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ON SOME ASYMPTOTIC EIGENVALUES OF HILL' S EQUATION

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Abstract. In this paper, we deal with Hill' s equation with symmetric single well potential. We find the lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of Hill' s equation. We also calculate the eigenvalues of Hill' s equation with two mixed problems, asymptotically.

1. Introduction

We consider the following differential equation

$$y''(t)+[\lambda-q(t)]\,y(t)=0 \tag{1.1}$$

where λ is a real parameter and $q(t)$ is a real-valued, continuous and periodic function with period a. We also accept that $q(t)$ is a symmetric single well potential with mean value zero. By a symmetric single well potential on $[0,a]$, we mean a continuous function $q(t)$ on $[0,a]$ which is symmetric about $t=\frac{a}{2}$ and non-increasing on $\left[0,\frac{a}{2}\right]$, so we can say that $q(t)=q(a-t)$ and $q'(t)$ exist because of monotony. In literature, a lot of researchers deal with this equation with various boundary conditions, various potentials and they find eigenfunctions, eigenvalues, the expression of Green' s function and instability intervals. Some of those are [1]-[14]. Here we calculate the lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of (1.1). We also obtain the eigenvalues of (1.1) with mixed problems.

Let us explain the these problems in the following section (More details can be seen in [11]):

2. Preliminaries

We begin with the general second-order equation

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$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{2.1}$$

in which the coefficients $a_r(x)$ are complex-valued, piecewise continuous, and periodic, all with the same period a. Thus

$$a_r(x+a) = a_r(x) \qquad (0 \leq r \leq 2)$$

where a is a non-zero real constant. It is also assumed that the left and right-hand limits of $a_0(x)$ at every point are non-zero, so that the usual theory of linear differential equations without singular points applies.

The name of Hill' s equation is given to the equation

$$\{P(x)y'(x)\}' + Q(x)y(x) = 0 \tag{2.2}$$

where $P(x)$ and $Q(x)$ are real-valued and have the same period a. In addition, it is assumed that $P(x)$ is continuous and nowhere zero and that $P'(x)$ and $Q(x)$ are piecewise continuous. Thus (2.2) is a particular case of (2.1) and it is named after G. W. Hill following his work on it 1877.

When we write $p(x)$ instead of $P(x)$ and $Q(x)$ involves a real parameter λ in the form

$$Q(x) = \lambda s(x) - q(x)$$

where $s(x)$ and $q(x)$ are piecewise continuous with period a and there is a constant $s > 0$ such that $s(x) \geq s$. (2.2) is now

$$\{p(x)y'(x)\}' + \{\lambda s(x) - q(x)\}\, y(x) = 0. \tag{2.3}$$

In order to indicate the depence on λ which occurs in (2.3), we write $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ for the solutions of (2.3) which satisfy the initial conditions

$$\phi_1(0,\lambda) = 1, \quad \phi_1'(0,\lambda) = 0; \qquad \phi_2(0,\lambda) = 0, \quad \phi_2'(0,\lambda) = 1.$$

Let us define the discriminant as

$$D(\lambda) := \phi_1(a,\lambda) + \phi_2'(a,\lambda). \tag{2.4}$$

Although the parameter λ is taken to be real here, it is sometimes necessary to allow it to be complex. Whether λ is real or complex, $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$, and their $x-$ derivatives are, for fixed x, analytic functions of λ. Hence, by (2.4), $D(\lambda)$ is an analytic function of λ. Since, in particular, $D(\lambda)$ is a continuous function of λ, the values of λ for which $|D(\lambda)| < 2$ for an open set on the real $\lambda-$axis. This set, which as we shall see is not empty, can be expressed as the union of a countable collection of disjoint open intervals. (2.3) is stable when λ lies in these intervals, and the intervals are therefore called the stability intervals of (2.3). Similarly, the intervals in which $|D(\lambda)| > 2$ are called the instability intervals of (2.3). Finally, the intervals formed by the closures of the stability intervals are, those in which $|D(\lambda)| \leq 2$the conditional stability intervals of are called (2.3). [11] establishes the existence of the stability and instability intervals and gives a precise description of them.

