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New Identities For Harmonic And Hyperharmonic Numbers Via Riordan Arrays

Fatma Yeşil BARAN [1,*] , **Burcu TUTKAVUL** [2]

[1] *Dept. of Computer Science, Amasya University, Amasya, 05100, Turkey ,fatma.yesilbaran@amasya.edu.tr.*
[2] *Dept. of Mathematics, Amasya University, Amasya, 05100, Turkey, brckiyak60@gmail.com*
**Corresponding author: fatma.yesilbaran@amasya.edu.tr*

ABSTRACT: This paper deals with obtaining new identities and equations for Harmonic and Hyperharmonic numbers.We get some matrices which defined by these numbers. We also derive some identities for these numbers with the aid of Riordan array. In conclusion, we get new identities related to harmonic and hyperharmonic numbers, enabling us to determine the row sums of these matrices.

Keywords – *Harmonic number; Hyperharmonic number; Riordan array; row sum; generating function*.

1. Introduction

Harmonic numbers are employed in many areas of number theory and have been studied since antiquity. There are various generalizations of harmonic numbers. Conway and Guy defined Hyperharmonic numbers and obtained some equations that allow Hyperharmonic numbers to be written in terms of Harmonic numbers (Conway and Guy, 1996). Definitions of these numbers are given as follows:

The harmonic number is defined as $\mathcal{H}_n = \sum_{k=1}^{n} \frac{1}{k}$ with $\mathcal{H}_0 = 0$ is called the $n-$th harmonic number. The $\alpha-$th order $n-$th hyperharmonic numbers, defined by $\mathcal{H}_n^{(\alpha)}$, is defined as $\mathcal{H}_n^{(\alpha)} = \sum_{k=1}^{n} \mathcal{H}_k^{(\alpha-1)}$ with $\alpha > 1$ integer and $\mathcal{H}_n^{(1)} = \mathcal{H}_n$. The generating functions for each of these numbers are provided,

$$\sum_{n=0}^{\infty} \mathcal{H}_n t^n = -\frac{\ln(1-t)}{1-t}\mathcal{H} \tag{1}$$

$$\sum_{n=0}^{\infty} \mathcal{H}_n^{(\alpha)} t^n = -\frac{\ln(1-t)}{(1-t)^{\alpha}} \tag{2}$$

Numbers to be written in terms of Harmonic numbers (Conway and Guy, 1996). By noting that the resulting infinite triangular matrix of some generalized Harmonic numbers follows a Riordan sequence and numerous polynomials with the Harmonic numbers, Cheon et al. were able to establish linkages between Stirling numbers of both sorts and additional generalized Harmonic numbers (Cheon et al., 2006; Cheon and El -Mikkawy 2008 ; Cheon et al., 2007). Wang applied generalized Harmonic numbers with combinatorial sequences (Wang, 2010). Harmonic number theory was further developed in the well-known studies of (Duran et al., 2020; Kızılates and Tuglu, 2015; Tuglu and Kızılates,, 2015; Tuglu et al., 2015; Cetin et al., 2021; Dil and Mezo, 2008; Tuglu et al., 2023). By using the operators and transforms, the authors investigate the generating function, Binet-like formula, summation formula, recurrence relation and other characteristics equations of new sequences which are named Horadam, Quadrapell and Tribonacci (Kızılateş et al., 2017; Kızılateş, 2021; Kızılateş et al., 2022).

Historically, there have been two versions of the Riordan series concept. The first version consists of a $\left(d_{i.j}\right)_{(i.j\in Z)}$ shaped infinite matrix. It is mostly used to examine algebraic properties and deals with Formal Laurent series. an $\left(d_{i.j}\right)_{(i.j\in Z)}$ shaped lower triangular matrix makes up the second version.. It was used to examine formal power series and combinatorial properties. Shapiro, Getu, Woan and Woodson are the researchers who made the first studies on the group and defined this structure. They also obtained the inverses of the matrices represented by Riordan notation by using the properties of the Riordan group (Shapiro et al., 1991). They defined Riordan matrix denoted by $(g(t), f(t))$, has generating function of the j −th column is $g(t)f(t)^j$ where for $j = 0,1,2, ...,$ $g(0) \neq 0, f(0) = 0$ and $f'(0) \neq 0$. The set of all Riordan matrices is a group under matrix multiplication, which operation is defined as

