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Diophantine equations on cross-multiplicative forms of the Pell and Pell-Lucas numbers

Ahmet EMİN[1,*], Ahmet DAŞDEMİR[2]

[1]*Karabük University Faculty of Science, Department of Mathematics, Central Campus, Karabük.*
[2]*Kastamonu University, Faculty of Science, Department of Mathematics, Kuzeykent Campus, Kastamonu.*

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Abstract

This study aims to explore all Pell numbers that are the product of two random Pell-Lucas numbers and all Pell-Lucas numbers that are the product of two random Pell numbers based on linear forms in logarithms of algebraic numbers using Matveev's theorem and Dujella - Pethő reduction lemma. Further, we find all the common terms of Pell and Pell-Lucas numbers and show that no Pell and no Pell-Lucas numbers can be written as a square of another.

Keywords: *Pell number, Pell-Lucas number, Diophantine equation, Matveev's theorem, Dujella-Pethő reduction lemma, linear forms in logarithms.*

Pell ve Pell-Lucas sayılarının karşılıklı çarpımsal eşitlikleri ile ilgili Diophantine denklemleri

Öz

Bu çalışma, logaritmalardaki lineer formlara dayalı olarak geliştirilen Matveev's teoremi ve Dujella-Pethő indirgeme lemması kullanılarak, iki rastgele Pell-Lucas sayısının çarpımı olan tüm Pell sayıları ve iki rastgele Pell sayısının çarpımı olan tüm Pell-Lucas sayılarını araştırmayı amaçlamaktadır. Ayrıca, Pell ve Pell-Lucas sayılarına ait ortak terimler incelenmiş ve hiçbir Pell sayısının bir Pell-Lucas sayısının karesi olamayacağı gibi, hiçbir Pell-Lucas sayısının da bir Pell sayısının karesi olarak yazılamayacağı gösterilmiştir.

Anahtar kelimeler: *Pell sayısı, Pell-Lucas sayısı, Diophantine denklemler, Matveev's teoremi, Dujella-Pethö indirgeme lemması, logaritmalarda lineer formlar.*

*Ahmet EMİN, ahmetemin@karabuk.edu.tr, http://orcid.org/0000-0001-7791-7181
Ahmet DAŞDEMİR, ahmetdasdemir37@gmail.com, http://orcid.org/0000-0001-8352-2020

1. Introduction

The Pell numbers, denoted by $\{P_n\}_{n\geq 0}$, create a second-order integer sequence, which is defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for all integers $n \geq 2$ with initial conditions $P_0 = 0$ and $P_1 = 1$. Further, the Pell-Lucas numbers, denoted by $\{Q_n\}_{n\geq 0}$, can be obtained by using the same recurrence relation but with initial conditions $Q_0 = 2$ and $Q_1 = 2$. On the other hand, the mentioned sequences can be generated by Binet's formula as follows:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \gamma^n + \delta^n, \tag{1}$$

where $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$, which are the roots of $x^2 - 2x - 1 = 0$. It can be said that the Pell and Pell-Lucas numbers are among the most fascinating integer sequences due to their numerous applications in fields such as number theory, cryptography, and group theory. These sequences are essential in solving Diophantine equations and understanding the structure of certain algebraic objects. In recent years, their applications in cryptography and group theory have been extensively explored. Examples of studies in [1-4] can be consulted, and the fundamental reference in [5] provides more precise examples.