The periodic eigenvalue problem comprises (2.3), considered to hold in $[0, a]$, and the periodic boundary conditions

$$y(a) = y(0), \quad y'(a) = y'(0)$$

and the eigenvalues λ_n of this problem satisfy

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \text{and} \quad \lambda_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Also, λ_n are the zeros of the function $D(\lambda) - 2$ and that a given λ_n is a double eigenvalue if and only if

$$\phi_2(a, \lambda_n) = \phi_1'(a, \lambda_n) = 0.$$

The semi-periodic (or called as anti-periodic) eigenvalue problem comprises (2.3), considered to hold in $[0, a]$, and the semi-periodic boundary conditions

$$y(a) = -y(0), \quad y'(a) = -y'(0)$$

and the eigenvalues μ_n of this problem satisfy

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots, \text{and} \quad \mu_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty.$$

Also, μ_n are the zeros of the function $D(\lambda) + 2$ and that a given μ_n is a double eigenvalue if and only if

$$\phi_2(a, \mu_n) = \phi_1'(a, \mu_n) = 0.$$

We also know [11]

$$\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_3 \leq \cdots.$$

We denote also by Λ_n and ν_n respectively the eigenvalues in the two eigenvalue problems which comprise (2.3), considered to hold in $[0, a]$, and the two sets of boundary conditions

$$y(0) = y(a) = 0 \tag{2.5}$$

and

$$y'(0) = y'(a) = 0. \tag{2.6}$$

The equation (2.5) is named as Dirichlet condition, whereas (2.6) is named as Neumann condition. Also from [11], $n = 0, 1, 2, \cdots$

$$\mu_{2n} \leq \Lambda_{2n} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \Lambda_{2n+1} \leq \lambda_{2n+2}, \tag{2.7}$$

$$\mu_{2n} \leq \nu_{2n+1} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \nu_{2n+2} \leq \lambda_{2n+2}. \tag{2.8}$$

Let us apply to (2.3) the Liouville transformation

$$t = \int_0^x [s(u)/p(u)]^{1/2}\, du, \quad z(t) = [p(x)s(x)]^{1/4}\, y(x).$$

The transformed equation is

$$z'' + [\lambda - Q(t)z(t)] = 0 \tag{2.9}$$

where

$$Q(t) = q(x) - [p(x)]^{1/4} [s(x)]^{-3/4} \frac{d}{dx} p(x) \frac{d}{dx} [p(x)s(x)]^{-1/4}.$$

It can be seen that the parameter λ is unchanged. Also, the periodic and semi-periodic boundary conditions for the $x-$ interval $[0, a]$ are transformed into boundary conditions of the same type for the corresponding $t-$ interval. Hence, the periodic (λ_n) and semi-periodic (μ_n) eigenvalues for (2.9) are the same as for (2.3). We note that $Q(t)$ is r times differentiable if $q^r(x)$, $p^{r+2}(x)$ and $s^{r+2}(x)$ all exist and we can't apply the Liouville transformation if p'' and q'' do not exist.

3. The Results

In this part, we provide our results. Firstly, let us give two mixed problem with Hill' s equation for $t \in [0, a/2]$:

The Mixed Problem 1

$$\begin{aligned} y''(t) + [\lambda - q(t)]\, y(t) = 0 \\ y'(0) = y(a/2) = 0, \end{aligned}$$

and its eigenvalue is denoted as λ^{M_1};

The Mixed Problem 2

$$\begin{aligned} y''(t) + [\lambda - q(t)]\, y(t) = 0 \\ y(0) = y'(a/2) = 0, \end{aligned}$$

and its eigenvalue is denoted as λ^{M_2}.