$$(g(t).f(t)).(u(t),v(t)) = \big(g(t)u\big(f(t)\big), v(f(t))\big) \tag{3}$$

The following theorem, established by (Sprugnoli, 2006), is known as the Fundamental Theorem of Riordan groups. Let $D = \left(d_{(n,k)}\right)$ is a Riordan matrix which is defined by pair or (g, f) and $h(x) = \sum_{k=0}^{n} h_k t^k$ the equation holds

$$\sum_{j=0}^{i} d_{i,j} h_j = \left[t^i g(t) h(f(t))\right] \tag{4}$$

Then Rogers generalized the properties of Pascal’s triangle and Riordan introduced the structure (Rogers, 1978). Hennesy combined Riordan arrays with continued fractions (Hennesy,2011). Also the authors have studied on identites of Riordan arrays (Luzon et al., 2012; Merlini and Verri, 2000, Merlini and Sprugnoli, 2002). The authors give the classical row sum of the Riordan sequence, the alternating row sum and the weighted row sums (Shapiro, 2003; He and Shapiro, 2016). Let D be Riordan matrix which can be expressed by (g, f).

- The sum of the $i.$ row entries of the Riordan array is defined by

$$\alpha_i = \sum_{j=0}^{\infty} d_{i,j} = \left[t^i\right] \frac{g(t)}{1 - f(t)} \tag{5}$$

- The Riordan array's $i.$row entries' alternating sum is

$$\beta_i = \sum_{j=0}^{\infty} (-1)^j d_{i,j} = \left[t^i\right] \frac{g(t)}{1 + f(t)} \tag{6}$$

We direct the reader to a few recent contributions for the history and applications of the hyperharmonic and harmonic numbers employing the Riordan array to various fields (Wang, 2010; Munarini, 2011; Koparal et al., 2021).

In this paper, our aim is to obtain new identites for Harmonic and hyperharmonic numbers by aid of Riordan matrix, which based on some special matrices defining by these numbers. Firstly, we defined a few unique matrices that include hyperharmonic and harmonic numbers. Then several theorems are given to confirm our results. So we have given new identites for these numbers. Lastly, the sums of these newly defined matrices in classical row and alternating row are presented, respectively.

2. Main Results

In this section, firstly, we describe the definition and some basic properties of matrices which defined by Harmonic and generalized Harmonic numbers. Then we obtain several theorems by using these numbers. Let us define the $\mathcal{H} = \left(h_{n,k}\right)_{n,k=1}^{\infty}$ and $Q = (q_{n.k})_{n,k=1}^{\infty}$ as follows:

$$\mathcal{H} = (h_{n.k})_{n,k=1}^{\infty} = \begin{cases} \mathcal{H}_{n-k+1}, & n \geq k \\ 0, & n < k \end{cases} \tag{7}$$

and

$$Q = (q_{n.k})_{n,k=1}^{\infty} = \begin{cases} q^{n-k}, & n \geq k \\ 0, & n < k \end{cases} \tag{8}$$

where $\mathcal{H}_n$ is the $n-th$ harmonic number. In other words, these matrices are presented as:

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & & & & \cdots \\ \mathcal{H}_2 & \mathcal{H}_1 & & & \cdots \\ \mathcal{H}_3 & \mathcal{H}_2 & \mathcal{H}_1 & & \cdots \\ \mathcal{H}_4 & \mathcal{H}_3 & \mathcal{H}_2 & \mathcal{H}_1 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

and

$$Q = \begin{bmatrix} q^0 & & & & \cdots \\ q^1 & q^0 & & & \cdots \\ q^2 & q^1 & q^0 & & \cdots \\ q^3 & q^2 & q^1 & q^0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

Lemma 1. Suppose that $\mathcal{H}$ is a matrix as in (7). Then the Riordan representation of the H matrix is given by

$$\left(-\frac{\ln(1-t)}{t(1-t)}, t\right) \tag{9}$$

Proof. The $0-th$ column of the H matrix is $\left[\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_{3,\ldots}\right]^T$. $0-th$ column consist of coefficients of a formal power series in the form of $\sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n$.

