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Research Article

A Note on k-Secant Lines of the Klein Cubic Threefold in $PG(4,3)$

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Abstract

In this paper, we developed an algorithm to classify and construct k-secant lines for the Klein cubic threefold $\mathcal{F}$ in the projective space $PG(4,3)$. The algorithm identifies secant lines based on their intersections with $\mathcal{F}$, revealing distinct categories of non-secant and k-secant lines. Additionally, we partitioned the point set of $\mathcal{F}$ into three subsets, uncovering geometric configurations and forming 5-gons related to perspectivity. This analysis provides new insights into the structure of the Klein cubic threefold and its geometric properties.

Keywords: Klein cubic three*f*old, Projective space, Galois field, k-secant lines.

$PG(4,3)$ deki Klein Kübiğin k-kesen Doğruları Üzerine Bir Not

Öz

Bu çalışmada, $PG(4,3)$ projektif uzayındaki Klein kübik $\mathcal{F}$ için k-sekant doğruları belirleyen ve sınıflandıran bir algoritma geliştirdik. Algoritma, $\mathcal{F}$ ile kesişimlerine dayanarak sekant doğrularını belirleyerek, kesen olmayan ve k-kesen doğruların farklı sınıflarını ortaya çıkarmaktadır. Ayrıca, $\mathcal{F}$ nin nokta kümesini üç alt kümeye ayırarak belirli geometrik konfigürasyonları inceledik ve perspektiflikle ilişkili beşgenler (5-gonlar) oluşturduk. Bu analiz, Klein kübik yapısına ve geometrik özelliklerine dair yeni bakış açıları sunmaktadır.

Anahtar Kelimeler: Klein kübik, Projektif uzay, Galois cismi, k-kesen doğrular.

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1. Introduction

The study of embedded structures in projective spaces has significantly advanced our understanding of geometric relationships and algebraic varieties, offering novel insights into higher-dimensional spaces [1-4, 7]. One important concept in projective geometry is that of k-secant lines, which intersect a given surface at precisely k distinct points. One of the earliest and most notable examples is the non-singular Klein cubic threefold, initially studied by Felix Klein in 1879 in the 4-dimensional projective space [11]. This surface, denoted by $\mathcal{F}$, is embedded in the 4-dimensional projective space. Its geometric properties, including the classification of its lines, remain a central topic of modern research. Over finite fields, the classification of non-singular cubic surfaces continues to be an active area of investigation. For example, it has been shown that a non-singular cubic surface over the finite field *GF(2)* can contain 15, 9, 5, 3, 2, 1, or 0 lines [9, 10]. Rosati further demonstrated that when the field size q is odd, the number of lines can be one of 27, 15, 9, 7, 5, 3, 2, 1, or 0 [12]. Hirschfeld's foundational work in the 1960s introduced a program to classify cubic surfaces with 27 lines over finite fields [8], and his research on classical configurations such as the double-six has advanced our understanding of these structures [5,9]. This line of research remains important, with algorithms being developed to classify surfaces with 27 lines using the classical theory of double-sixes [6].

The Klein cubic defined by $x^2y + y^2z + z^2v + v^2w + w^2x = 0$ over $\mathbb{C}$ contains neither Eckardt points nor triple lines [13]. But, over $q = 2$, the Klein cubic threefold is a quadric because $x^2 = x$ for each element of $GF(2)$. This quadric has a nucleus, six spreads with five lines, and each point is an Eckardt point [3]. So, we need to investigate these properties over arbitrary fields. For $q = 3$, the aim of the article is to investigate the Klein cubic threefold is structural and geometric properties when extended to a field with three elements. We introduce an algorithm to classify all k-secant lines of the Klein cubic threefold $\mathcal{F}$ in $PG(4,3)$. By applying this algorithm, we identify and classify lines based on their intersections with $\mathcal{F}$ at 0, 1, 2, 3, or 4 points. Additionally, we partition the points of $\mathcal{F}$ into three disjoint subsets, each exhibiting unique secant configurations. This analysis of k-secant lines and the formation of three distinct 5-gons, based on the points of $\mathcal{F}$ and their perspectivity properties, provided deeper insights into the geometric structure of the Klein cubic threefold.

