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GEOMETRIC PHASES, MAGNETIC CURVES FOR DARBOUX FRAMES ON LIGHTLIKE AND TIMELIKE SURFACES

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Abstract. In this paper, we obtain ${}^{D}\mathbf{E}_{\mathcal{T}}$, ${}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$, ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, ${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, magnetic curves, Lorentz force equations and geometric phases for Darboux frame of a spacelike curve with non-lightlike principal normal lying on a lightlike surface, null Darboux frame on a timelike surface, 1GDF and 2GDF in the tangential direction. Later, we derive intrinsic directional derivatives in $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface. Finally, we present geometric phases and magnetic curves in $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface.

1. Introduction

Geometric phase, also known as Berry phase, is occurs when a quantum system undergoes cyclic variation, and the final state of the system depends not only on the initial and final conditions but also on the path taken [1]. The interaction between the electric field and geometric phase has important implications in quantum computing, condensed matter physics, and quantum information processing and optik. This concept has gained crucial interest in last years. The investigation of the electric field change has contributed to the development of materials of science, condensed matter physics and plasma physics [2-9].

In recent times, numerous authors have presented new Darboux frames. Balakrishnan presented certain moving space curves are endowed with a geometric phase for the Darboux frame in Euclidean 3-space [10]. Later, Ertuğ presented the variation of electric field with respect to Darboux triad in Euclidean and Minkowski 3-spaces [11,12]. Alessio et al. [13] have studied null Darboux frame $\{\mathbf{T},\mathbf{V},\mathbf{N}\}$ derivative formulas on a timelike surface:

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{V} \\ \mathbf{N} \end{bmatrix}_{\mathcal{T}} = \begin{bmatrix} \kappa_g^* & 0 & \kappa_n^* \\ 0 & -\kappa_g^* & -\tau_g^* \\ \tau_g^* & -\kappa_n^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{V} \\ \mathbf{N} \end{bmatrix} \tag{1.1}$$

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where the geodesic curvature $\kappa_g^* = \langle \mathbf{T}_{\mathcal{T}}, \mathbf{V} \rangle$, the geodesic torsion $\tau_g^* = \langle \mathbf{N}_{\mathcal{T}}, \mathbf{V} \rangle$, the normal curvature $\kappa_n^* = \langle \mathbf{T}_{\mathcal{T}}, \mathbf{N} \rangle$, tangential direction $\mathcal{T}$ and
$\langle \mathbf{T}, \mathbf{T} \rangle = 0 = \langle \mathbf{V}, \mathbf{V} \rangle = \langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{V} \rangle = \mathbf{0}, \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{T}, \mathbf{V} \rangle = 1,\ \mathbf{T} \times \mathbf{V} = \mathbf{N}$, $\mathbf{V} \times \mathbf{N} = \mathbf{V}$, $\mathbf{N} \times \mathbf{T} = \mathbf{T}$:

Topbaş et al. [14] introduced Darboux frame $\{\mathbf{T}, \mu, \mathbf{U}\}$ derivative equations of a spacelike curve on null surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ for Darboux frame on lightlike surface [14]:

$$(1.2)\qquad \begin{pmatrix} \mathbf{T} \\ \mu \\ \mathbf{U} \end{pmatrix}_{\mathcal{T}} = \begin{pmatrix} 0 & \varepsilon_0 \kappa_n & \varepsilon_0 \kappa_g \\ -\kappa_g & \varepsilon_0 \tau_g & 0 \\ -\kappa_n & 0 & -\varepsilon_0 \tau_g \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mu \\ \mathbf{U} \end{pmatrix}$$

where the geodesic curvature $\kappa_g = \langle \mu, \mathbf{T}_{\mathcal{T}} \rangle$, the geodesic torsion $\tau_g = \langle \mathbf{U}, \mu_{\mathcal{T}} \rangle$ and the normal curvature $\kappa_n = \langle \mathbf{U}, \mathbf{T}_{\mathcal{T}} \rangle$ and
$\langle \mathbf{T}, \mathbf{T} \rangle = 1$, $\langle \mathbf{U}, \mathbf{U} \rangle = \langle \mu, \mu \rangle = \langle \mathbf{T}, \mu \rangle = \langle \mathbf{T}, \mathbf{U} \rangle = \mathbf{0}$, $\langle \mathbf{U}, \mu \rangle = \varepsilon_0 = \pm 1$ and $\mathbf{T} \times \mu = \varepsilon_0 \mu$, $\mu \times \mathbf{U} = \mathbf{T}$, $\mathbf{U} \times \mathbf{T} = \varepsilon_0 \mathbf{U}$.