An investigation of Diophantine equations of various configurations consisting of integer sequences such as Fibonacci, Lucas, Pell, or Jacobsthal numbers has become very popular in recent years. In [6], Alekseyev investigated the needed conditions under which two various generalized sequences can have common terms. In [7], Bravo and Luca identified all common terms of two different generalized *k*-Fibonacci sequences. In [8], Bensella and Behloul found the Leonardo numbers that are also the Jacobsthal numbers. In [9], Chalebgwa and Ddamulira studied all the Padovan numbers in the form of palindromic combinations of two distinct repdigits in the usual base. In [10], Daşdemir and Varol examined the Jacobsthal numbers that are expressible with the product of two Modified Pell numbers. The authors of [11] studied Fibonacci and Lucas numbers that are a product of their arbitrary terms. In [12], Ddamulira et al. conducted a solution process to determine whether Fibonacci or Pell numbers can be expressed as the product of two Pell or Fibonacci numbers. One of the key contributions of this work is that it represents one of the earliest studies to investigate whether the product of terms from two well-known integer sequences equals a term from another integer sequence. Additionally, this study provides valuable guidance on how to effectively apply several important lemmas and theorems presented within. In [13], Emin investigated Pell numbers that can be expressed as the sum of two Mersenne numbers. Next, the same author studied Mersenne numbers that are expressible as the summation of two Fibonacci numbers in [14] and powers of two written as the sums of the squares of two Lucas numbers in [15]. Unlike the studies referenced earlier, these works extend the exploration to include sums in exponential forms within integer sequences, inspired by the idea of investigating whether the sum of terms from two integer sequences equals a term from another sequence. In [16], Erduvan and Keskin found all repdigits that are the products of two Fibonacci or Lucas numbers. In [17], Luca and Togbé studied all the Pell equations whose solutions are the usual Fibonacci numbers. In [18], Marques and Togbé proved that the sum of only the first powers or square of two consecutive Fibonacci numbers can be a Fibonacci number. In [19], Chaves and Marques developed an investigation into the problem originally studied by Marques and Togbé [18] for generalized k-Fibonacci numbers. In [20], Sahukar and Panda searched all solutions to the Brocard-Ramanujan-type equations consisting of balancing-like and associated balancing-like numbers.

To the best of our knowledge, the Pell numbers that can be expressible in terms of the product of two random Pell-Lucas numbers or vice versa, the Pell-Lucas numbers that are the product of two random Pell numbers, have not yet been investigated. In this paper, our problems are as follows:

$$P_k = Q_m Q_n \tag{2}$$

and

$$Q_k = P_m P_n , \tag{3}$$

where $k \geq 1$ and $1 \leq m \leq n$. There is a lack of mathematical investigation to provide fundamental insights regarding the solutions to the above-mentioned Diophantine equations. Further, it is noteworthy that the integer sequences on both sides of the above equations have the same algebraic equation, i.e., $x^2 - 2x - 1 = 0$. This is a factor that beclouds the solution within the framework of the usual method in the current literature. To address the issue, we put forth a mathematical approach to investigate the problems considered herein based on the Matveev's theorem [21, p. 1219] and Dujella-Pethő reduction lemma [22, p. 303].

2. Basic tools

This section presents essential definitions, some results, and the notations used from algebraic number theory.

Let η be an algebraic number, and let

$$f(x) = t_0 \prod_{s=1}^{l} \left(x - \eta^{(s)} \right)$$

be its minimal polynomial of degree l, where $t_0 > 0$ and $\eta^{(s)}$ is the *sth* conjugate of η in $\mathbb{Z}[x]$. Furthermore, let $h(\eta)$ denotes the logarithmic height of the algebraic number η; it is given by

$$h(\eta) = l^{-1} \left(\log|t_0| + \sum_{s=1}^{l} \log\left(\max\left\{ \left|\eta^{(s)}\right|, 1 \right\} \right) \right).$$

Several properties related to logarithmic height, which can also be found in numerous references such as [12, 21- 22], are presented below:

$$h(\eta_1+\eta_2) \leq h(\eta_1)+h(\eta_2)+\log 2,$$
$$h(\eta_1\eta_2^{\pm 1}) \leq h(\eta_1)+h(\eta_2),$$
$$h(\eta^p)=|p|h(\eta).$$

Let η_i be an algebraic number in the real number field $\mathcal{F}$ for $i=1,2,...,l$ where the degree of $\mathcal{F}$ is *D*. Additionally, let $b_1,b_2,...,b_l$ represent nonzero rational integers. Now we will introduce the following notations:

$$\Lambda := \eta_1^{b_1}\eta_2^{b_2}...\eta_l^{b_l}-1 \text{ and } B:=\max\{|b_1|,|b_2|,...,|b_l|\}.$$

In the following, with the aid of the notations provided above, we will present the theorem published by Matveev in [21], which will be frequently used throughout this paper.