2. Preliminaries

Let $GF(q)$ denote Galois field of order $q = p^k$, where p is a prime. If any $(n + 1)$-dimensional vector space V, the n-dimensional projective space $PG(4,q)$ over $GF(q)$ is the set of all subspaces of V distinct from the trivial subspaces. The 1-dimensional subspaces are called the points of $PG(n,q)$, the 2-dimensional subspaces are called the (projective) lines and the 3-dimensional ones are called (projective) planes. We note that by going from a vector space to the associated projective space, the dimension decreases by one. Hence, an $(n+1)$-dimensional vector space V gives rise to an n-dimensional projective space $PG(4,q)$. The points in projective space $PG(4,q)$ are defined by equivalence classes of non-zero vectors in the vector space *V*. For example, for 4-dimensional vector space, the associated 3-dimensional projective is defined where points are represented by equivalence classes of vectors (v, w, x, y, z) reducing the dimension by one unit.

A point in projective space $PG(n,q)$ is defined by an equivalence class of non-zero vectors in the vector space *V*. In projective geometry, two non-zero vectors v and w in V are considered equivalent if they differ by a scalar multiple:

$$v \sim w \Leftrightarrow w = \lambda v,\ \lambda \in GF(q)^*.$$

Here, $GF(q)^*$ denotes the set of non-zero elements in $GF(q)$. For example, for $4-$ dimensional vector space, the associated $3-$ dimensional projective is defined where points are represented by equivalence classes of vectors (w, x, y, z), reducing the dimension by one unit.

The $4-$dimensional projective space $PG(4,q)$ over $GF(q)$ contains $q^4+q^3+q^2+q+1$ points. The total number of lines and the number of lines passing through a given point in $PG(4,q)$ are given by the formulas $\frac{(q^5-1).(q^4-1)}{(q^2-1).(q-1)}$ and $\frac{q^4-1}{q-1}$, respectively. Thus, $PG(4,3)$ has 121 points and 1210 lines. In $PG(4,3)$, every line contains 4 points, and 40 lines pass through each point.

We use the combinatorial definition, where a line is considered as a subsets of points in $PG(4,3)$. A line passing through the points $P(a_0,a_1,a_2,a_3,a_4)$ and $Q(b_0,b_1,b_{2,}b_3,b_4)$ in $PG(4,3)$, where a_i, $b_i \in GF(3)$, $i = 0,1,2,3,4$, is denoted by:

$$\langle P,Q \rangle = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}.$$

Definition 1 The Klein cubic threefold $\mathcal{F}$ is a projective variety given by the equation:

$$\mathcal{F}:\quad x^2y + y^2z + z^2v + v^2w + w^2x = 0 \qquad (1)$$

where x, y, z, v and w are the coordinates of a point (v, w, x, y, z) in 4-dimensional projective space. The Klein cubic threefold $\mathcal{F}$ over the field $GF(q)$ can be identified with a set *S* of the points $PG(4,q)$ that satisfies the equation (1). In particular, for $q = 3$, the set *S* consists of 40 point.

3. A Computational Strategy for *k*-Secant Lines of $\mathcal{F}$

In projective space, geometric structures are defined by the relationships a points, lines, and planes, extending beyond the limitations of Euclidean geometry. One intriguing aspect of projective geometry involves the study of *k*-secant lines, which are lines that intersect a given geometric object, such as a curve or surface, at exactly *k* distinct points. The classification and enumeration of these *k*-secant lines offer deep insights into the underlying geometry of the object. Understanding these relationships is crucial in various fields, including algebraic geometry, where the concept of *k*-secant lines plays a significant role in exploring the configuration and distribution of points within the projective space.

In this section, we focus on classifying the *k*-secant lines that intersect a given cubic threefold $\mathcal{F}$ in $PG(4,3)$. By systematically analyzing these lines, we can uncover patterns and properties that reveal the complexity of the surface's geometry. This classification not only enhances our understanding of the structure of the surface but also contributes to broader applications in algebraic geometry, where the arrangement of *k*-secant lines is instrumental in exploring the deeper aspects of geometric configurations in the projective space. So, we illustrate the algorithm that used to construct *k*-secants of $\mathcal{F}$ in $PG(4,3)$.

Algorithm: Finding all *k*- secants of $\mathcal{F}$

The goal of these algorithms is to identify and classify all *k*-secant lines of the Klein cubic threefold $\mathcal{F}$ in $PG(4,3)$. A k -secant line intersects $\mathcal{F}$ at exactly *k* points. The strategy involves constructing lines using points in $PG(4,3)$, tracking their intersections with $\mathcal{F}$, and ensuring each line is unique. This systematic approach helps reveal the geometric structure of $\mathcal{F}$ and its intersection properties.

