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A Tauberian theorem for the logarithmic summability in ordered spaces

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Abstract

The present manuscript aims to extend a Tauberian theorem previously established for the Cesàro and weighted mean summability methods of single sequences in ordered spaces to the logarithmic summability method, also known as the (ℓ,1,1) method, for double sequences. In order to achieve this, we present several Tauberian conditions which address the O_L-oscillatory behavior of a double sequence (s_{mn}) with respect to logarithmic summability in certain senses. These conditions facilitate the transition from (ℓ,1,1), (ℓ,1,0), and (ℓ,0,1) summability to P-convergence in ordered spaces.

Keywords: *Double sequences, ordered linear spaces, slowly decreasing sequences with respect to (ℓ,1,1), Tauberian conditions, Tauberian theorems, logarithmic summability method*

Sıralı uzaylarda logaritmik toplanabilme için bir Tauber tipi teorem

Öz

Bu çalışma daha önce sıralı uzaylardaki tek katlı dizilerin Cesàro ve ağırlıklı ortalama toplanabilirlik yöntemleri için oluşturulmuş Tauber tipi teoremleri, iki katlı diziler için logaritmik toplanabilirlik yöntemine, diğer adıyla (ℓ,1,1) yöntemine genişletmeyi amaçlar. Bu amaçla, çeşitli anlamlarda logaritmik toplanabilirliğe göre iki katlı bir (s_{mn}) dizinin O_L-salınım davranışını ele alan birkaç Tauber tipi koşul sunuyoruz. Bu koşullar, sıralı uzaylarda dizinin (ℓ,1,1), (ℓ,1,0) ve (ℓ,0,1) toplanabilirliğinden P-yakınsaklığına geçişine olanak sağlar.

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Anahtar kelimeler: *Çift diziler, sıralı doğrusal uzaylar, (ℓ,1,1) metoduna göre yavaş azalan diziler, Tauber koşullar, Tauber teoremler, logaritmik toplanabilme metodu*

1. Introduction

The study of summability methods has played a significant role in the advancement of mathematical analysis, particularly in understanding the convergence behavior of sequences and series. Among these methods, logarithmic summability has come to attract the attention of many researchers due to the influence of the pioneering work of Ishiguro [1] in the early 1960s, as well. His foundational papers, such as [1] and [2], laid the groundwork for further explorations into the properties and implications of logarithmic summability. With these seminal works, Ishiguro initiated a series of investigations that have examined Tauberian theorems, which provide equivalences between convergence and logarithmic summability method $(\ell, 1)$ for the single sequences or method $(L, 1)$ for the power series. In [1], Ishiguro proved that every summable sequence (s_n) in terms of method $(\ell, 1)$ is also summable in terms of method (*L, 1*) to the same value besides the converse of that is not always true. In the following article, Ishiguro [2] offered a condition $s_n = O_L(1)$ for that to be true. Drawing inspiration from Hardy's [3] and Szász's [4] works in 1963, Ishiguro [5] indicated that if a sequence (s_n) is logarithmic summable $(\ell, 1)$ (or (L, 1)) to ξ and $\omega_n^{(0)} = o(1)$, where

$$\omega_n^{(0)}(s) = (n+1)\ell_{n-1}(s_n - s_{n-1}) \sim n \log n \,(s_n - s_{n-1}),$$

then it also converges to same value. In the latter of his consecutive papers, Ishiguro [6] insured the equivalence of the Cesàro and $(\ell, 1)$ methods, provided that $\log n\,(\sigma_n - \xi) = o(1)$, where σ_n denotes the logarithmic means of sequence (s_n). In sequel, Kwee [7] demonstrated that the necessary condition for convergence of a sequence that is logarithmic summable $(\ell, 1)$ is

$$lim\ inf_{m\to\infty}(s_n - s_m) \geq 0 \quad \text{whenever } n > m \to \infty \text{ and } \frac{\log n}{\log m} \to 1. \tag{1}$$

To construct equivalence of the logarithmic methods $(\ell, 1)$ and $(L, 1)$, Kwee [8] presented some Tauberian theorems dealing with implication from the method $(L, 1)$ to the method $(\ell, 1)$ under conditions such as $s_n = O_L(1), s_n - \sigma_n = O_L(1)$, the condition (1)

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k = O(1) \quad \text{or} \quad v_n := \sum_{k=0}^{n} \frac{\sigma_k}{k+1} = o(\log^2 n), \tag{2}$$

independently from each other. In the following article, Kwee [9] generalized some Tauberian results, obtained by Ishiguro [1] and Kwee [7-8], from the logarithmic methods $(\ell, 1)$ and $(L, 1)$ to (ℓ, α) and (L, α). In subsequent years, researchers such as Rangachari and Sitaraman [10], Kaufman [11], Kohanovskiĭ [12-13], and Burljai [14] broadened Tauberian theorems' peculiar to logarithmic methods scope in certain senses supporting from Ishiguro's and Kwee's contributions. These findings fostered a deeper understanding of its applications in mathematical analysis, illuminating conditions under which logarithmic summability could yield convergence.