Since using equation of (1), we get

$$\sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n = -\frac{\ln(1-t)}{t(1-t)}$$

It is seen that the generating function of the $0-th$ column is $-\frac{\ln(1-t)}{t(1-t)}$. Under the lighting under the equation of (3), the other columns of the$\mathcal{H}$matrix are the t unit shifted of the $0-th$ column, so the Riordan representation of the $\mathcal{H}$ matrix is obtained as

$$\left(-\frac{\ln(1-t)}{t(1-t)},t\right).$$

On the other hand, let D matrix, denoted by $D=\left(-\frac{\ln(1-t)}{t(1-t)},t\right)$, from equation of (3), we obtain

$$\begin{aligned} d_{i,j} &= [t^i]\left[-\frac{\ln(1-t)}{t(1-t)}t^j\right] \\ &= [t^i]\left(\sum_{n=0}^{\infty}\mathcal{H}_{n+1}t^n\right)t^j \\ &= [t^{i-j}]\sum_{n=0}^{\infty}\mathcal{H}_{n+1}t^n \\ &= H_{i-j+1}. \end{aligned}$$

Since the entries of the $\left(-\frac{\ln(1-t)}{t(1-t)},t\right)$ matrix is $\mathcal{H}_{n-k+1}$. Then the matrix defined by $\left(-\frac{\ln(1-t)}{t(1-t)},t\right)$ is Riordan representatiton is of $\mathcal{H}$ matrix.

Lemma 2. Assume that Q is a matrix as in (8). Then the Riordan representation of the Q matrix is

$$\left(\frac{1}{1-qt},t\right). \tag{10}$$

Proof. The evidence for this lemma is that it resembles Lemma 1.

Theorem 1. Let $\mathcal{H}$ and Q are as in (7) and (8). The Riordan representation of the HQ matrix is

$$\left(-\frac{\ln(1-t)}{t(1-t)(1-qt)},t\right). \tag{11}$$

Proof. From equation of (9) and (10),it can be written as

$$\mathcal{H}Q=\left(-\frac{\ln(1-t)}{t(1-t)},t\right)\left(\frac{1}{1-qt},t\right).$$

By using (3), the Riordan representations of $\mathcal{H}Q$ is obtained as

$$\left(-\frac{\ln(1-t)}{t(1-t)(1-qt)},t\right).$$

Theorem 2. The entries of $\mathcal{H}Q=\left(d_{i,j}\right)_{i,j=1}^{\infty}$ matrix is obtained as

$$d_{i,j} = \sum_{k=1}^{i-j+1} \mathcal{H}_k q^{i-j-k+1} \tag{12}$$

where the $\mathcal{H}$ and Q matrices defined in (7) and (8), respectively.

Proof. The Riordan representation of the $\mathcal{H}Q$ matrix is in the form of $\left(-\frac{\ln(1-t)}{t(1-t)(1-qt)}, t\right)$, and the $j - th$ column consists of the elements of the array which produced by the $-\frac{\ln(1-t)}{t(1-t)(1-qt)} t^j$ function. Hence

$$\begin{aligned} d_{i,j} &= [t^i]\left(-\frac{\ln(1-t)}{t(1-t)(1-qt)} t^j\right) \\ &= [t^i]\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{H}_{k+1} q^{n-k} t^n\right) t^j \\ &= [t^i]\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n-j} \mathcal{H}_{k+1} q^{n-j-k} t^n\right) \\ &= [t^i]\sum_{k=0}^{n-j} \mathcal{H}_{k+1} q^{n-j-k} \\ &= \sum_{k=1}^{i-j+1} \mathcal{H}_k q^{i-j-k+1}. \end{aligned}$$

Theorem 3. Let the matrix γ be defined by the array of

$$\left(-\frac{\ln(1-t)}{t(1-t)(1-qt)}, \frac{t}{1-qt}\right). \tag{13}$$

The entries of the γ matrix, with $\gamma = (d_{i,j})_{i,j=1}^{\infty}$ is

$$d_{i,j} = \sum_{k=1}^{i+1} \binom{i-k+1}{j} \mathcal{H}_k q^{i-j-k+1} \tag{14}$$