Theorem 3.1. *The lower and upper boundaries of the difference between Dirichlet and Neumann eigenvalues of (1.1) on $[0, a]$ satisfy, as $n \to \infty$*

i)

$$\begin{aligned} & -\frac{a}{(2n+1)^2 \pi^2} \left| \int_0^{a/2} q'(t) \sin\left(\frac{2(2n+1)\pi}{a} t\right) dt \right| + o\left(n^{-3}\right) \\ & \leq \Lambda_{2n} - \nu_{2n+1} \\ & \leq \frac{a}{(2n+1)^2 \pi^2} \left| \int_0^{a/2} q'(t) \sin\left(\frac{2(2n+1)\pi}{a} t\right) dt \right| + o\left(n^{-3}\right), \end{aligned}$$

ii)

$$-\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|+o\left(n^{-3}\right)$$
$$\leq\Lambda_{2n+1}-\nu_{2n+2}$$
$$\leq\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|+o\left(n^{-3}\right).$$

Proof. If we subtract (2.8) from (2.7), we reach that

$$\mu_{2n}-\mu_{2n+1}\leq\Lambda_{2n}-\nu_{2n+1}\leq\mu_{2n+1}-\mu_{2n}, \tag{3.1}$$

$$\lambda_{2n+1}-\lambda_{2n+2}\leq\Lambda_{2n+1}-\nu_{2n+2}\leq\lambda_{2n+2}-\lambda_{2n+1}. \tag{3.2}$$

We also have from [1] that the periodic and semi-periodic eigenvalues of (1.1) on $[0,a]$ satisfy, as $n\to\infty$

$$\begin{array}{c}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{array}=\frac{2\left(n+1\right)\pi}{a}\mp\frac{a}{8\left(n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right|$$
$$-\frac{a^{2}}{64\left(n+1\right)^{3}\pi^{3}}$$
$$\times\left[aq^{2}\left(a\right)+2a\int_{0}^{a/2}q\left(t\right)q'\left(t\right)dt-4\int_{0}^{a/2}tq\left(t\right)q'\left(t\right)dt\right]+o\left(n^{-3}\right)$$

and

$$\begin{array}{c}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{array}=\frac{\left(2n+1\right)\pi}{a}\mp\frac{a}{2\left(2n+1\right)^{2}\pi^{2}}\left|\int_{0}^{a/2}q'\left(t\right)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right|$$
$$-\frac{a^{2}}{8\left(2n+1\right)^{3}\pi^{3}}$$
$$\times\left[aq^{2}\left(a\right)+2a\int_{0}^{a/2}q\left(t\right)q'\left(t\right)dt-4\int_{0}^{a/2}tq\left(t\right)q'\left(t\right)dt\right]+o\left(n^{-3}\right).$$

If we use this results and equations (3.1) and (3.2), we prove the theorem.

Notice that, the problems are on $[0,a]$, but we can write our solutions on $[0,a/2]$ because of symmetric single well potential q.

□

Theorem 3.2. *The eigenvalues of the Mixed Problem 1 and the Mixed Problem 2 satisfy, as* $n\to\infty$

i)

$$\begin{aligned}\left[\lambda_{2n}^{M_1}\right]^{1/2}=\left[\lambda_{2n}^{M_2}\right]^{1/2} &= \frac{(2n+1)\pi}{a}\\ &- \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right|\\ &- \frac{a^2}{8(2n+1)^3\pi^3}\\ &\times \left[aq^2(a)+2a\int_0^{a/2} q(t)q'(t)\,dt-4\int_0^{a/2} tq(t)q'(t)\,dt\right]\\ &+ o\left(n^{-3}\right),\end{aligned}$$

ii)

$$\begin{aligned}\left[\lambda_{2n+1}^{M_1}\right]^{1/2}=\left[\lambda_{2n+1}^{M_2}\right]^{1/2} &= \frac{(2n+1)\pi}{a}\\ &+ \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right|\\ &- \frac{a^2}{8(2n+1)^3\pi^3}\\ &\times \left[aq^2(a)+2a\int_0^{a/2} q(t)q'(t)\,dt-4\int_0^{a/2} tq(t)q'(t)\,dt\right]\\ &+ o\left(n^{-3}\right).\end{aligned}$$