In the later years, the works proceeded with contributions from Móricz [15-16] based upon the finding of necessary and sufficient conditions for the logarithmic summability of sequences and its interaction with statistical methods. Móricz [15] established two results dealing with implication from the statistical logarithmic summability $(\ell, 1)$ to convergence under conditions controlling O and O_L-oscillatory behavior of sequence.

After Móricz [16] introduced the concept of slow oscillation of a sequence with respect to summability $(\ell, 1)$, and indicated to be equivalent to (1), the author presented necessary and sufficient Tauberian conditions under which the convergence follows from its logarithmic summability $(\ell, 1)$.

More recent studies such as Alghamdi et al. [17], Totur and Okur [18], Sezer and Çanak [19-20], Çınar and Çanak [21], Okur [22] enlarged the scope of logarithmic summability from various aspects. In [17, 21], Alghamdi et al. and Çınar and Çanak discussed the relation between statistical logarithmic summability and statistical logarithmic convergence. Totur and Okur [18] investigated the logarithmic summability methods of numerical sequences and their applications such as Tauberian theorems. Using the sequence $\left(\omega_n^{(r)}(s)\right)$ defined recursively instead of $\left(\omega_n^{(0)}(s)\right)$, Sezer and Çanak [19] generated some Tauberian results based on this sequence. These investigations not only reaffirm the standing of logarithmic summability in classical mathematical analysis but also emphasize its applicability to current topics through statistical extensions and the development of new Tauberian conditions.

In the present paper, our aim is to expand a Tauberian theorem for the Cesàro method due to Maddox [23] and the weighted mean method due to Çanak [24] in ordered spaces to the $(\ell, 1, 1)$, summability method of double sequences. These researchers formulate the related results as follows, respectively:

Theorem 1. *([23])* Let $(X, \leq)$ be an ordered linear space over $\mathbb{R}$ and suppose that a sequence (s_n) is Cesàro summable to $\xi \in X$, relative to $\tau \in X$. If (s_n) is slowly decreasing, relative to $\tau \in X$, then (s_n) is convergent to ξ, relative to $\tau \in X$.

Theorem 2. *([24])* Let $(X, \leq)$ be an ordered linear space over $\mathbb{R}$ and let

$$lim\,inf_{m\to\infty} \frac{P_{\lambda m}}{P_m} > 1 \quad \text{for all} \quad \lambda > 1, \tag{3}$$

be satisfied. Suppose that a sequence (s_n) is summable by the weighted mean method to $\xi \in X$, relative to $\tau \in X$. If (s_n) is slowly decreasing, relative to $\tau \in X$, then (s_n) is convergent to ξ, relative to $\tau \in X$.

In case $p_n = 1/(n+1)$, the weighted mean method, or known as the $\left(\overline{N}, p\right)$ method, leads to the logarithmic method $(\ell, 1)$ for single sequences where $P_n \sim \log n$ for all $n \in N$.

Accordingly, we present several Tauberian conditions, addressing O_L-oscillatory behavior of a double sequence (s_{mn}) with respect to logarithmic summability in certain senses, from the $(\ell, 1, 1)$, $(\ell, 1, *)$, and $(\ell, *, 1)$ summability to P-convergence in ordered spaces.

This article highlights key findings from the existing literature while proposing a new technique for research on the topic of logarithmic summability and its associated Tauberian theorems, as a consequently aims to synthesize these developments. By gathering the mentioned studies from historical and current perspectives, we seek to contribute to the ongoing process on investigating the relationships between logarithmic summability method and convergence.

2. Preliminaries

A double sequence $s = (s_{mn})$ is defined as a function from $\mathbb{N} \times \mathbb{N}$ into either the set of $\mathbb{R}$ or $\mathbb{C}$. The number s_{mn} represents the value of the function s at a point $(m, n) \in \mathbb{N} \times \mathbb{N}$ and is known as the (m, n)-term of the double sequence.