Step(1): Finding the points and lines of $PG(4,3)$ by the formula defining in equation (1). In this case, we get 121 points and 1210 lines.

Step(2): Finding the points and lines of $\mathcal{F}$ by the formula defining in equation (1). In this case, we get 40 points as shown the set *S*.

Algorithm for Identifying all *k*-secants of $\mathcal{F}$ in projective space $PG(4,3)$.

1. Initialization

Input Sets:

P: A set of points in the projective geometry $PG(4,3)$.

S: A subset of points from *P*.

2. Define Auxiliary Functions

Galois Field Arithmetic:

gf3_add(v_1, v_2). Adds two vectors v_1 and v_2 component-wise under $GF(3)$ arithmetic (i.e., modulo 3).

gf3_mul(scalar, *v*): Multiplies a vector *v* by a scalar under $GF(3)$ arithmetic.

Find Line Function:

find_line (P, P_1, P_2) . Given two points P_1 and P_2, this function computes two more points P_3 and P_4 on the line defined by P_1 and P_2 using the addition and multiplication rules of $GF(3)$. The line $\{P_1, P_2, P_3, P_4\}$ is then returned.

3. Main Function to Find 40 Lines

Step 1: Initialize an empty list lines to store the 40 lines.

Step 2: Initialize a set used points to track the points that have been used, starting with P_1.

Step 3: Create the first line l_1.

Select the first point P_2 from *P* that is not in used points.

Use find line (P, P_1, P_2) to generate the line l_1.

Add l_1 to the lines list and update used points with all points in l_1.

Step 4: Create the next 38 lines (l_2 to l_{39}):

For each line, select the next point P_n from *P* that is not in the set of points used.

Use find line (P, P_1, P_n) to generate the line l_n.

Add l_n to the list of lines and update the set of points used with all points in l_n.

Step 5: Create the final line l_{40}:

Identify the remaining points in *P* that are not in the set of points used.
If there are two or more remaining points, form the line l_{40} using P_1 and the first three remaining points.

Add l_{40} to the list of lines.

4. Iterating Over Subset *S*

Step 6: Initialize an empty set unique lines to store unique lines as frozen sets.

Step 7: Iterate over each point in subset *S*:

For each point P_1 in *S*, call find 40 lines (P, P_1) to generate 40 lines.

For each line, check if there are exactly 0, 1, 2, or 3 points missing in subset *S*.

If such a line is found, convert the line to a frozen set and add it to unique lines if it is not already present.

5. Output

Step 8: Print all unique lines stored in unique lines.

This algorithm methodically identifies and records lines in the projective geometry $PG(4,3)$ that have exactly 0, 1, 2, or 3 missing points from a given subset *S*, ensuring that each line is unique.

Theorem 2 Let $\mathcal{F}$ be the Klein cubic threefold with exactly 40 points. For a given surface $\mathcal{F}$ in the projective space $PG(4,3)$, there are 240 lines that are 0-secant (not intersecting the surface), 480 lines that are 1-secant (intersecting the surface at a single point), 360 lines that are 2-secant (intersecting the surface at exactly two distinct points), 120 lines that are 3-secant (intersecting the surface at exactly three distinct points), and 10 lines that are 4-secant (intersecting the surface at exactly four distinct points).

Proof. Using the algorithm for identifying all k-secants of $\mathcal{F}$ in projective space $PG(4,3)$, we observe the following: There are exactly 10 lines that are 4-secant, meaning they intersect $\mathcal{F}$ at four points, with no missing points from a given subset *S*. There are exactly 120 lines that are 3-secant, intersecting $\mathcal{F}$ at three points, with one missing point from a given subset *S*. Furthermore, there are exactly 360 lines that are 2-secant, intersecting $\mathcal{F}$ at two points, with two missing points from *S*. There are also 480 lines that are 1-secant, intersecting $\mathcal{F}$ at one point, with three missing points from a given subset *S*. Therefore, the number of lines in $PG(4,3)$ that do not intersect $\mathcal{F}$ (0-secant) is found to be 240.