Proof. The $j - th$ column of the γ matrix consists of the elements of the array produced by the function as

$$-\frac{\ln(1-t)}{t(1-t)(1-qt)} \frac{t^j}{(1-qt)^j} = \frac{-\ln(1-t)}{t(1-t)} \frac{t^j}{(1-qt)^{j+1}}$$

So

$$-\frac{\ln(1-t)}{t(1-t)} \frac{t^j}{(1-qt)^{j+1}} = \sum_{n=j}^{\infty}\left(\sum_{k=0}^{n} \mathcal{H}_{k+1} \binom{n-k}{n-j-k} q^{n-j-k}\right) t^n.$$

From here, the entires of the matrix is obtained as

$$d_{i,j} = \sum_{k=0}^{i} \mathcal{H}_{k+1} \binom{i-k}{i-j-k} q^{i-j-k}$$

By equation of $\binom{i-j}{i-j-k} = \binom{i-k}{j}$, then we get

$$d_{i,j} = \sum_{k=0}^{i} \mathcal{H}_{k+1} \binom{i-k}{j} q^{i-j-k}.$$

With the help of index change , which is $k \to k-1$, we obtain

$$d_{i,j} = \sum_{k=0}^{i+1} \mathcal{H}_{k} \binom{i-k+1}{j} q^{i-j-k+1}.$$

Theorem 4. Suppose that γ is a matrix as in (13). Then we get

$$\sum_{j=0}^{i} \sum_{k=1}^{i-j+1} \mathcal{H}_k q^{i-k+1} \binom{s+j-1}{s-1} = \sum_{k=0}^{i} \mathcal{H}_{k+1} \binom{s+i-k}{s} q^{i-k}$$

Proof. Let $h(t)$ is a function as defined by $h(t) = \frac{1}{(1-qt)^s} = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} q^n t^n$. From equalities of (4) and (13), we get

$$\begin{aligned}\sum_{j=0}^{i} \sum_{k=1}^{i-j+1} \mathcal{H}_k q^{i-j-k+1} \binom{s+j-1}{s-1} q^j &= [t^i] \frac{-\ln(1-t)}{t(1-t)(1-qt)} \frac{1}{(1-qt)^s} \\ &= [t^i] \frac{-\ln(1-t)}{t(1-t)(1-qt)^{s+1}} \\ &= [t^i] \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1} \binom{s+n-k}{s} q^{n-k} \right) t^n \\ &= \sum_{k=0}^{i} \mathcal{H}_{k+1} \binom{s+i-k}{s} q^{i-k}\end{aligned}$$

and so we get

$$\sum_{j=0}^{i} \sum_{k=1}^{i-j+1} \mathcal{H}_k q^{i-k+1} \binom{s+j-1}{s-1} = \sum_{k=0}^{i} \mathcal{H}_{k+1} \binom{s+i-k}{s} q^{i-k}.$$

In particular, if $q = 1$ and $s = 1$ are taken, the equation in Theorem 4 turns to

$$\sum_{j=0}^{i} \sum_{k=1}^{i-j+1} \mathcal{H}_k = \sum_{k=0}^{i} \mathcal{H}_{k+1} (i-k).$$

Theorem 5. Let the matrix γ be defined as in (13). The we obtain

$$\sum_{j=0}^{i}\sum_{k=1}^{i+1}(-1)^j\binom{i-k+1}{j}\mathcal{H}_k q^{i-k+1}=\mathcal{H}_{i+1}.$$

Proof. Assume that $h(t)$ is a function as in $h(t)=\frac{1}{1+qt}=\sum_{n=0}^{\infty}(-1)^n q^n t^n$. From (4) and (13), we have

$$\sum_{j=0}^{i}\sum_{k=1}^{i+1}(-1)^j q^j\binom{i-k+1}{j}\mathcal{H}_k q^{i-j-k+1}=[t^i]\left(\frac{-\ln(1-t)}{t(1-t)(1-qt)}\frac{1}{1+q\left(\frac{t}{1-qt}\right)}\right)$$

$$=[t^i]\frac{-\ln(1-t)}{t(1-t)}$$

$$=[t^i]\sum_{n=0}^{\infty}\mathcal{H}_{n+1}t^n$$

$$=\mathcal{H}_{i+1}.$$

and so

$$\sum_{j=0}^{i}\sum_{k=1}^{i+1}(-1)^j\binom{i-k+1}{j}\mathcal{H}_k q^{i-k+1}=\mathcal{H}_{i+1}.$$