A double sequence $s = (s_{mn})$ is convergent in the sense of Pringsheim, or simply, P-convergent to ξ if for all $\varepsilon > 0$ there exists a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|s_{mn} - \xi| <$ whenever $m, n \geq n_0$ (see [25]). The number ξ is referred to as the P −limit of s, denoted by $P - \lim_{m,n\to\infty} s_{mn} = \xi$, where both m and n approach to ∞ independently.

A double sequence (s_{mn}) is considered bounded *(or one-sided bounded)* if there exists a positive constant M such that $|s_{mn}| \leq M$ *(or* $s_{mn} \geq -M$*)* for all non-negative values of m and n.
It is important to note that a double sequence (s_{mn}) may converge even if it is not a bounded function of m and n. In other words, P-convergence of (s_{mn}) does not necessarily imply that its terms are bounded, which contrasts with the behavior observed in single sequences. For example, the sequence (s_{mn}) defined by

$$s_{mn} = \begin{cases} 9^{n+1}, & if\ m = 3, n \in \mathbb{N}, \\ 9^{m+3}, & if\ n = 5, m \in \mathbb{N}, \\ 1, & otherwise \end{cases}$$

is P-convergent, but it is unbounded.

Notation 3. Let (s_{mn}) be a double sequence.

(i) The symbols $s_{mn} = O(1)$ and $s_{mn} = O_L(1)$ mean that $|s_{mn}| \leq H$ and $s_{mn} \geq M$ for some constants $H, M > 0$ and each $m, n \geq n_0$.

(ii) The symbol $s_{mn} = o(1)$ means that $s_{mn} \to 0$ $as\ m, n \to \infty$.

A double sequence (s_{mn}) is said to be the logarithmic summable *(of order* 1*)*, or briefly, $(\ell, 1, 1)$ summable to ξ if the double sequence (σ_{mn}^{11}) defined by

$$\sigma_{mn}^{11} := \frac{1}{\ell_m \ell_n} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{s_{ij}}{(i+1)(j+1)} \quad \text{where} \quad \ell_m = \sum_{i=0}^{m} \frac{1}{i+1} \sim \log m, \tag{4}$$

is P-convergent to ξ and it is denoted by $\mathrm{P} - \lim \sigma_{mn}^{11} = \xi$ or equivalently, $P - \lim s_{mn} = \xi\ (\ell, 1, 1)$. Similarly, $(\ell, 1, *)$ and $(\ell, *, 1)$ summable sequences are respectively defined via double sequences (σ_{mn}^{10}) and (σ_{mn}^{01}) as

$$\sigma_{mn}^{10} := \frac{1}{\ell_m} \sum_{i=0}^{m} \frac{s_{in}}{(i+1)} \quad \text{and} \quad \sigma_{mn}^{01} := \frac{1}{\ell_n} \sum_{j=0}^{n} \frac{s_{mj}}{(j+1)}$$

for all $m, n \in \mathbb{N}$. If a bounded double sequence is P-convergent to ξ, then it is also $(\ell, 1, 1)$ summable to the same value. However, the opposite of this implication is not generally true. The question comes to mind if there are certain conditions on the term s_{mn} under which its $(\ell, 1, 1)$ summability implies its P-convergence. The condition $T\{s_{mn}\}$ which makes such a situation possible is called a *Tauberian condition*. The resulting theorem, which states that P-convergence follows from its $(\ell, 1, 1)$ summability and $T\{s_{mn}\}$, is called a *Tauberian theorem* (see [26]).

For a double sequence (s_{mn}), we define
$$\Delta_{11}s_{mn} := \Delta_{10}\Delta_{01}s_{mn} = \Delta_{10}(\Delta_{01}s_{mn}) = \Delta_{01}(\Delta_{10}s_{mn})$$
$$= s_{mn} - s_{m,n-1} - s_{m-1,n} + s_{m-1,n-1},$$
$$\Delta_{10}s_{mn} := s_{mn} - s_{m-1,n},$$
$$\Delta_{01}s_{mn} := s_{mn} - s_{m,n-1}$$
for all $m, n \in \mathbb{N}$.