Theorem 2. 1 (Matveev's Theorem). Let Λ be non-zero, and let $\mathcal{F}$ be a real number field of degree *D*. Then,

$$\log(|\Lambda|) > -1.4\cdot 30^{l+3}\cdot l^{4.5}\cdot D^2\cdot(1+\log D)\cdot(1+\log B)\cdot A_1\cdot A_2\cdot...\cdot A_l$$

where A_i are positive real numbers and satisfy $A_j \geq \max\{Dh(\eta_j),|\log\eta_j|,0.16\}$ for $j=1,2,...,l$.

To tighten the bounds when applying Theorem 2.1, the following lemma, developed by Dujella and Pethő as Lemma 5(a) in [22], is used.

Lemma 2. 2 (Dujella and Pethő). Let *M* be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction expansion of the irrational number τ such that $6M<q$, and let *A*, *B*, and μ be real numbers where A>0 and B>1. Define $\varepsilon := \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon>0$, then there is no solution to the inequality

$$0<k\tau-n+\mu<AB^{-\phi}$$

where $k,\mu,$ and ϕ are positive integers with

$$k\leq M \text{ and } \phi \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Main results

The main result of the paper is given below:

Theorem 3. 1. Let *k*, *m*, and *n* be a non-zero integer such that $k \geq 1$ and $1 \leq m \leq n$. Then, all the solutions to Equation (2) are given only by $(k,m,n)=(4,1,2)$, and all the solutions to Equation (3) are given only by $(k,m,n)=(1,1,2)$.

Proof. Here, we will consider both target equations simultaneously but divulge details only for Equation (2).

First, one can prove the following inequalities by the induction method:

$$\gamma^{n-2} \leq P_n \leq \gamma^{n-1}, \quad \gamma^{n-1} \leq Q_n \leq 2\gamma^n, \text{ and } |\delta|^{-n+1} \leq Q_n \leq |\delta|^{-n-1}. \tag{4}$$

From (4), we can write

$$\gamma^{k-2} \leq P_k = Q_n Q_m \leq |\delta|^{-n-m-2}. \tag{5}$$

So, we get

$$(k-2)\log\gamma \leq -(n+m+2)\log|\delta| \Rightarrow k \leq 2-(n+m+2)\frac{\log|\delta|}{\log\gamma} < 4n,$$

where we used the fact that $m \leq n$ and $\frac{\log|\delta|}{\log\gamma} \cong -1$. Using Binet's formulas in Equation (1), we can write

$$\begin{aligned} P_k = Q_m Q_n &\Rightarrow \frac{\gamma^k - \delta^k}{2\sqrt{2}} = \left(\gamma^m + \delta^m\right)\left(\gamma^n + \delta^n\right) \\ &\Rightarrow \gamma^k - 2\sqrt{2}\cdot\delta^{m+n} = 2\sqrt{2}\cdot\gamma^{n+m} + 2\sqrt{2}\cdot\gamma^n\cdot(-1)^m\gamma^{-m} \\ &\qquad + 2\sqrt{2}\cdot\gamma^m\cdot(-1)^n\gamma^{-n} + (-1)^k\gamma^{-k}. \end{aligned}$$

Taking the absolute value of both sides under $k < 4n$ and $m \leq n$ after dividing the last equation by γ^k, we conclude that

$$\begin{aligned} \left|1-\gamma^{-k}\cdot\delta^{n+m}\cdot 2\sqrt{2}\right| &\leq \left|2\sqrt{2}\gamma^{n+m-k} + 2\sqrt{2}\gamma^{n-m-k} + 2\sqrt{2}\gamma^{m-n-k} + \gamma^{-k}\right| \\ &\leq \left|6\sqrt{2}\gamma^{-2m} + \gamma^{-m}\right| = \left|6\sqrt{2}\cdot\frac{1}{\gamma^{2m}} + \frac{1}{\gamma^m}\right| < \frac{2}{\gamma^m} \end{aligned}$$

for $m \geq 3$. So, we have

$$|\Lambda_1| < \frac{2}{\gamma^m}, \quad \Lambda_1 := \gamma^{-k}\delta^{n+m}2\sqrt{2} - 1. \tag{6}$$