Theorem 6. Let the matrix $\mathcal{H}$ be defined as in (7). Then the row sums of $\mathcal{H}$ matrix are given by

(i) Classic row sum:

$$\alpha_i=\sum_{k=0}^{i}\mathcal{H}_{k+1}$$

(ii) Alternating row sum:

$$\beta_i=\sum_{k=0}^{i}\mathcal{H}_k(-1)^{i-k}$$

Proof. (i) If the equation of (5) is applied to $\mathcal{H}$ matrix, we obtain

$$\alpha_i=[t^i]\frac{\frac{-\ln(1-t)}{t(1-t)}}{1-t}$$

$$=[t^i]\sum_{n=0}^{\infty}\mathcal{H}_{n+1}t^n\sum_{n=0}^{\infty}t^n$$

$$= [t^i] \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1} \right) t^n$$

$$= \sum_{k=0}^{i} \mathcal{H}_{k+1.}$$

(ii) If the equation of (6) is applied to $\mathcal{H}$ matrix, we get

$$\beta_i = [t^i] \frac{\frac{-\ln(1-t)}{t(1-t)}}{1+t}$$

$$= [t^i] \sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n \sum_{n=0}^{\infty} (-1)^n t^n$$

$$= [t^i] \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1} (-1)^{n-k} \right) t^n$$

$$= \sum_{k=0}^{i} (-1)^{i-k} \mathcal{H}_{k+1.}$$

Theorem 7. The row sums of $\mathcal{H}$Q matrix is given by

(i) Classic row sum:

$$\alpha_i = \sum_{k=0}^{i} \sum_{l=0}^{i-k} \mathcal{H}_{k+1} q^{i-k-l}$$

(ii) Alternating row sum:

$$\beta_i = \sum_{k=0}^{i} \sum_{l=0}^{i-k} (-1)^i \mathcal{H}_{k+1} q^{i-k-l}$$

Proof. (i) If the equation of (5) is applied to $\mathcal{H}Q$ matrix, we find that

$$\alpha_i = [t^i] \frac{\frac{-\ln(1-t)}{t(1-t)(1-qt)}}{1-x}$$

$$= [t^i] \frac{-\ln(1-t)}{t(1-t)} \frac{1}{1-t} \frac{1}{1-qt}$$

$$= [t^i] \sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n \sum_{n=0}^{\infty} \sum_{l=o}^{n} q^{n-l} t^n$$

$$= [t^i] \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1} \sum_{l=0}^{n-k} q^{n-k-l} \right) t^n$$

$$= [t^i] \sum_{k=0}^{i} \sum_{l=0}^{i-k} \mathcal{H}_{k+1} \, q^{i-k-l}$$

(ii) If the equation of (6) is apllied to $\mathcal{H}Q$ matrix,

$$\beta_i = [t^i] \frac{q(t)}{1+f(t)}$$

$$= [t^i] \frac{\frac{-\ln(1-t)}{t(1-t)(1-qt)}}{1+x}$$

$$= [t^i] \frac{-\ln(1-t)}{t(1-t)} \frac{1}{1+t} \frac{1}{1-qt}$$

$$= [t^i] \sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n \sum_{n=0}^{\infty} (-1)^n t^n \sum_{n=o}^{\infty} q^n t^n$$

$$= [t^i] \sum_{n=0}^{\infty} \mathcal{H}_{n+1} t^n \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} (-1)^l q^{n-k-l} \right) t^n$$

$$= \sum_{k=0}^{i} \sum_{l=0}^{i-k} (-1)^l \mathcal{H}_{k+1} q^{i-k-l}$$

is obtained.

Let us define the $\mathbb{H} = (h_{i,j})_{i,j=1}^{\infty}$ as follows

$$\mathbb{H} = (h_{i,j})_{i,j=1}^{\infty} = \begin{cases} \mathcal{H}_{i-j+1}^{(\alpha)}, & i \geq j \\ 0, & i < j \end{cases} \tag{15}$$

where $\mathcal{H}_n^{(\alpha)}$ is the $n-th$ Hyperharmonic number.

