Using Special Rules for Transformation of the Finding Exact Solutions of the Singular Schrödinger Differential Equation

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ABSTRACT: In this article we give a very brief outline of one way of carrying out the spectral analysis of a boundary value problem with specified singularities and investigating the corresponding inverse problem. We find out the solutions of equation

$$y'' + \left[V(x) + 2\frac{(a^2+1)\lambda^2}{(1-\sqrt{a^2+1}\lambda x)^2} \right] y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty)$$

satisfying the boundary condition

$$y'(0) - a\lambda y(0) = 0$$

where V is a real valued function, λ is a spectral parameter and a is a natural number. As the mention above, these solutions of a singular boundary value problem were made of our premises which results came out the solutions of a non singular boundary value problem

$$y'' - V(x)y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty),$$
$$y'(0) - a\lambda y(0) = 0.$$

Key Words: Schrödinger equation, spectrum, spectral operators, singularity.

Singülerliğe Sahip Schrödinger Diferansiyel Denkleminin Çözümlerinin Özel Bir Dönüşüm Yardımı ile Bulunması

ÖZET: Bu makalede özel bir singüleriteye sahip sınır değer probleminin, spektral analizinin incelenmesinde kısa bir yöntemi verilecektir.

$$y'' + \left[V(x) + 2\frac{(a^2+1)\lambda^2}{(1-\sqrt{a^2+1}\lambda x)^2} \right] y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty)$$

singüler diferansiyel denkleminin

$$y'(0) - a\lambda y(0) = 0$$

sınır koşulunu gerçekleyen çözümlerini

$$y'' - V(x)y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty),$$
$$y'(0) - a\lambda y(0) = 0$$

singüler olmayan diferansiyel denkleminin sınır değer problemini gerçekleyen çözümleri yardımı ile bulunduğu gösterilecektir. Burada V kompleks değerli bir fonksiyon, a bir doğal sayı ve λ bir parametredir.

Anahtar Kelimeler: Schrödinger denklemi, spektrum, spektral operator, singülerlik.

INTRODUCTION

Let us consider the following boundary value problems

$$y'' - V(x)y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty), \tag{1}$$

$$y(0) = 0$$

and

$$y'' - \left[V(x) + \frac{n(n+1)}{x^2} \right] y + \lambda^2 y = 0, \ \ x \in R_+ = [0,\infty), \tag{2}$$

where V is an absolutely continuous real valued function in each finite subinterval of R_+, n is a natural number and satisfying

$$\int_0^\infty |V(x)| \, dx < \infty. \tag{3}$$

Under the condition (3), by using the special transformation Z.S. Agranovich and V.A. Marchenko found out the solutions of equation (2) by means of using the solutions (1) (Agranovich and Marchenko, 1963).

MATERIAL and METHOD

Now let us consider the following boundary value problems giving by

$$y'' - V(x)y + \lambda^2 y = 0, \quad x \in R_+ = [0, \infty), \quad (4)$$

$$y'(0) - a\lambda y(0) = 0. \quad (5)$$

and

$$y' - \left[V(x) + 2\frac{(a^2+1)\lambda^2}{(1-\sqrt{a^2+1}\lambda x)^2} \right] y + \lambda^2 y = 0, \quad x \in R_+ = [0, \infty) \quad (6)$$

where V is an absolutely continuous real valued function in each finite subinterval of R_+ and satisfying the condition (3), λ is a spectral parameter and a is a natural number.

Under the condition (3), the equation (1) has the solutions such as

$$e^{\pm}(x, \lambda) = e^{\pm iw(x) \pm i\lambda x} + \int_x^\infty K^{\pm}(x, t) e^{\pm i\lambda t} dt, \quad (7)$$

and

$$g^{\pm}(x, \lambda) = e^{\mp iw(x) \mp i\lambda x} + \int_x^\infty K^{\mp}(x, t) e^{\mp i\lambda t} dt, \quad (8)$$

for λ in the closed upper and lower half-planes, respectively, where

$$w(x) = \int_x^\infty V(t) dt$$

and kernels $K^{\pm}(x, t)$ are expressed in term of V(x) and $K^{\pm}(x, t)$ are the solutions of Volterra type integral equations and they satisfy the following inequalities (Bairamov et al., 1999).