The double logarithmic Kronecker identity for a sequence (s_{mn}) are defined via $\left(V_{mn}^{11}(\Delta_{11}s)\right)$ as follows:
$$s_{mn} - \sigma_{mn}^{10}(s) - \sigma_{mn}^{01}(s) + \sigma_{mn}^{11}(s) = V_{mn}^{11}(\Delta_{11}s),$$
where
$$V_{mn}^{11}(\Delta_{11}s) := \frac{1}{\ell_m\ell_n}\sum_{i=1}^{m}\sum_{j=1}^{n}\ell_{i-1}\ell_{j-1}\,\Delta_{11}s_{ij}$$
for all $m, n \in \mathbb{N}$ (see [27, 28]). The double sequence $\left(V_{mn}^{11}(\Delta_{11}s)\right)$ is the $(\ell, 1,1)$ mean of $\left((m+1)(n+1)\ell_{m-1}\ell_{n-1}\Delta_{11}s_{mn}\right)$ and it is called the logarithmic generator sequence of (s_{mn}) in the sense $(1,1)$.

Throughout this work, we assume an ordered linear space $(X, \leq)$ over $\mathbb{R}$ denoting the zero element by o and a given non-negative element by τ. We also consider that (s_{mn}) is an element of double sequences class in X.

Now, we give concepts of P-convergence and slow decrease with respect to logarithmic summability in certain senses for double sequences in X. In the literature, the term "slow decrease" was defined by Schmidt [29] in the case of the Cesàro summability of real sequences. Motivated by the definition of "slow decrease" with respect to Cesàro summability, Móricz [10] introduced the concept of slow decrease of a real sequence with respect to summability $(\ell, 1)$.

A double sequence (s_{mn}) in X is said to be P-convergent to $\xi \in X$, relative to $\tau \in X$, if for all $\varepsilon > o$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that
$$-\varepsilon\tau \leq s_{mn} - \xi \leq \varepsilon\tau \quad \text{whenever} \quad m, n > n_0.$$

A double sequence (s_{mn}) in X is said to be slowly decreasing with respect to logarithmic summability in sense $(1,1)$, relative to $\tau \in X$, if for all $\varepsilon > o$ there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$, and $\lambda = \lambda(\varepsilon) > 1$ such that
$$s_{ij} - s_{in} - s_{mj} + s_{mn} \geq -\varepsilon\tau \quad \text{whenever} \quad n_0 \leq m < i \leq m^{\lambda} \text{ and } n_0 \leq n < j \leq n^{\lambda},$$
and slowly decreasing with respect to logarithmic summability in sense $(1,0)$, relative to $\tau \in X$, if
$$s_{in} - s_{mn} \geq -\varepsilon\tau \quad \text{whenever} \quad n_0 \leq m < i \leq m^{\lambda} \text{ and } n_0 \leq n,$$
and slowly decreasing with respect to logarithmic summability in sense $(0,1)$, relative to $\tau \in X$, if
$$s_{mj} - s_{mn} \geq -\varepsilon\tau \quad \text{whenever} \quad n_0 \leq m \text{ and } n_0 \leq n < j \leq n^{\lambda}.$$
Notice that when X is the real linear space $\mathbb{R}$ with its usual order, relative to 1, then these definitions reduce to the classical definitions of P-convergence and slow decrease of double sequences with respect to logarithmic summability in certain senses.

In a general ordered linear space $(X,\leq)$, we consider a given series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}$ with its double sequence of partial sums (s_{mn}). The logarithmic means of (s_{mn}) are defined by (4).

3. Main results

In this section, we formulate our main result for the $(\ell,1,1)$ summable double sequences in $(X,\leq)$ as follows:

Theorem 4. Let $(X,\leq)$ be an ordered linear space over the real numbers. Suppose that a sequence (s_{mn}) is $(\ell,1,1),(\ell,1,*)$ and $(\ell,*,1)$ summable to $\xi\in X$, relative to $\tau\in X$. If (s_{mn}) is slowly decreasing with respect to logarithmic summability in senses $(1,1),(1,0)$, and $(0,1)$, relative to $\tau\in X$, then (s_{mn}) is P-convergent to ξ, relative to $\tau\in X$.