In other words, these matrices are presented as:

$$\mathbb{H} = \begin{bmatrix} \mathcal{H}_1^{(\alpha)} & & & & \cdots \\ \mathcal{H}_2^{(\alpha)} & \mathcal{H}_1^{(\alpha)} & & & \cdots \\ \mathcal{H}_3^{(\alpha)} & \mathcal{H}_2^{(\alpha)} & \mathcal{H}_1^{(\alpha)} & & \cdots \\ \mathcal{H}_4^{(\alpha)} & \mathcal{H}_3^{(\alpha)} & \mathcal{H}_2^{(\alpha)} & \mathcal{H}_1^{(\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Lemma 3. Assume that $\mathbb{H}$ is a matrix as in (15). Then the Riordan representations of the $\mathbb{H}$ matrix is

$$\left(\frac{-\ln(1-t)}{t(1-t)^{\alpha}}, t \right). \tag{16}$$

Proof. $0-th$ column of the $\mathbb{H}$ matrix is $\left[\mathcal{H}_1^{(\alpha)}, \mathcal{H}_2^{(\alpha)}, \mathcal{H}_3^{(\alpha)}, \cdots \right]^T$.$0-th$ column consist of the coefficients of a formal power series in the form of $\sum_{n=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha)} t^n$.

Since using equation of (2), we obtain

$$\sum_{n=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha)} t^n = \frac{-\ln(1-t)}{t(1-t)^\alpha}.$$

It is seen that the generating function of the $0 - th$ column is $\frac{-\ln(1-t)}{t(1-t)^\alpha}$. Considering the equation of (3), the other columns of the $\mathbb{H}$ matrix are the x unit shifted of the $0 - th$ column, then the Riordan representation of the $\mathbb{H}$ matrix is obtained as

$$\left(\frac{-\ln(1-t)}{t(1-t)^\alpha}, t\right).$$

On the other hand, let D is a matrix defined by $D = \left(\frac{-\ln(1-t)}{t(1-t)^\alpha}, t\right)$, from (3), we get

$$\begin{aligned} d_{i,j} &= [t^i]\left[\frac{-\ln(1-t)}{t(1-t)^\alpha} t^j\right] \\ &= [t^i]\left(\sum_{n=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha)}\right) t^j \\ &= [t^{i-j}] \sum_{n=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha)} t^n \\ &= \mathcal{H}_{i-j+1}^{(\alpha)} \end{aligned}$$

Since thr entries of the $\left(\frac{-\ln(1-t)}{t(1-t)^\alpha}, t\right)$ matrix is $\mathcal{H}_{i-j+1}^{(\alpha)}$, then the matrix which is define by $\left(\frac{-\ln(1-t)}{t(1-t)^\alpha}, t\right)$ is Riordan representation of $\mathbb{H}$ matrix.

Theorem 8. The Riordan representation of the $\mathbb{H}$Q matrix is

$$\left(\frac{-\ln(1-t)}{t(1-t)^\alpha} \frac{1}{1-qt}, t\right) \tag{17}$$

where $\mathbb{H}$ and the Q matrices are defined in (15) and (8), respectively.

Proof. Using the equations of (10) and (16), it can be written as

$$\mathbb{H}Q = \left(\frac{-\ln(1-t)}{t(1-t)^\alpha}, t\right)\left(\frac{1}{1-qt}, t\right).$$

By using equation of (3), the Riordan representation of $\mathbb{H}Q$ is obtained as

$$\left(\frac{-\ln(1-t)}{t(1-t)^\alpha} \frac{1}{1-qt}, t\right).$$

Theorem 9. The entries of the $\mathbb{H}Q = (d_{i,j})_{i,j=1}^{\infty}$ is given by

$$d_{i,j} = \sum_{k=0}^{i-j} \mathcal{H}_{k+1}^{(\alpha)} q^{i-j-k} \tag{18}$$

where ℍ and the Q matrices are defined in (15) and (8), respectively.

