Z}_{>0} = \{1,2,3, \ldots\}$ is a positive integer, $k \in \mathbb{Z}_{>0} = \{1,2,3, \ldots\}$ is an arbitrary but fixed positive integer, while $r \in \mathbb{Z}_{\geqslant 0}$ is an arbitrary but fixed non-negative integer.

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References

[1] Andrica, D., Bagdasar, O., "Recurrent Sequences", 2020, Springer, Berlin.
[2] Bernhart, F., "Catalan, Motzkin, and Riordan numbers", Discrete Math. 204 (1999) : 73-112.
[3] Dana-Picard, T., "Parametric integrals and Catalan numbers", International Journal of Mathematical Education in Science and Technology 36(4) (2005) : 410-414.
[4] Dana-Picard, T., Zeitoun., D.G., "Closed forms for 4-parameter families of integrals", International Journal of Mathematical Education in Science and Technology 40(6) (2009) : 828-837.
[5] Dana-Picard, T., "Integral presentations of Catalan numbers", International Journal of Mathematical Education in Science and Technology 41(1) (2010) : 63-69.
[6] Dana-Picard, T., "Integral presentations of Catalan numbers and Wallis formula", International Journal of Mathematical Education in Science and Technology 42(1) (2011) : 122-129.
[7] Dana-Picard, T., Zeitoun, D.G., "Parametric improper integrals, Wallis formula and Catalan numbers", International Journal of Mathematical Education in Science and Technology 43(4) (2012) : 515-520.
[8] Deza, E., "Mersenne numbers and Fermat numbers (Vol. 1)", World Scientific, Singapore, 2021.
[9] Glasser, M.L., Zhou, Y., "An integral representation for the Fibonacci numbers and their generalization", Fibonacci Quart. 53(4) (2015) : 313-318.
[10] Grimaldi, R., "Fibonacci and Catalan Numbers: an introduction", John Wiley Sons, 2012.
[11] Horadam, A.F., "Generating functions for powers of a certain generalised sequence of numbers", Duke Mathematical Journal 32(3) (1965) : 437-446.
[12] Horadam, A.F., "Basic properties of a certain generalized sequence of numbers", The Fibonacci Quarterly 3 (1965) : 161-176.
[13] Horadam, A.F., "Special properties of the sequence Wn(a; b; p; q)", The Fibonacci Quarterly 5 (1967) : 424-434.
[14] Mccalla, P., Nkwanta, A., "Catalan and Motzkin integral representations. in The Golden Anniversary Celebration of the National Association of Mathematicians, Contemporary Mathematics", American Mathematical Society 759 (2020) : 125134.
[15] Li, W., Cao, J., Niu, D., Zhao, J., Qi, F., "A brief survey and an analytic generalization of the Catalan numbers and their integral representations", Mathematics 11(8) (2023) : 1870.
[16] Keskin, R., Siar, Z., "Some new identities concerning the Horadam sequence and its companion sequence", Communications of the Korean Mathematical Society 34(1) (2019) : 1-16.

[17] Koshy, T., "Catalan numbers with applications", Oxford University Press, 2008.
[18] Penson, K.A., Sixdeniers, J.M., "Integral representations of Catalan and related numbers", J. Integer Seq. 4(2) (2001) : Article 01.2.5.
[19] Gi, F., Guo, B.N., "Integral representations of the Catalan numbers and their applications", Mathematics 5(3) (2017) : 40.
[20] Stewart, S.M., "Simple integral representations for the Fibonacci and Lucas Numbers", Aust. J. Math. Anal. Appl. 19(2) (2022) : 2-5.