Proof: Without loss of generality, we suppose that $\xi=o$. Otherwise, we consider the series

$$(a_{00}-\xi)+\sum_{m=1}^{\infty}a_{m0}+\sum_{n=1}^{\infty}a_{0n}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}.$$

Set the double sequence (t_{mn}^{11}) as

$$t_{mn}^{11}:=\sum_{i=1}^{m}\sum_{j=1}^{n}\ell_{i-1}\ell_{j-1}\,\Delta_{11}s_{ij}\tag{5}$$

for all $m,n\geq 1$. Since we have

$$\Delta_{11}\sigma_{mn}^{11}=\sigma_{mn}^{11}-\sigma_{m-1,n}^{11}-\sigma_{m,n-1}^{11}+\sigma_{m-1,n-1}^{11}=\frac{t_{mn}^{11}}{(m+1)\ell_m\ell_{m-1}(n+1)\ell_n\ell_{n-1}}\tag{6}$$

for $m,n\geq 1$, it follows from (6) that

$$\begin{aligned}\sigma_{[m^\lambda],[n^\lambda]}^{11}-\sigma_{[m^\lambda],n}^{11}-\sigma_{m,[n^\lambda]}^{11}+\sigma_{mn}^{11}&=\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\Delta_{11}\,\sigma_{ij}^{11}\\&=\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}t_{ij}^{11}\end{aligned}\tag{7}$$

for $\lambda>1$ where $[m^\lambda]$ denotes the integer part of m^λ. Let $\varepsilon>o$ be given. Define $\varepsilon'=\varepsilon/\zeta$, where $\zeta=[2\lambda^2/(\lambda-1)]^2$ for $\lambda>1$. It is known that (s_{mn}) is slowly decreasing with respect to logarithmic summability in senses $(1,1),(1,0)$, and $(0,1)$, relative to $\tau\in X$, there exist $n_1=n_1(\varepsilon'),n_2=n_2(\varepsilon')\in\mathbb{N}$ and $\lambda>1$ such that

$$s_{in}-s_{mn}\geq-\left(\frac{\varepsilon'}{40}\right)\tau\quad\text{whenever } n_1<m<i\leq m^\lambda \text{ and } n_1\leq n,$$

$$s_{mj}-s_{mn}\geq-\left(\frac{\varepsilon'}{40}\right)\tau\quad\text{whenever } n_2<n<j\leq n^\lambda \text{ and } n_2\leq m,$$

and additionally there exist $n_0=n_0(\varepsilon')=\min\{n_1,n_2\}\in\mathbb{N}$ and $\lambda>1$ such that

$$s_{ij}-s_{in}-s_{mj}+s_{mn}\geq-\left(\frac{\varepsilon'}{40}\right)\tau\text{ whenever } n_0<m<i\leq m^\lambda,\ n_0<n<j\leq n^\lambda.$$

Since (σ_{mn}^{11}) is P-convergent to o, relative to $\tau\in X$, it follows from (7) that

$$-\left(\frac{\varepsilon'}{40}\right)\tau \leq \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} t_{ij}^{11} \leq \left(\frac{\varepsilon'}{40}\right)\tau \tag{8}$$

for sufficiently large m, n. Define the double sequence (γ_{mn}) as

$$\gamma_{mn} := \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}$$

for sufficiently large m, n. Then, we obtain

$$\gamma_{mn} t_{mn}^{11} = \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} t_{mn}^{11}$$

$$= \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} t_{ij}^{11}$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{mj}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{in}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11}\right)$$

$$\leq \left(\frac{\varepsilon'}{20}\right)\tau - \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{mj}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{in}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[m^\lambda]} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} \left(t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11}\right) \tag{9}$$

for sufficiently large m, n. It is clear that

$$t_{mn}^{11} = \sum_{i=1}^{m} \sum_{j=1}^{n} \ell_{i-1}\, \ell_{j-1} \Delta_{11} s_{ij} = \sum_{i=1}^{m} \ell_{i-1} \sum_{j=1}^{n} \ell_{j-1}\, \Delta_{01}\left(\Delta_{10} s_{ij}\right)$$

$$= \sum_{i=1}^{m} \ell_{i-1} \left(\ell_n \Delta_{10} s_{in} - \sum_{j=0}^{n} \frac{1}{(j+1)} \Delta_{10}\, s_{ij} \right)$$

$$= \ell_n \sum_{i=1}^{m} \ell_{i-1}\, \Delta_{10} s_{in} - \sum_{j=0}^{n} \frac{1}{(j+1)} \sum_{i=1}^{m} \ell_{i-1}\, \Delta_{10} s_{ij}$$

$$= \ell_n\left(\ell_m s_{mn} - \sum_{i=0}^{m}\frac{1}{(i+1)}s_{in}\right) - \sum_{j=0}^{n}\frac{1}{(j+1)}\left(\ell_m s_{mj} - \sum_{i=0}^{m}\frac{1}{(i+1)}s_{ij}\right)$$