We can consider the case where $l=3$, $\eta_1 = \gamma$, $\eta_2 = \delta$, $\eta_3 = 2\sqrt{2}$, $b_1 = -k$, $b_2 = n+m$, and $b_3 = 1$. Here, $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $\mathcal{F} = \mathbb{Q}(\sqrt{2})$ of degree $D = 2$. Further, $\Lambda_1 \neq 0$

. On the contrary, let us assume that $\Lambda_1 = 0$. Then, it must be $\gamma^k \cdot \delta^{-n-m} = 2\sqrt{2}$. However, this leads to a contradiction because the square of $\gamma^k \cdot \delta^{-n-m}$ does not equal an integer, whereas the square of $2\sqrt{2}$ is an integer. Therefore, it follows that $\gamma^k \cdot \delta^{-n-m} \neq 2\sqrt{2}$.

On the other hand, to apply Matveev's the famous theorem to Equation (6), we compute the following:

$$h(\eta_1) = h(\eta_2) = \frac{1}{2}\log\gamma,\ h(\eta_3) = \frac{3}{2}\log 2,\ A_1 = A_2 = \log\gamma, \text{ and } A_3 = 3\log 2,$$

where $h(\eta_j)$ denotes the logarithmic height of η_j and A_j's are a positive real number satisfying that $A_j \geq \max\{Dh(\eta_j), |\log\eta_j|, 0.16\}$ for $j = 1,2,3$. Further, when $B = 4n$, then $B \geq \max\{|-k|, n+m, 1\}$. In this case, according to Matveev's theorem, we can write

$$\log(|\Lambda_1|) > -1.57\times 10^{12} \times (1+\log 4n). \tag{7}$$

Also, from Inequality (6), we have

$$\log(|\Lambda_1|) < \log 2 - m\log\gamma. \tag{8}$$

From Inequalities (7) and (8), we deduce that

$$m < 1.79\times 10^{12} \times (1+\log 4n). \tag{9}$$

Now, let us return to Equation (2). Using a procedure similar to the one used to derive Inequality (6), we can obtain the following:

$$|\Lambda_2| < \frac{17}{\gamma^n},\ \Lambda_2 := \gamma^{-k}\delta^n 2\sqrt{2}Q_m - 1. \tag{10}$$

So, we can write

$$\log(|\Lambda_2|) < \log 17 - n\log\gamma. \tag{11}$$

Also, we can consider the case where $l = 3$, $\eta_1 = \gamma$, $\eta_2 = \delta$, $\eta_3 = 2\sqrt{2}Q_m$, $b_1 = -k$, $b_2 = n$, and $b_3 = 1$ this time. So, $\eta_1, \eta_2, \eta_3 \in \mathbb{Q}(\sqrt{2})$ and $\mathcal{F} = \mathbb{Q}(\sqrt{2})$ of degree $D = 2$. As can be seen, since $\gamma^k \cdot \delta^{-n} = 2\sqrt{2}Q_m$ is never satisfied, $\Lambda_2 \neq 0$. Then, we can write

$$h(\eta_1) = h(\eta_2) = \frac{1}{2}\log\gamma, \text{ and } A_1 = A_2 = \log\gamma.$$