Djordjević and Nesovic introduced the first kind generalized Darboux frame (1GDF) derivative formulas of 1GDF on a lightlike surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ as the following [15]:

$$(1.3)\qquad \begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mu} \\ \widetilde{\mathbf{U}} \end{pmatrix}_{\mathcal{T}} = \begin{pmatrix} 0 & \varepsilon_1 \widetilde{\kappa}_n & \varepsilon_1 \widetilde{\kappa}_g \\ -\widetilde{\kappa}_g & \varepsilon_1 \widetilde{\tau}_g & 0 \\ -\widetilde{\kappa}_n & 0 & -\varepsilon_1 \widetilde{\tau}_g \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{T}} \\ \widetilde{\mu} \\ \widetilde{\mathbf{U}} \end{pmatrix}$$

where $\left\{ \widetilde{\mathbf{T}} = \mathbf{T} + \varpi \mathbf{U},\ \widetilde{\mu} = -\varepsilon_1 \frac{\varpi}{\zeta} \mathbf{T} + \frac{1}{\zeta} \mu - \varepsilon_1 \frac{\varpi^2}{2\zeta} \mathbf{U},\ \widetilde{\mathbf{U}} = \zeta \mathbf{U} \right\}$, $\varpi \neq 0$, $\zeta \neq 0$ are differentiable functions, generalized geodesic curvature $\widetilde{\kappa}_g = \frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi \tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2 \kappa_n}{2\zeta}$, the generalized geodesic torsion $\widetilde{\tau}_g = \tau_g - \varepsilon_1 \varpi \kappa_n - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}$, the generalized normal curvature $\widetilde{\kappa}_n = \zeta \kappa_n$,
$\left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{T}} \right\rangle = 1$, $\left\langle \widetilde{\mathbf{U}}, \widetilde{\mathbf{U}} \right\rangle = \langle \widetilde{\mu}, \widetilde{\mu} \rangle = \left\langle \widetilde{\mathbf{T}}, \widetilde{\mu} \right\rangle = \left\langle \widetilde{\mathbf{T}}, \widetilde{\mathbf{U}} \right\rangle = \mathbf{0}$, $\left\langle \widetilde{\mathbf{U}}, \widetilde{\mu} \right\rangle = \varepsilon_1 = \pm 1$, $\widetilde{\mathbf{T}} \times \widetilde{\mu} = \varepsilon_1 \widetilde{\mu}$, $\widetilde{\mu} \times \mathbf{U} = \widetilde{\mathbf{T}}$, $\widetilde{\mathbf{U}} \times \widetilde{\mathbf{T}} = \varepsilon_1 \widetilde{\mathbf{U}}$.

Furthermore, Djordjević and Nesovic introduced the second kind generalized Darboux frame $\{\widetilde{\mathbf{T}} = \mathbf{T}^* = \mathbf{T},\ \mu^* = \frac{1}{\zeta} \mu,\ \mathbf{U}^* = \zeta \mathbf{U}\}$ derivative formulas of 2GDF on lightlike surface in the tangential direction $\mathcal{T}$ in $\mathbb{R}_1^3$ [15]:

$$(1.4)\qquad \begin{pmatrix} \mathbf{T}^* \\ \mu^* \\ \mathbf{U}^* \end{pmatrix}_{\mathcal{T}} = \begin{pmatrix} 0 & \varepsilon_1 \zeta \kappa_n & \varepsilon_1 \frac{\kappa_g}{\zeta} \\ -\frac{\kappa_g}{\zeta} & \varepsilon_1 (\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) & 0 \\ -\zeta \kappa_n & 0 & -\varepsilon_1 (\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}) \end{pmatrix} \begin{pmatrix} \mathbf{T}^* \\ \mu^* \\ \mathbf{U}^* \end{pmatrix}$$