4. Disjoint Subsets and Secant Line Configurations in Klein Cubic Threefold $\mathcal{F}$

To reveal the distinguishing features of $\mathcal{F}$, we write the set *S* of points of $\mathcal{F}$ as the union of three disjoint subsets S_1, S_2 and S_3 such that

$$\begin{aligned} S_1 = \{&(1,0,0,0,0),\ (0,1,0,0,0), (0,0,1,0,0),\ (0,0,0,1,0),\ (0,0,0,0,1),\\ &(0,0,1,0,2),\ (0,1,0,0,1),\ (0,1,0,0,2),\ (0,1,0,1,0),\ (0,1,0,2,0),\\ &(1,0,1,0,0),\ (1,0,2,0,0),\ (1,0,0,1,0),\ (1,0,0,2,0),\ (0,0,1,0,1)\},\end{aligned}$$

$$S_2 = \{\ A_1 = (0,1,1,1,1),\ A_2 = (1,0,1,1,1),\ A_3 = (1,1,0,1,1),\ A_4 = (1,1,1,0,1), A_5 = (1,1,1,1,0)\}$$

and $S_3 = S \setminus (S_1 \cup S_2)$. Let the subset E of S_1 be the set of points E_i such that $E_i = (e_1, e_2, e_3, e_4, e_5)$, where $e_j = 1$ for $i = j$, otherwise $e_j = 0,\ i, j \in \{1,2,3,4,5\}$. Also, the points of S_3 that lie on the lines spanned by E_i and A_i , $i = 1,2,...,5$ be C_i such that $C_1 = (1,2,2,2,2)$, $C_2 = (1,2,1,1,1)$, $C_3 = (1,1,2,1,1)$, $C_4 = (1,1,1,2,1)$, and $C_5 = (1,1,1,1,2)$. Let's denote the subset of S_3 consisting of these points by C.

The following theorem gives the *k*-secant line numbers of $\mathcal{F}$ for any point selected in the sets S_1, S_2 and S_3.

Theorem 3 Let $\mathcal{F}$ be the Klein cubic threefold in $PG(4,3)$.

(1) Two of the lines passing through any point of the subset *E* of S_1 intersect $\mathcal{F}$ at four points; six of these lines intersect $\mathcal{F}$ at three points; twenty-one lines of these lines intersect $\mathcal{F}$ at two points, and eleven of these lines intersect $\mathcal{F}$ at one point.

(2) None of the lines passing through any point A_j, $j = 1,2,...,5$ in the set S_2 intersect $\mathcal{F}$ at four points; twelve lines of these lines intersect $\mathcal{F}$ at three points; fifteen lines of these lines intersect $\mathcal{F}$ at two points, and thirteen lines of these lines intersect with $\mathcal{F}$ at one point.

(3) One of the lines passing through any point in the set S_3 or $S_1 \setminus E$ intersects $\mathcal{F}$ at four points; nine of these lines intersect $\mathcal{F}$ at three points; eighteen of these lines intersect $\mathcal{F}$ at two points, and nine of these lines intersect $\mathcal{F}$ at one point.

Proof. We consider an algorithm for identifying all *k*-secants of $\mathcal{F}$ for each point P_1 in *S*, in projective space $PG(4,3)$. If this algorithm is applied to any point E_i in the set *E* instead of P_1, the *k*-secants of $\mathcal{F}$ passing through each E_i point are found, and the *k*-secant numbers passing through the E_i points, as stated in the theorem, is obtained. Similarly, if this algorithm is applied to any point in the set S_2, or S_3, or $S_1 \setminus \mathrm{E}$ instead of P_1, the number *k*-secants passing through these points, as described in the theorem, is also obtained.

Theorem 4 Consider the sets *E*, S_2 and *C* in $\mathcal{F}$. The lines spanned by the pairs of points (E_i, E_{i+1}), (A_i, A_{i+1}), and (C_i, C_{i+1}), where $i = 1,2,...,5$ (mod5), are the 2-secants of the Klein cubic threefold $\mathcal{F}$. Furthermore, if the line spanned by two points E_i and E_j, where *i* and *j* are non-consecutive indices, is 4-secant of $\mathcal{F}$, then the lines spanned by A_i and A_j, or C_i and C_j, for nonconsecutive indices *i* and *j,* are 3-secant of $\mathcal{F}$.