Also, since η_3 is a root of $x^2 - 8Q_m^2$, we have $h(\eta_3) = \log\left(2\sqrt{2}Q_m\right)$. Further, using the fact that $h(\eta_1\eta_2) \le h(\eta_1) + h(\eta_2)$ and $Q_m \le 2\gamma^m$, we can conclude that

$$h(\eta_3) = \log\left(2\sqrt{2}Q_m\right) = \log 2\sqrt{2} + \log Q_m \le \log 2\sqrt{2} + \log 2 + m\log\gamma \le 3m\log\gamma$$

for all $m \ge 1$. Therefore, we can write

$$6m\log\gamma = 2(3m\log\gamma) \ge Dh(\eta_3) \ge \max\left\{Dh(\eta_3), \left|\log(\eta_3)\right|, 0.16\right\}.$$

So, we have that $A_3 = 6m\log\gamma$. In addition, $B \ge \max\{|-k|, n, 1\}$ for $B = 4n$. According to Matveev's theorem, we have

$$\log(|\Lambda_2|) > -3.99\times 10^{12} \times (1 + \log 4n) \times m. \tag{12}$$

From Inequalities (11) and (12), we have

$$n < 4.53\times 10^{12}\left(1 + \log(4n)\right) \times m. \tag{13}$$

By comparing Inequality (13) with Inequality (9), we obtain

$$n < 3.77\times 10^{28}. \tag{14}$$

By the same token, after making the same mathematical consideration for Equation (3), we can attain the following definition and results:

$$|\Lambda_3| < \frac{2}{\gamma^{2m}},\ \Lambda_3 := \gamma^{-k}\delta^{n+m}2^{-3} - 1,\ k < 4n,\ \ m < 6.25\times 10^{11} \times \left(1 + \log(4n)\right),$$

$$|\Lambda_4| < \frac{3}{\gamma^{n}},\ \Lambda_4 := \gamma^{-k}\delta^{n}Q_m - 1,\ n < 1.6\times m \times 10^{14},\ n < 6.7\times 10^{27}.$$

As a result, we can outline the above results as follows.

Lemma 3.2. Let the triple (k, m, n) be a solution of Equation (2) or Equation (3). Then, $k < 4n$, $1 \le m \le n$, and $n < 3.77\times 10^{28}$.

As can be seen, we found that there are a finite number of solutions to our problems but the bounds are quite rough. To obtain a more suitable case, we will use the Dujella -Pethő reduction lemma.

First, we take a look at the notation

$$\Gamma_1 := -k\log\gamma + (n+m)\log|\delta| + \log 2\sqrt{2}. \tag{15}$$

Then, we get

$$\Lambda_1 := \left|e^{\Gamma_1} - 1\right| < \frac{2}{\gamma^m}.$$

It is clear that $|\Lambda_1| < \frac{2}{\gamma^m} < \frac{1}{2}$ for all $m \geq 2$. Additionally, we know from Ddamulira et al. [10, p. 16] that $|x| \leq 2|e^x - 1|$ whenever $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Therefore, we can write $|\Gamma_1| < 2\left|e^{\Gamma_1} - 1\right| < \frac{4}{\gamma^m}$. Considering this inequality together with (15) and dividing both sides by $\log|\delta|$, we obtain

$$\left|k\frac{\log\gamma}{\log|\delta|} - (n+m) + \frac{\log\left(1/2\sqrt{2}\right)}{\log|\delta|}\right| < \frac{5}{\gamma^m}.$$

Then, according to the Dujella-Pethő reduction lemma for $M = 1.51\times 10^{29}$ $(M > 4n > k)$ and $\tau = \frac{\log\gamma}{\log|\delta|}$, *66th* convergent of the continued fraction expansion of τ is

$$\frac{p_{66}}{q_{66}} = \frac{438580154598432582940942547204 4}{1623644029634121411967368101 4241}$$

and so $6M < q_{66} = 16236440296341214119673681014241$. From this, we have

$$\varepsilon := \|\mu q_{66}\| - M\,\|\tau q_{66}\|,\ \varepsilon > 0.05,\ \mu = \frac{\log\left(1/2\sqrt{2}\right)}{\log|\delta|}.$$

So, taking $A := 5$, $B := \gamma$, and $k := m$ into account, we deduce that $m \leq 87$.