where generalized geodesic curvature $\frac{\kappa_g}{\zeta}$, generalized geodesic torsion $\tau_g - \varepsilon_1 \frac{\zeta_{\mathcal{T}}}{\zeta}$ and generalized normal curvature $\zeta \kappa_n$.
Magnetic curves which a divergence free vector field were studied in [16-20]. Intrinsic directional derivatives have been investigated in [21-28].

In section 1, we give introduction. In Section 2, 3 and 4, we obtain ${}^{D}\mathbf{E}_{\mathcal{T}}$, ${}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$, ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$,

${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$, Lorentz force equations and magnetic curves in the tangential direction. We derive intrinsic directional derivatives in the $\widetilde{\mu}, \widetilde{U}-lines$ directions for 1GDF on a lightlike surface. Later, we present $\widetilde{\mu}, \widetilde{\mathbf{U}}-$magnetic curves and geometric phases in the $\widetilde{\mu}$, $\widetilde{U}-lines$ directions for 1GDF on a lightlike surface .

2. ${}^{D}\mathbf{E}_{\mathcal{T}}$, ${}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$, ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ AND ${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$

${}^{D}\mathbf{E}_{\mathcal{T}}$ for Darboux frame on a lightlike surface in the tangential direction

In general form, the change of the electric field ${}^{D}\mathbf{E}_{\mathcal{T}}$ for Darboux frame of a spacelike curve with non-lightlike principal normal lying on a lightlike surface can be expressed

$$ {}^{D}\mathbf{E}_{\mathcal{T}} = a_1\mathbf{T} + a_2\mu + a_3\mathbf{U}. \tag{2.1}$$

Assume that

$$\left\langle {}^{D}\mathbf{E}, \mathbf{T} \right\rangle = 0. \tag{2.2}$$

$$\left\langle {}^{D}\mathbf{E}, {}^{D}\mathbf{E} \right\rangle = const. \tag{2.3}$$

From Eqs.(1.2), (2.1), (2.2) and (2.3), the followings are obtained:

$$a_1 = -\varepsilon_0\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle - \varepsilon_0\kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \ . \tag{2.4}$$

$$a_2 = \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle, \quad a_3 = -\varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \tag{2.5}$$

Here, ς_1 is a parameter. Assume that, $\left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \neq 0$, $\left\langle {}^{D}\mathbf{E}, \mu \right\rangle \neq 0$. If Eqs.(2.4) and (2.5) are substituted in Eq.(2.1), then

$$\begin{aligned} \frac{\delta({}^{D}\mathbf{E})}{\delta\mathcal{T}} &= {}^{D}\mathbf{E}_{\mathcal{T}} = -\varepsilon_0(\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle + \kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle)\mathbf{T} \\ &\quad + \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \mu - \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \mathbf{U} \end{aligned} \tag{2.6}$$

is obtained. $\varsigma_1 \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle \mu - \varsigma_1 \left\langle {}^{D}\mathbf{E}, \mu \right\rangle \mathbf{U}$ denotes the rotation around $\mathbf{T}$ for Darboux frame with a nonnull principal normal lying on a lightlike surface. For $\varsigma_1 = 0$,

$${}^{D}\mathbf{E}_{\mathcal{T}} = -\varepsilon_0(\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle + \kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle)\mathbf{T} \tag{2.7}$$

Lorentz force equation ${}^{D}\Phi$ of the electric field vector ${}^{D}\mathbf{E}$ for Darboux frame of a spacelike curve with a nonlightlike principal normal lying on a lightlike surface is described by

$${}^{D}\Phi({}^{D}\mathbf{E}) = {}^{D}\ \mathbf{E}_{T} = \mathcal{A}_1 \times {}^{D}\mathbf{E}. \tag{2.8}$$