Proof. Let *l* be the line by E_i and E_{i+1}, or A_i and A_{i+1}, or C_i and C_{i+1}, where $i = 1,2,...,5$ (mod5) in $PG(4,3)$. Since the other two points on *l* do not satisfy the equation (1), the line *l* is 2-secant of $\mathcal{F}$. The other two points on the line spanned by E_1 and E_3 are (1, 0, 1, 0) and (1, 0, 2, 0, 0), on the line spanned by E_1 and E_4 are (1, 0, 0, 1, 0) and (1, 0, 0, 2, 0); on the line spanned by E_2 and E_4 are (0, 1, 0, 1, 0) and (0, 1, 0, 2, 0), on the line spanned by E_2 and E_5 are (0, 1, 0, 0, 1) and (0, 1, 0, 0, 2), and on the line spanned by E_3 and E_5 are (0, 0, 1, 0, 1) and (0, 0, 1, 0, 2). Since all these points are in *S*, any line spanned by two points on S_1 with non-consecutive indices forms a 4-secant of $\mathcal{F}$. Similarly, Since there is only one point on the line spanned by A_i and A_j, or C_i and C_j, where *i* and *j* are non-consecutive indices, this line is 3-secant of $\mathcal{F}$: The proof for S_2 is done similarly.

Theorem 5 In the projective space $PG(4,3)$, in the set $\mathcal{F}$, the sets E, S_2, and C determine a 5-gon, while the set S_3 forms a spread with five lines.

Proof. Since the sets E, S_2, and C consist of five points no three of which are collinear, these sets determine a 5-gon. Moreover, since all the points of S_3 lie on the non-intersecting 4-secant lines l_i of $\mathcal{F}$, where $i = 1,2,...,5$ and the lines are given by:

l_1 = {(1, 1, 2, 0, 0), (0, 0, 1, 2, 1), (1, 1, 1, 1, 2), (1, 1, 0, 2, 1)}

l_2 = {(1, 2, 1, 1 1), (1, 2, 2, 2, 0), (1, 2, 0, 0, 2), (0, 0, 1, 1, 2)}

l_3 = {(0, 1, 1, 2, 0), (1, 1, 1, 0, 2), (1, 2, 2, 2, 2), (1, 0, 0, 1, 2)}

l_4 = {(0, 1, 2, 2, 2), (1, 0, 0, 2, 2), (1, 1, 2, 1, 1), (1, 2, 1, 0, 0)}

l_5 = {(1, 0, 2, 1, 1), (1, 2, 0, 0, 1), (1, 1, 1, 2, 1), (0, 1, 2, 1, 0)}

the set S_3 forms a 5-linear spread.

5. Corollary In the context of the theorem, the three distinct 5-gons determined by the points of $\mathcal{F}$ are perspective from a point not contained in $\mathcal{F}$. Moreover, the intersection points of the opposite sides of these 5-gons form a 5-gon that lies outside of $\mathcal{F}$.

Proof. Since the intersection point of the lines connecting the opposite corners of three different 5-gons is (1, 1, 1, 1, 1), this point is the perspectivity center. Additionally, the intersection points of the opposite sides of these pentagons,

$(1,2,0,0,0), (0,1,2,0,0), (0,0,1,2,0), (0,0,1,2,0), (0,0,0,1,2)$, and $(1,0,0,0,2)$

form a 5-gon that lies outside of $\mathcal{F}$.

6. Conclusion

We developed and implemented an algorithm to classify and construct *k*-secant lines for the Klein cubic threefold $\mathcal{F}$ in the projective space $PG(4,3)$. Our algorithm successfully identified and categorized secant lines based on their intersection properties with $\mathcal{F}$. We determined that there are 240 lines that do not intersect $\mathcal{F}$ (0-secant), 480 lines that intersect at a single point (1-secant), 360 lines that intersect at two points (2-secant), 120 lines that intersect at three points (3-secant), and 10 lines that intersect at four points ur points (4-secant).

Additionally, we divided the point set of $\mathcal{F}$ into three distinct subsets, S_1, S_2, and S_3, revealing their specific geometric configurations. This analysis of *k*-secant lines and the formation of three distinct 5-gons, based on the points of $\mathcal{F}$ and their perspectivity properties, provided deeper insights into the geometric structure of the Klein cubic threefold. Our findings contribute to the understanding of projective and algebraic geometry, offering a framework for classifying *k*-secant lines in cubic threefolds. The algorithm developed here can be used as a tool for future studies on higher-dimensional geometric objects in projective spaces.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

All authors have investigated and studied no the published version of the manuscript.

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