$$= \ell_m\ell_n s_{mn} - \ell_n\sum_{i=0}^{m}\frac{1}{(i+1)}s_{in} - \ell_m\sum_{j=0}^{n}\frac{1}{(j+1)}s_{mj} + \sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{(i+1)(j+1)}s_{ij} \quad (10)$$

for all $m, n \geq 1$. From this point of view, we find

$$t_{mj}^{11} - t_{mn}^{11} = \ell_m\ell_n\left(s_{mj} - s_{mn}\right) + \ell_m\sum_{k=n+1}^{j-1}\frac{1}{(k+1)}\left(s_{mj} - s_{mk}\right)$$
$$+\sum_{r=0}^{m}\sum_{k=n+1}^{j}\frac{1}{(r+1)(k+1)}\left(s_{rk} - s_{rn}\right) + \ell_m\ell_j\left(\sigma_{mn}^{10} - \sigma_{mj}^{10}\right), \quad (11)$$

$$t_{in}^{11} - t_{mn}^{11} = \ell_m\ell_n\left(s_{in} - s_{mn}\right) + \ell_n\sum_{r=m+1}^{i-1}\frac{1}{(r+1)}\left(s_{in} - s_{rn}\right)$$

$$+\sum_{k=0}^{n}\sum_{r=m+1}^{i}\frac{1}{(r+1)(k+1)}\left(s_{rk} - s_{mk}\right) + \ell_i\ell_n\left(\sigma_{mn}^{01} - \sigma_{in}^{01}\right), \quad (12)$$

and

$$t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11} = \ell_m\ell_n\left(s_{ij} - s_{in} - s_{mj} + s_{mn}\right)$$
$$+\sum_{r=m+1}^{i-1}\sum_{k=n+1}^{j-1}\frac{1}{(r+1)(k+1)}\left(s_{ij} - s_{rj} - s_{ik} + s_{rk}\right)$$
$$+\ell_n\sum_{r=m+1}^{i-1}\frac{1}{(r+1)}\left(s_{ij} - s_{rj} - s_{in} + s_{rn}\right)$$
$$+\ell_m\sum_{k=n+1}^{j-1}\frac{1}{(k+1)}\left(s_{ij} - s_{ik} - s_{mj} + s_{mk}\right) \quad (13)$$

for sufficiently large m, n. From slow decrease with respect to logarithmic summability of (s_{mn}) in senses $(1,1)$, $(1,0)$, and $(0,1)$, relative to $\tau \in X$, we have

$s_{in} - s_{\mu n} \geq -\left(\frac{\varepsilon'}{40}\right)\tau$ whenever $n_1 < m < i \leq m^{\lambda}$, $m < \mu < i$ and $n_1 \leq n$,

$s_{mj} - s_{m\nu} \geq -\left(\frac{\varepsilon'}{40}\right)\tau$ whenever $n_2 < n < j \leq n^{\lambda}$, $n < \nu < j$ and $n_2 \leq m$,

$s_{ij} - s_{i\nu} - s_{\mu j} + s_{\mu\nu} \geq -\left(\frac{\varepsilon'}{40}\right)\tau$ whenever $n_0 < m < i \leq m^{\lambda}$, $m < \mu < i$ and $n_0 < n < j \leq n^{\lambda}$, $n < \nu < j$.

Since (s_{mn}) is $(\ell, 1, *)$ and $(\ell, *, 1)$ summable to $o \in X$, relative to $\tau \in X$, the differences $\left(\sigma_{mn}^{10} - \sigma_{mj}^{10}\right)$ and $\left(\sigma_{mn}^{01} - \sigma_{in}^{01}\right)$ are P-convergent to o, relative to $\tau \in X$, as $m, n \to \infty$. Hence, considering these situations, we attain from (11)-(13) that

$$t_{mj}^{11}-t_{mn}^{11}\geq -\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_n-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_{j-1}-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_j-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_j$$

$$\geq -\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_n-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_j-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_j-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_j$$

$$=-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_n-2\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_j-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_j$$

$$=-\left(\frac{\varepsilon'}{40}\right)\tau(3\ell_m\ell_j-\ell_m\ell_n), \tag{14}$$

$$t_{in}^{11}-t_{mn}^{11}\geq -\left(\frac{\varepsilon'}{40}\right)\tau(3\ell_i\ell_n-\ell_m\ell_n) \tag{15}$$