Now, consider $2 \leq m \leq 87$. Then, we can write

$$\Gamma_2 := -k\log\gamma + n\log|\delta| - \log\left(\frac{1}{2\sqrt{2}Q_m}\right)$$

and

$$\Lambda_2 := \left|e^{\Gamma_2} - 1\right| < \frac{17}{\gamma^n}.$$

It is clear that $|\Lambda_2| < \frac{17}{\gamma^n} < \frac{1}{2}$ for all $n > 4$. According to the above result of Ddamulira et al. [12], we get

$$\left| k\frac{\log\gamma}{\log|\delta|} - n + \frac{\log\left(1/2\sqrt{2}Q_m\right)}{\log|\delta|} \right| < \frac{39}{\gamma^n}.$$

Based on the Dujella-Pethő reduction lemma for $M = 1.51\times 10^{29}$ $(M > 4n > k)$ and $\tau = \frac{\log\gamma}{\log|\delta|}$, *72th* convergent of the continued fraction expansion of τ is

$$\frac{p_{72}}{q_{72}} = \frac{1344141359139361157238581738156910}{4976073517847943047090943729455893}$$

and $6M < q_{72} = 4976073517847943047090943729455893$. As a result, for $m \in \{2,\ldots,87\}$, we have

$$\varepsilon_m := \|\mu_m q_{72}\| - M\,\|\tau q_{72}\|, \varepsilon_m > 0.008, \mu_m = \frac{\log\left(1/2\sqrt{2}Q_m\right)}{\log|\delta|}.$$

Considering $A := 39$, $B := \gamma$, and $k := n$ it follows that $n \leq 98$.

Repeating the above approaches for Λ_3 and Λ_4, we get that $m \leq 96$ and $n \leq 195$. Organizing a looping algorithm in Mathematica© for Equations (2) and (3) over the range $m \leq 96$ and $n \leq 195$ demonstrates the validity of Theorem 3.1.

The outcomes of Theorem 3.1 bestow the following salient features additionally.

Corollary 3. 3. Just as no Pell number can be written as the square of the Pell-Lucas number, no Pell-Lucas number can be written as the square of the Pell number.

Proof. For the case where $m = n$, Equation (2) implies $P_k = Q_m^2$. From [23] and [24], it is known that there is no perfect powers among Pell numbers, except for $1^2 = 1$ and $13^2 = 169$. However, since 1 and 13 are not Pell-Lucas numbers, Equation (2) has no solution when $m = n$.

Similarly, for the case where $m = n$, Equation (3) implies $Q_k = P_m^2$. From [25], it is known that there are no perfect squares among Pell-Lucas numbers. Therefore, there is no integer k that satisfies the equation $Q_k = P_m^2$ when $m = n$.

Corollary 3. 4. The only coincidence of Pell and Pell-Lucas numbers is $P_2 = Q_1 = 2$.

Proof. We know from Theorem 3.1 that the only solution (k,m,n) to the equation $Q_k = P_m P_n$ is $(1,1,2)$. Substituting these values into the equation $Q_k = P_m P_n$, we get $Q_1 = P_1 P_2$, and from this, it is clear that the only solution is $Q_1 = P_2 = 2$.

4. Conclusions

In this study, we investigated the relationships between Pell and Pell-Lucas numbers through the lens of linear forms in logarithms of algebraic numbers. Specifically, we determined all Pell numbers that can be expressed as the product of two arbitrary Pell-Lucas numbers, as well as all Pell-Lucas numbers that are the product of two arbitrary Pell numbers. Our findings demonstrated that there are no Pell or Pell-Lucas numbers that can be expressed as the square of another number in their respective sequences. Additionally, we identified all common terms shared by the Pell and Pell-Lucas sequences, further enriching the understanding of these fundamental integer sequences.

An open question arising from this work is whether the product of a Pell number and a Pell-Lucas number can itself be a term in the Pell or Pell-Lucas sequence. Exploring this question could provide new insights into the structural properties and interrelations of these sequences. Future research might also consider generalizations to other related integer sequences or investigate higher-order recurrence relations under similar frameworks.

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