Proof. The Riordan representation of the $\mathbb{H}Q$ matrix is in the form of $\left(\frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{1}{1-qt}, t\right)$ and the $j-th$ column consist of the elements of the array which produced by the $\frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{1}{1-qt}t^{j}$ function.

$$\frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{1}{1-qt}t^{j} = \sum_{n=j}^{\infty}\left(\sum_{k=0}^{n-j}\mathcal{H}_{k+1}^{(\alpha)}q^{n-k-j}\right)t^{n}$$

İs in the form.

From here, the general term of the matrix

$$d_{i,j} = \sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}q^{i-j-k},$$

is obtained.

Theorem 10. Let the matrix τ be defined by the array of

$$\left(\frac{-\ln(1-t)}{t(1-t)^{\alpha}(1-qt)}, \frac{t}{1-qt}\right). \tag{19}$$

Therefore, the entries of τ matrix, with $\tau = \left(d_{i,j}\right)_{i,j=1}^{\infty}$, *is*

$$d_{i,j} = \sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}\binom{i-k}{j}q^{i-j-k}. \tag{20}$$

Proof. The $j-th$ column of the τ matrix consist of the elements of the array produced by the function as

$$\frac{-\ln(1-t)}{t(1-qt)}\frac{t^{j}}{(1-qt)^{j}} = \frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{t^{j}}{(1-qt)^{j+1}}.$$

So

$$\frac{-\ln(1-t)}{t(1-t)^{\alpha}(1-qt)^{j+1}}t^{j} = \sum_{n=0}^{\infty}\mathcal{H}_{n+1}^{(\alpha)}t^{n}\sum_{n=0}^{\infty}\binom{j+n}{n}q^{n}t^{n+j}$$

$$= \sum_{n=j}^{\infty}\sum_{k=0}^{n-j}\mathcal{H}_{k+1}^{(\alpha)}\binom{n-k}{j}q^{n-j-k}$$

From here, the entries of the matrix is obtained as

$$d_{i,j} = \sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}\binom{i-k}{j}q^{i-j-k}.$$

Theorem 11. Suppose that ℍQ is a matrix as in (17). Then we get

$$\sum_{j=0}^{i}\sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}q^{i-j-k}\binom{s+j-1}{j}=\sum_{k=0}^{i}\mathcal{H}_{k+1}^{(\alpha+s)}q^{i-k}.$$

Proof. Let $h(t)$ is a function as $\frac{1}{(1-t)^s}=\sum_{n=0}^{\infty}\binom{s+n-1}{s-1}t^n$. From (4) and (17), we get

$$\sum_{j=0}^{i}\sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}q^{i-j-k}\binom{s+j-1}{s-1}=[t^i]\frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{1}{1-qt}\frac{1}{(1-t)^s}$$

$$=[t^i]\frac{-\ln(1-t)}{t(1-t)^{\alpha+s}}\frac{1}{1-qt}$$

$$=[t^i]\sum_{n=0}^{\infty}\mathcal{H}_{n+1}^{(\alpha+s)}t^n\sum_{n=0}^{\infty}q^nt^n$$

$$=[t^i]\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mathcal{H}_{k+1}^{(\alpha+s)}q^{i-k}\right)t^n$$

$$=\sum_{k=0}^{i}\mathcal{H}_{k+1}^{(\alpha+s)}q^{i-k}$$

and so

$$\sum_{j=0}^{i}\sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}q^{i-j-k}\binom{s+j-1}{j}=\sum_{k=0}^{i}\mathcal{H}_{k+1}^{(\alpha+s)}q^{i-k}.$$

Theorem 12. Let the matrix τ be defined as in (19). Following equation holds

$$\sum_{j=0}^{i}\sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}\binom{i-k}{j}q^{i-k}(-1)^j=\mathcal{H}_{i+1}^{(\alpha)}.$$