From Eq.(2.8), Lorentz force equations of $\{\mathbf{T}, \mu, \mathbf{U}\}$ are given by

$$\begin{aligned} {}^{D}\Phi(\mathbf{T}) &= \varepsilon_0\kappa_n\mu + \varepsilon_0\kappa_g\mathbf{U} && (2.9) \\ {}^{D}\Phi(\mu) &= -\kappa_g\mathbf{T} + \varepsilon_0\varsigma_1\mathbf{U} && (2.10) \\ {}^{D}\Phi(\mathbf{U}) &= -\kappa_n\mathbf{T} - \varepsilon_0\varsigma_1\mu && (2.11) \end{aligned}$$

From Eq.(2.9), (2.10), (2.11), the magnetic vector field is obtained:

$$\mathcal{A}_1 = \varsigma_1 \mathbf{T} - \kappa_n \mu + \kappa_g \mathbf{U}$$

In the general form, it can given by

$$^{D}\mathbf{E} = \varepsilon_0 \langle E, U \rangle \mu + \varepsilon_0 \langle E, \mu \rangle \mathbf{U} \tag{2.12}$$

Via Eqs.(2.12), it is derived

$$\begin{aligned} {}^{D}\mathbf{E}_{\mathcal{T}} &= \mu(\varepsilon_0 \langle E, U \rangle_{\mathcal{T}} + \tau_g \langle E, U \rangle\,) \\ &\quad + \mathbf{U}(\varepsilon_0 \langle E, \mu \rangle_{\mathcal{T}} - \tau_g \langle E, \mu \rangle\,) \\ &\quad - \varepsilon_0 \mathbf{T}(\kappa_n \left\langle {}^{D}\mathbf{E}, \mu \right\rangle + \kappa_g \left\langle {}^{D}\mathbf{E}, \mathbf{U} \right\rangle) \end{aligned} \tag{2.13}$$

Comparing Eqs.(2.7) and (2.13), it can be obtained

$$\langle E, \mu \rangle_{\mathcal{T}} = \varepsilon_0 \tau_g \langle E, \mu \rangle \tag{2.14}$$

$$\langle E, U \rangle_{\mathcal{T}} = -\varepsilon_0 \tau_g \langle E, U \rangle \tag{2.15}$$

Geometric phase around $\mathbf{T}$ for the frame $\{\mathbf{T}, \mu, \mathbf{U}\}$ on lightlike surface via Eqs.(2.14) and (2.15) is $\varepsilon_0 \tau_g$.

$^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$ for null Darboux frame on timelike surface in the tangential direction

The change of electric field $^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}}$ for null Darboux frame on a timelike surface in the $\mathcal{T} - lines$ direction is given by

$$\frac{\delta(^{\mathcal{ND}}\mathbf{E})}{\delta \mathcal{T}} = {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = b_1 \mathbf{T} + b_2 \mathbf{V} + b_3 \mathbf{N}. \tag{2.16}$$

Assume that

$$\left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{N} \right\rangle = 0, \tag{2.17}$$

$$\left\langle {}^{\mathcal{ND}}\mathbf{E}, {}^{\mathcal{ND}}\mathbf{E} \right\rangle = const. \tag{2.18}$$

From Eqs.(1.1), (2.16), (2.17) and (2.18), it can be obtained

$$b_1 = \varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle, \; b_2 = -\varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle \tag{2.19}$$

$$b_3 = \kappa_n^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle - \tau_g^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle \tag{2.20}$$

where ς_2 is a parameter. Assume that $\left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle \neq 0$, $\left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle \neq 0$. If Eqs.(2.19), (2.20) are substituted in Eq.(2.16), then

$$\begin{aligned} {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} &= \varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle \mathbf{T} - \varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle \mathbf{V} \\ &\quad + (\kappa_n^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle - \tau_g^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle\,)\mathbf{N} \end{aligned} \tag{2.21}$$