$$\left| K^{\pm}(x, t) \right| \le \frac{1}{2} \sigma \left(\frac{x+t}{2} \right) \exp\{\sigma_1(x)\}$$

where

$$\sigma(x) = \int_x^\infty V(t) dt$$

According to (7) and (8), the Wronskian of the solutions $e^{\pm}(x, \lambda)$ and $g^{\pm}(x, \lambda)$ are

$$W\left[e^{\pm}(x, \lambda), g^{\pm}(x, \lambda) \right] = \mp 2i\lambda \quad \text{(Bairamov et al., 1999)}$$

for λ in the closed upper and lower half-planes, respectively. So the pairs $e^{\pm}(x, \lambda)$, $g^{\pm}(x, \lambda)$ and $e^-(x, \lambda)$, $g^-(x, \lambda)$ form two fundamental systems solutions of equation (4) in the closed upper and lower half-planes, respectively. Hence, the equation (4) has the solutions satisfying the condition (5) such as

$$\varphi^{\pm}(x, \lambda) = \frac{d_1^{\pm}(x, \lambda)}{2i\lambda} e^{\pm}(x, \lambda) - \frac{d^{\pm}(x, \lambda)}{2i\lambda} g^{\pm}(x, \lambda) \quad (9)$$

where

$$d_1^{\pm}(\lambda) = (g^{\pm})'(0, \lambda) - a\lambda g^{\pm}(0, \lambda)$$

and

$$d^{\pm}(\lambda) = (e^{\pm})'(0, \lambda) - a\lambda e^{\pm}(0, \lambda).$$

As it is known, the solutions of $\varphi^+(x, \lambda)$ and $\varphi^-(x, \lambda)$ given by (9) are important in the investigation of spectral analysis and scattering theories of the boundary value problem (4) - (5) (Bairamov et al., 1997; Bairamov and Çelebi, 1997). But the equation (6) has no solutions represented as the solutions (9) due to the having a singularity of term

$$(a^2 + 1)\lambda^2 / (1 - \sqrt{a^2+1}\lambda x)^2 \quad \text{in the interval } [0, \infty).$$

In this study, our purpose is to find that the equation (6) has the similar solutions to (9) by using the solution of the equation (4).

The similar problem has been studied for Sturm-Liouville and Klein-Gordon equation in (Agranovich and Marchenko, 1963; Karaman and Yanık, 2000).

CONCLUSION

Let us consider the following boundary value problem

$$y'' - V(x)y = 0, \quad x \in R_+ \quad (10)$$

and

$$y'(0) + \sqrt{a^2+1}\,\lambda y(0) = 0. \quad (11)$$

Then we get

Theorem 1. For all λ, equation (10) has the solution $f(x, \lambda)$ which satisfies the boundary condition (11) and $f(x, \lambda)$ has the representation

$$f(x, \lambda) = 1 - \sqrt{a^2+1}\,\lambda x + \int_0^x (x-t)V(t)f(t, \lambda)dt \quad (12)$$

moreover the following asymptotic equalities

$$f(x, \lambda) = 1 - \sqrt{a^2+1}\,\lambda x + o(1),$$

$$f(x, \lambda) = -\sqrt{a^2+1}\,\lambda + o(1), \quad x \to 0 \quad (13)$$

are valid.

Proof. If we integrate equation (10) twice and we use the boundary condition (11), we get the equation (12) and (13) (Naimark, 1968) (p145).

Let $\varphi(x,\lambda)$ be the normalized eigen-function of the boundary value problem (4) - (5). If we consider the following transformation

$$y(x,\lambda) = \frac{f(x,\lambda)\varphi'(x,\lambda) - f'(x,\lambda)\varphi(x,\lambda)}{\lambda f(x,\lambda)} \quad (14)$$

we can give the following

Theorem 2. If the function $f(x,\lambda)$ is not vanished in the interval $(0,\infty)$, then the function $y(x,\lambda)$ defined by (14) satisfies the equation

$$y'' - V_1(x,\lambda)y + \lambda^2 y = 0 \quad (15)$$

where

$$V_1(x,\lambda) = V(x) - 2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]'.$$

Proof. Let us write the first and second derivatives of $y(x,\lambda)$

$$y'(x,\lambda) = -\lambda\varphi(x,\lambda) - \frac{f'(x,\lambda)}{f(x,\lambda)}\left\{\frac{f(x,\lambda)\varphi'(x,\lambda) - f'(x,\lambda)\varphi(x,\lambda)}{\lambda f(x,\lambda)}\right\}$$

and

$$y''(x,\lambda) = -\lambda^2 \frac{f(x,\lambda)\varphi'(x,\lambda)}{\lambda f(x,\lambda)} - \left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)' y(x,\lambda) + \left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)^2 y(x,\lambda)$$

$$+ \lambda \frac{f'(x,\lambda)\varphi(x,\lambda)}{\lambda f(x,\lambda)}$$

$$= -\lambda^2 \left\{\frac{f(x,\lambda)\varphi'(x,\lambda) - f'(x,\lambda)\varphi(x,\lambda)}{\lambda f(x,\lambda)}\right\} + \left\{\left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)^2 - \left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)'\right\} y(x,\lambda)$$

$$= -\lambda^2 y(x,\lambda) + V(x)y(x,\lambda) + 2\left[\frac{f'^2(x,\lambda)}{f^2(x,\lambda)} - \frac{f''(x,\lambda)}{f(x,\lambda)}\right] y(x,\lambda).$$