Proof. Firstly, it can be note that μ is the eigenvalues of Hill's equation on the interval $[0,a]$ but the eigenvalues of mixed problems is for Hill' s equation on the interval $[0,a/2]$. [4] doesn' t entirely give mixed eigenvalues but it gives some properties for the mixed eigenvalues and proves that, if you have a symmetric potential

$$\lambda_k^{M_1}=\lambda_k^{M_2}=\mu_k$$

is satisfied. Our potential is symmetric single well, so we can use this equality. From this equality and $\mu_{2n}^{1/2}$ and $\mu_{2n+1}^{1/2}$ (above given from [1]), we prove the theorem and hence, we can give asymptotic eigenvalues. □

4. Conclusions

In this study, we find some asymptotic eigenvalues of Hill's equation. Our potential is symmetric single well, so we show that we can write our results on the half interval, we don't need to give asymptotic eigenvalues on the whole interval of the problem, the half interval is enough.

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References

[1] E. Başkaya, Periodic and semi-periodic eigenvalues of Hill' s equation with symmetric double well potential, Turkic World Mathematical Society Journal of Applied and Engineering Mathematics, vol.10, N.2, pp.346-352 (2020).
[2] E. Başkaya, Asymptotics of Eigenvalues for Regular Sturm-Liouville Problems with Spectral Parameter-Dependent Boundary Conditions and Symmetric Single Well Potential, Turkish Journal of Mathematics and Computer Science, vol.13, N.1, pp.44-50 (2021).
[3] E. Başkaya, On the asymptotics of eigenvalues for a Sturm-Liouville problem with symmetric single well potential, Demonstratio Mathematica, vol.57, N. 1, pp. 20230129 (2024).
[4] A. Cabada, J. A. Cid and L. López-Somoza, Green' s functions and spectral theory for the Hill' equation, Applied Mathematics and Computation, vol.286, pp. 88-105 (2016).
[5] H. Coşkun and E. Başkaya, Asymptotics of eigenvalues of regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential, Mathematica Scandinavica, vol.107, N.2, pp.209–223 (2010).
[6] H. Coşkun and N. Bayram, Asymptotics of eigenvalues for regular Sturm–Liouville problems with eigenvalue parameter in the boundary condition, Journal of Mathematical Analysis and Applications, vol.306, N.2, pp.548-566 (2005).
[7] H. Coşkun and A. Kabataş, Asymptotic approximations of eigenfunctions for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential, Mathematica Scandinavica, vol.113, N.1, pp.143-160 (2013).
[8] H. Coşkun and A. Kabataş, Green's Function of Regular Sturm-Liouville Problem Having Eigenparameter in One Boundary Condition, Turkish Journal of Mathematics and Computer Science, vol.4, pp.1–9 (2016).
[9] H. Coşkun, A. Kabataş and E. Başkaya, On Green's function for boundary value problem with eigenvalue dependent quadratic boundary condition, Boundary Value Problems, vol.71, (2017).
[10] H. Coşkun, E. Başkaya and A. Kabataş, Instability intervals for Hill's equation with symmetric single well potential, Ukrainian Mathematical Journal, vol.71, N. 6, pp.977-983 (2019).
[11] M. S. P. Eastham, The spectral theory of periodic differential equations, Scottish Academic Press, Edinburgh and London, (1973).
[12] A. Kabataş, On eigenfunctions of Hill' s equation with symmetric double well potential, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, vol.71, N.3, pp.634-649 (2022).

[13] A. Kabataş, Eigenfunction and Green' s Function Asymptotics for Hill' s Equation with Symmetric Single-Well Potential, Ukrainian Mathematical Journal, vol.74, N.2, pp.218-231 (2022).
[14] A. Kabataş, One boundary value problem including a spectral parameter in all boundary conditions, Opuscula Mathematica, vol.43, N.5, pp.651-661 (2023).

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