$\varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle \mathbf{T} - \varsigma_2 \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle \mathbf{V}$ denotes the rotation around $\mathbf{N}$ for null Darboux frame on a timelike surface. For $\varsigma_2 = 0$,

$$^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = (\kappa_n^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{V} \right\rangle - \tau_g^* \left\langle {}^{\mathcal{ND}}\mathbf{E}, \mathbf{T} \right\rangle)\mathbf{N} \tag{2.22}$$

Null Darboux Lorentz force equation $^{\mathcal{ND}}\Phi$ of the electric field vector for null Darboux frame on timelike surface is described by

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{E}) = {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = \mathcal{A}_2 \times {}^{\mathcal{ND}}\mathbf{E} \tag{2.23} $$

Via Eqs.(2.21) and (2.23), Lorentz force equations of the frame $\{\mathbf{T},\mathbf{V},\mathbf{N}\}$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{T}) = \varsigma_2\mathbf{T}+\kappa_n^*\mathbf{N} \tag{2.24} $$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{V}) = -\tau_g^*\mathbf{N}-\varsigma_2\mathbf{V} \tag{2.25} $$

$$ {}^{\mathcal{ND}}\Phi^{(N)}(\mathbf{N}) = \tau_g^*\mathbf{T}-\kappa_n^*\mathbf{V} \tag{2.26} $$

Via Eqs.(2.23), (2.24), (2.25) and (2.26), null Darboux magnetic field vector is derived

$$ \mathcal{A}_2 = -\tau_g^*\mathbf{T}+\varsigma_2\mathbf{N}-\kappa_n^*\mathbf{V} $$

In the general form,

$$ {}^{\mathcal{ND}}\mathbf{E} = \left\langle {}^{\mathcal{ND}}E, V\right\rangle \mathbf{T} + \left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle \mathbf{V} \tag{2.27} $$

With the aid (2.27), it is obtained

$$ \begin{aligned} {}^{\mathcal{ND}}\mathbf{E}_{\mathcal{T}} = {} & \mathbf{T}(\left\langle {}^{\mathcal{ND}}E, V\right\rangle_{\mathcal{T}} + \kappa_g^* \left\langle {}^{\mathcal{ND}}E, V\right\rangle) \\ & +\mathbf{V}(\left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle_{\mathcal{T}} - \kappa_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle) \\ & +\mathbf{N}(\kappa_n^* \left\langle {}^{\mathcal{ND}}E, V\right\rangle - \tau_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle) \end{aligned} \tag{2.28} $$

Comparing Eqs.(2.22) and (2.28), the followings are obtained

$$ \begin{aligned} \left\langle {}^{\mathcal{ND}}E, V\right\rangle_{\mathcal{T}} &= -\kappa_g^* \left\langle {}^{\mathcal{ND}}E, V\right\rangle \\ \left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle_{\mathcal{T}} &= \kappa_g^* \left\langle {}^{\mathcal{ND}}E, \mathcal{T}\right\rangle \end{aligned} $$

Geometric phase around $\mathbf{N}$ for the frame $\{\mathbf{T},\mathbf{V},\mathbf{N}\}$ is κ_g^*.

${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 1GDF on lightlike surface in the tangential direction

The change of electric field ${}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 1GDF on lightlike surface in the tangential direction $\mathcal{T}$ is given by

$$ {}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = c_1\widetilde{\mathbf{T}} + c_2\widetilde{\mu} + c_3\widetilde{\mathbf{U}}. \tag{2.29} $$

Consider

$$ \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}}\right\rangle = 0 \tag{2.30} $$

$$ \left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\mathbf{E}\right\rangle = const. \tag{2.31} $$

From Eqs.(1.3), (2.29), (2.30) and (2.31), the followings are obtained :

$$ c_1 = -\varepsilon_1 \left(\zeta\kappa_n \left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu}\right\rangle + (\frac{\kappa_g}{\zeta} + \varepsilon_1\frac{\varpi\mathcal{T}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1\frac{\varpi^2\kappa_n}{2\zeta}) \left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}}\right\rangle \right) \tag{2.32} $$

$$ c_2 = \mathfrak{z}_1^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mathbf{U}}\right\rangle, \quad c_3 = -\mathfrak{z}_1^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E},\widetilde{\mu}\right\rangle \tag{2.33} $$

where $\mathfrak{z}_1^1$ is a parameter. For $\mathfrak{z}_1^1 = 0$, Eqs.(2.32), (2.33) are rewritten in Eq.(2.29),