Hence, we find that

$$y''(x,\lambda) + \left[V(x) - 2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]'\right]y(x,\lambda) + \lambda^2 y(x,\lambda) = 0. \quad (16)$$

Now we show that the transformation $y(x,\lambda)$ satisfies the boundary value condition (5). If we write the first derivative of $y(x,\lambda)$ and we get it in the boundary condition (5), then we have

$$y'(0,\lambda) - a\lambda y(0,\lambda) = -\lambda\varphi(0,\lambda) - \frac{f'(0,\lambda)}{f(0,\lambda)}\frac{f(0,\lambda)\varphi'(0,\lambda)}{\lambda f(0,\lambda)} + \frac{f'^2(0,\lambda)\varphi(0,\lambda)}{\lambda f^2(0,\lambda)}$$

$$- a\lambda\left\{\frac{f(0,\lambda)\varphi'(0,\lambda) - f'(0,\lambda)\varphi(0,\lambda)}{\lambda f(0,\lambda)}\right\}$$

$$= -\lambda\varphi(0,\lambda) + \frac{f'^2(0,\lambda)}{\lambda f^2(0,\lambda)}\varphi(0,\lambda) - a\lambda\varphi'(0,\lambda)$$

$$+ \frac{f'(0,\lambda)}{\lambda f(0,\lambda)}\left\{\varphi'(0,\lambda) - a\lambda\varphi(0,\lambda)\right\}$$

$$= \frac{\varphi(0,\lambda)}{\lambda f^2(0,\lambda)}\left\{f'^2(0,\lambda) - (a^2+1)\lambda^2 f^2(0,\lambda)\right\}$$

$$+ \frac{f'(0,\lambda)}{\lambda f(0,\lambda)}\left\{\varphi'(0,\lambda) - a\lambda\varphi(0,\lambda)\right\}$$

from the conditions (5) and (11) we get

$$y'(0,\lambda) - a\lambda y(0,\lambda) = 0$$

and from the asymptotic equalities (13), as $x \to 0$, we have

$$-2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]' = -2\left(-\frac{\sqrt{(a^2+1)}\,\lambda}{1-\sqrt{(a^2+1)}\,\lambda x} + o(1)\right)'$$

$$= 2\frac{(a^2+1)\lambda^2}{(1-\sqrt{(a^2+1)}\,\lambda x)^2} + o(1)$$

Hence, by (16) it follows that the potential $V(x,\lambda)$ behaves like

$(a^2+1)\lambda^2 / (1-\sqrt{a^2+1}\,\lambda x)^2$ in the interval $(0,\infty)$. In this way we use the function (14) in the non-singular boundary value problem (4)-(5) and we find the singular boundary value problem (16).

$$\varphi''(x,\lambda) - \left[V_1(x,\lambda) + 2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]'\right]\varphi(x,\lambda) + \lambda^2\varphi(x,\lambda) = 0. \qquad (18)$$

If we substitute the potential $V_1(x,\lambda)$ defined in (15) in the last statement, then we find the non-singular equation

So we get $\quad y'' - V(x)y + \lambda^2 y = 0.$

the following theorem

Theorem 3. If the function $\varphi(x,\lambda)$ defined by (17) is the solution of (18), then the function $y(x,\lambda)$ defined by (14) is the solution of (15).

Now we find the inverse transformation of transformation (14). Since

$$y(x,\lambda) = \frac{f(x,\lambda)\varphi'(x,\lambda) - f'(x,\lambda)\varphi(x,\lambda)}{\lambda f(x,\lambda)}$$

then we get $\quad \lambda\dfrac{y(x,\lambda)}{f(x,\lambda)} = \left[\dfrac{\varphi(x,\lambda)}{f(x,\lambda)}\right]'$

and hence, we also find

$$\varphi(x,\lambda) = \lambda f(x,\lambda)\int_0^x \frac{y(t,\lambda)}{f(t,\lambda)}\,dt. \qquad (17)$$

Now in a similar way, we find the differential equation

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