$$t_{ij}^{11}-t_{in}^{11}-t_{mj}^{11}+t_{mn}^{11}\geq -\left(\frac{\varepsilon'}{40}\right)\tau\ell_m\ell_n-\left(\frac{\varepsilon'}{40}\right)\tau(\ell_{i-1}-\ell_m)(\ell_{j-1}-\ell_n)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_n(\ell_{i-1}-\ell_m)-\left(\frac{\varepsilon'}{40}\right)\tau\ell_m(\ell_{j-1}-\ell_n)$$
$$=-\left(\frac{\varepsilon'}{40}\right)\tau\ell_{i-1}\ell_{j-1}, \tag{16}$$

respectively, and by (14)-(16)

$$-\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}(t_{mj}^{11}-t_{mn}^{11})$$
$$\leq\left(\frac{\varepsilon'}{40}\right)\tau\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{3\ell_m}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_{j-1}}$$
$$-\left(\frac{\varepsilon'}{40}\right)\tau\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{\ell_m\ell_n}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}$$
$$\leq\left(\frac{\varepsilon'}{40}\right)\tau\left(3\left(\frac{\ell_{[m^\lambda]}}{\ell_m}-1\right)\left(\frac{\ell_{[n^\lambda]}}{\ell_n}-1\right)-\frac{\ell_m\ell_n\left(\ell_{[m^\lambda]}-\ell_m\right)\left(\ell_{[n^\lambda]}-\ell_n\right)}{\ell_{[m^\lambda]}\ell_{[m^\lambda]-1}\ell_{[n^\lambda]}\ell_{[n^\lambda]-1}}\right)$$
$$\leq\left(\frac{\varepsilon'}{40}\right)3\tau\left(\frac{\ell_{[m^\lambda]}}{\ell_m}-1\right)\left(\frac{\ell_{[n^\lambda]}}{\ell_n}-1\right), \tag{17}$$

$$-\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}(t_{in}^{11}-t_{mn}^{11})$$
$$\leq\left(\frac{\varepsilon'}{40}\right)3\tau\left(\frac{\ell_{[m^\lambda]}}{\ell_m}-1\right)\left(\frac{\ell_{[n^\lambda]}}{\ell_n}-1\right), \tag{18}$$

and

$$-\sum_{i=m+1}^{[m^\lambda]}\sum_{j=n+1}^{[n^\lambda]}\frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}}(t_{ij}^{11}-t_{in}^{11}-t_{mj}^{11}+t_{mn}^{11})$$

$$\leq \left(\frac{\varepsilon'}{40}\right)\tau \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i (j+1)\ell_j}$$

$$\leq \left(\frac{\varepsilon'}{40}\right)\tau \left(\frac{\ell_{[m^\lambda]}}{\ell_m} - 1\right)\left(\frac{\ell_{[n^\lambda]}}{\ell_n} - 1\right), \tag{19}$$

for sufficiently large m, n, respectively. From (9) together with (17)-(19), we get

$$\gamma_{mn} t_{mn}^{11} \leq \left(\frac{\varepsilon'}{40}\right)\tau - \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i \ell_{i-1} (j+1)\ell_j \ell_{j-1}} \left(t_{mj}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i \ell_{i-1} (j+1)\ell_j \ell_{j-1}} \left(t_{in}^{11} - t_{mn}^{11}\right)$$

$$- \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i \ell_{i-1} (j+1)\ell_j \ell_{j-1}} \left(t_{ij}^{11} - t_{in}^{11} - t_{mj}^{11} + t_{mn}^{11}\right)$$

$$\leq \left(\frac{\varepsilon'}{40}\right)\tau + \left(\frac{\varepsilon'}{40}\right) 7\tau \left(\frac{\ell_{[m^\lambda]}}{\ell_m} - 1\right)\left(\frac{\ell_{[n^\lambda]}}{\ell_n} - 1\right)$$

$$\leq \left(\frac{\varepsilon'}{40}\right)\tau + \left(\frac{\varepsilon'}{40}\right) 28\tau(\lambda - 1)^2$$