(2.34)
$$ {}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 \left(\zeta\kappa_n \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta}) \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \right) \widetilde{\mathbf{T}} $$

Lorentz force equation of electric field vector for 1GDF on lightlike surface in the tangential direction is described

$$ {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})}({}^{1\mathcal{GD}}\mathbf{E}) = {}^{1\mathcal{GD}}\,\mathbf{E}_{\mathcal{T}} = \mathcal{A}_3 \times {}^{1\mathcal{GD}}\mathbf{E} \tag{2.35} $$

$\mathcal{A}_3$ is the magnetic vector field. From Eq.(2.35), Lorentz force equations of 1GDF in the tangential direction can be given by

$$ {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mathbf{T}}) = \varepsilon_1\zeta\kappa_n\widetilde{\mu} + \varepsilon_1(\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{U}} \tag{2.36} $$

$$ {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mu}) = -(\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{T}} + \varepsilon_1\mathfrak{z}_1^1\widetilde{\mathbf{U}} \tag{2.37} $$

$$ {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})}(\widetilde{\mathbf{U}}) = -\zeta\kappa_n\widetilde{\mathbf{T}} - \varepsilon_1\mathfrak{z}_1^1\widetilde{\mu} \tag{2.38} $$

In the general form,

$$ {}^{1\mathcal{GD}}\mathbf{E} = \varepsilon_1 \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \widetilde{\mu} + \varepsilon_1 \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \widetilde{\mathbf{U}} \tag{2.39} $$

With the aid (2.39), it is obtained

$$ \begin{aligned} {}^{1\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = {} & \widetilde{\mu} \left(\varepsilon_1 \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle_{\mathcal{T}} + (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \right) \\ & + \widetilde{\mathbf{U}} \left(\varepsilon_1 ({}^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu})_{\mathcal{T}} - (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \right) \\ & - \varepsilon_1\widetilde{\mathbf{T}} \left(\zeta\kappa_n \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + (\frac{\kappa_g}{\zeta} + \varepsilon_1 \frac{\varpi_{\mathcal{T}}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1 \frac{\varpi^2\kappa_n}{2\zeta}) \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \right) \end{aligned} \tag{2.40} $$

Comparing Eqs.(2.34) and (2.40), it can be derived

$$ \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle_{\mathcal{T}} = \varepsilon_1(\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{\mu} \right\rangle \tag{2.41} $$

$$ \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle_{\mathcal{T}} = -\varepsilon_1(\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1 \frac{\zeta_T}{\zeta}) \left\langle {}^{1\mathcal{GD}}\widetilde{E}, \widetilde{U} \right\rangle \tag{2.42} $$

From Eqs.(2.41) and (2.42), geometric phase around $\widetilde{\mathbf{T}}$ in the $\mathcal{T} - lines$ direction for 1GDF on lightlike surface is $(\varepsilon_1\tau_g - \varpi\kappa_n - \frac{\zeta_T}{\zeta})$.

${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 2GDF in the tangential direction on lightlike surface

The change of the electric field ${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}}$ for 2GDF on lightlike surface in the $\mathcal{T} - lines$ direction can be written by

$$ {}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = d_1\mathbf{T}^* + d_2\mu^* + d_3\mathbf{U}^*. \tag{2.43} $$

Assume that

$$\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{T}^{*} \right\rangle = 0 \tag{2.44}$$

$$\left\langle {}^{2\mathcal{GD}}\mathbf{E}, {}^{2\mathcal{GD}}\mathbf{E} \right\rangle = const. \tag{2.45}$$

From Eqs.(1.4), (2.43), (2.44) and (2.45), it can be derived:

$$d_1 = -\varepsilon_1 \zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle - \varepsilon_1 \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle \tag{2.46}$$

$$d_2 = \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle, \quad d_3 = -\varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle \tag{2.47}$$