Proof. Let $h(t)=\frac{1}{1+qt}=\sum_{n=0}^{\infty}(-1)^nq^nt^n$. From (4) and (19)

$$\sum_{j=0}^{i}\sum_{k=0}^{i-j}\mathcal{H}_{k+1}^{(\alpha)}\binom{i-k}{j}q^{i-k}(-1)^j=[t^i]\frac{-\ln(1-t)}{t(1-t)^{\alpha}}\frac{1}{1-qt}\frac{1}{1+q\,\frac{t}{1-qt}}$$

$$=[t^i]\frac{-\ln(1-t)}{t(1-t)^{\alpha}}$$

$$=[t^i]\sum_{n=0}^{\infty}\mathcal{H}_{n+1}^{(\alpha)}t^n$$

$$=\mathcal{H}_{i+1}^{(\alpha)}.$$

Theorem 13. Let the matrix ℍ be defined as in (15). In this case,

(i) Classic row sum:

$$\alpha_i = \mathcal{H}_{i+1}^{(\alpha+1)}$$

(ii) Alternating row sum:

$$\beta_i = \sum_{k=0}^{i} \mathcal{H}_{k+1}^{(\alpha)} (-1)^{i-k}$$

Proof. (i) If the equation of (5) is applied to $\mathbb{H}$ matrix, we obtain

$$\begin{aligned}\alpha_i &= [t^i] \frac{\frac{-\ln(1-t)}{t(1-t)^{\alpha}}}{1-t} \\ &= [t^i] \frac{-\ln(1-t)}{t(1-t)^{\alpha+1}} \\ &= [t^i] \sum_{k=0}^{\infty} \mathcal{H}_{k+1}^{(\alpha)} t^k \\ &= \mathcal{H}_{i+1}^{(\alpha+1)}.\end{aligned}$$

(ii) If the equation of (6) is applied to $\mathbb{H}$ matrix

$$\begin{aligned}\beta_i &= [t^i] \frac{\frac{-\ln(1-t)}{t(1-t)^{\alpha}}}{1+t} \\ &= [t^i] \sum_{k=0}^{\infty} \mathcal{H}_{k+1}^{(\alpha)} t^n \sum_{n=0}^{\infty} (-1)^n t^n \\ &= [t^i] \sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1}^{(\alpha)} (-1)^{n-k} \right) t^n \\ &= \sum_{k=0}^{i} \mathcal{H}_{k+1}^{(\alpha)} (-1)^{i-k}\end{aligned}$$

Theorem 14. The row sums of $\mathbb{H}$Q matrix are given by

(i) Classic row sum:

$$\alpha_i = \sum_{k=0}^{i} \mathcal{H}_{k+1}^{(\alpha+1)} q^{i-k}$$

(ii) Alternating row sum:

$$\beta_i = \sum_{k=0}^{i} \sum_{l=0}^{i-k} (-1)^l q^{i-k-l} \mathcal{H}_{k+1}^{(\alpha)}$$

Proof. (i) If the equation of (5) is applied $\mathbb{H}Q$ matrix, we obtain

$$\alpha_i = \left[t^i\right] \frac{\dfrac{-\ln(1-t)}{t(1-t)^{\alpha}(1-qt)}}{1-x}$$

$$= \left[t^i\right] \frac{-\ln(1-t)}{t(1-t)^{\alpha+1}} \frac{1}{1-qt}$$

$$= \left[t^i\right] \sum_{k=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha+1)} t^n \sum_{n=0}^{\infty} q^n t^n$$

$$= \left[t^i\right] \sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1}^{(\alpha+1)} q^{n-k} \right) t^n$$

$$= \sum_{k=0}^{i} \mathcal{H}_{k+1}^{(\alpha+1)} q^{i-k}$$

(ii)) If the equation of (6) is applied $\mathbb{H}Q$ matrix, we obtain

$$\beta_i = \left[t^i\right] \frac{\dfrac{-\ln(1-t)}{t(1-t)^{\alpha}(1-qt)}}{1+t}$$

$$= \left[t^i\right] \sum_{k=0}^{\infty} \mathcal{H}_{n+1}^{(\alpha)} t^n \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k q^{n-k} \right) t^n$$

$$= \left[t^i\right] \sum_{k=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{H}_{k+1}^{(\alpha)} \sum_{l=0}^{n-k} (-1)^l q^{n-k-l} \right) t^n$$

$$= \sum_{k=0}^{i} \mathcal{H}_{k+1}^{(\alpha)} \sum_{l=0}^{i-k} (-1)^l q^{i-k-l}$$

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