$$\leq \frac{3\varepsilon'\tau(\lambda - 1)^2}{4} \tag{20}$$

for sufficiently large m, n. When we simplify γ_{mn}, we obtain that

$$\gamma_{mn} = \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \frac{1}{(i+1)\ell_i \ell_{i-1} (j+1)\ell_j \ell_{j-1}} = \sum_{i=m+1}^{[m^\lambda]} \sum_{j=n+1}^{[n^\lambda]} \left(\frac{1}{\ell_{i-1}} - \frac{1}{\ell_i}\right)\left(\frac{1}{\ell_{j-1}} - \frac{1}{\ell_j}\right)$$

$$= \left(\frac{1}{\ell_m} - \frac{1}{\ell_{[m^\lambda]}}\right)\left(\frac{1}{\ell_n} - \frac{1}{\ell_{[n^\lambda]}}\right)$$

$$= \left(\frac{\ell_{[m^\lambda]} - \ell_m}{\ell_m \ell_{[m^\lambda]}}\right)\left(\frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n \ell_{[n^\lambda]}}\right),$$

and so,

$$\gamma_{mn} \ell_m \ell_n = \left(1 - \frac{\ell_m}{\ell_{[m^\lambda]}}\right)\left(1 - \frac{\ell_n}{\ell_{[n^\lambda]}}\right) \to \left(\frac{\lambda - 1}{\lambda}\right)\left(\frac{\lambda - 1}{\lambda}\right) \quad \text{as } m, n \to \infty.$$

Therefore, we reach

$$\frac{t_{mn}^{11}}{\ell_m \ell_n} = \frac{\gamma_{mn} t_{mn}^{11}}{\ell_m \ell_n \gamma_{mn}} \leq \frac{3\varepsilon'\tau\lambda^2}{4}\left(\frac{2\lambda}{\lambda - 1}\right)^2 \leq \frac{3\varepsilon'\tau\zeta}{4} \leq \frac{3\varepsilon\tau}{4} \tag{21}$$

for sufficiently large m, n. If we consider the double weighted Kronecker identity for (s_{mn}), we find

$$s_{mn} = V_{mn}^{11}(\Delta_{11} s) + \sigma_{mn}^{10} + \sigma_{mn}^{01} - \sigma_{mn}^{11} = \frac{t_{mn}^{11}}{\ell_m \ell_n} + \sigma_{mn}^{10} + \sigma_{mn}^{01} - \sigma_{mn}^{11} \tag{22}$$

for sufficiently large m, n. Since (s_{mn}) is $(\ell, 1,1)$, $(\ell, 1,*)$ and $(\ell,*,1)$ summable to $o \in X$, relative to $\tau \in X$, the sequences $(\sigma_{mn}^{11}), (\sigma_{mn}^{10})$, and (σ_{mn}^{01}) are P-convergent to o, relative to $\tau \in X$. As a result, we conclude by (21) and (22) that

$$s_{mn} = \frac{t_{mn}^{11}}{\ell_m \ell_n} + \sigma_{mn}^{10} + \sigma_{mn}^{01} - \sigma_{mn}^{11} \leq \frac{3\varepsilon\tau}{4} + \frac{\varepsilon\tau}{12} + \frac{\varepsilon\tau}{12} + \frac{\varepsilon\tau}{12} = \varepsilon\tau$$

for sufficiently large m, n. To indicate that $s_{mn} \geq -\varepsilon$ ultimately in m, n, we consider

$$\sum_{i=[m^\lambda]}^{m} \sum_{j=[n^\lambda]}^{n} \frac{1}{(i+1)\ell_i\ell_{i-1}(j+1)\ell_j\ell_{j-1}} t_{ij}^{11}$$

for all $m, n \geq 1$. As a result of the calculations made in parallel with that made in the first part of the proof, we complete the second part of the proof and we find $s_{mn} \geq -\varepsilon$ for sufficiently large m, n. Therefore, (s_{mn}) is P-convergent to ξ, relative to $\tau \in X$.

4. Conclusion

In this paper, we extended a Tauberian theorem for the Cesàro summability method due to Maddox [23] and the weighted mean summability method due to Çanak [24] in ordered spaces to the $(\ell, 1,1)$, summability method of double sequences. In an ordered linear space $(X, \leq)$ over the real numbers, we proved that if a double sequence (s_{mn}) is $(\ell, 1,1), (\ell, 1,*)$ and $(\ell,*,1)$ summable to $\xi \in X$, relative to a $\tau \in X$ and slowly decreasing with respect to logarithmic summability in senses $(1,1), (1,0)$, and $(0,1)$, relative to $\tau \in X$, then it is P-convergent to ξ, relative to $\tau \in X$.

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