ς_3 is a parameter. Assume that $\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle \neq 0$, $\left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle \neq 0$. If Eqs.(2.46) and (2.47) are substituted in Eq.(2.43), then

$$\begin{aligned} {}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} &= -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle) \mathbf{T}^{*} \\ &\quad + \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle \mu^{*} - \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle \mathbf{U}^{*} \end{aligned} \tag{2.48}$$

$\varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle \mu^{*} - \varsigma_3 \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle \mathbf{U}^{*}$ denotes the rotation around $\mathbf{T}^{*}$. For $\varsigma_3 = 0$,

$${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle) \mathbf{T}^{*} \tag{2.49}$$

Lorentz force equation ${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}$ of the electric field vector ${}^{2\mathcal{GD}}\mathbf{E}$ in the tangential direction can be described by

$${}^{2\mathcal{GD}}\Phi^{(T)}({}^{2\mathcal{GD}}\mathbf{E}) = {}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = \mathcal{A}_4 \times {}^{2\mathcal{GD}}\mathbf{E}_2 \tag{2.50}$$

From Eq.(2.50), Lorentz force equations of 2GDF in the tangential direction are given by:

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mathbf{T}^{*}) = \varepsilon_1 \zeta \kappa_n \mu^{*} + \varepsilon_1 \frac{\kappa_g}{\zeta} \mathbf{U}^{*} \tag{2.51}$$

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mu^{*}) = -\frac{\kappa_g}{\zeta} \mathbf{T}^{*} + \varepsilon_1 \varsigma_3 \mathbf{U}^{*} \tag{2.52}$$

$${}^{2\mathcal{GD}}\Phi^{(\mathcal{T})}(\mathbf{U}^{*}) = -\zeta \kappa_n \mathbf{T}^{*} - \varepsilon_1 \varsigma_3 \mu^{*} \tag{2.53}$$

$$\mathcal{A}_4 = \varepsilon_1 \varsigma_3 \mathbf{T}^{*} - \zeta \kappa_n \mu^{*} + \frac{\kappa_g}{\zeta} \mathbf{U}^{*} \tag{2.54}$$

is the magnetic vector field satisfying Eqs.(2.51), (2.52) and (2.53). Also,

$${}^{2\mathcal{GD}}\mathbf{E} = \varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, U^{*} \right\rangle \mu^{*} + \varepsilon_1 \left\langle {}^{2\mathcal{GD}}E, \mu^{*} \right\rangle \mathbf{U}^{*} \tag{2.55}$$

Via Eq.(2.55), it can be obtained

$${}^{2\mathcal{GD}}\mathbf{E}_{\mathcal{T}} = -\varepsilon_1 (\zeta \kappa_n \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mu^{*} \right\rangle + \frac{\kappa_g}{\zeta} \left\langle {}^{2\mathcal{GD}}\mathbf{E}, \mathbf{U}^{*} \right\rangle) \mathbf{T}^{*}$$

$+\mu^{*}\left(\varepsilon_{1}\left\langle{}^{2\mathcal{GD}}E,U^{*}\right\rangle_{\mathcal{T}}+(\tau_{g}-\varepsilon_{1}\frac{\zeta_{\mathcal{T}}}{\zeta})\left\langle{}^{2\mathcal{GD}}E,\mu^{*}\right\rangle\right)$ (2.56)

$+\mathrm{U}^{*}\left(\varepsilon_{1}\left\langle{}^{2\mathcal{GD}}E,\mu^{*}\right\rangle_{\mathcal{T}}-(\tau_{g}-\varepsilon_{1}\frac{\zeta_{\mathcal{T}}}{\zeta})\left\langle{}^{2\mathcal{GD}}E,\mu^{*}\right\rangle\right)$

Comparing Eqs.(2.49) and(2.56) ,

$$\left\langle{}^{2\mathcal{GD}}E,\mu^{*}\right\rangle_{\mathcal{T}} = \varepsilon_{1}(\tau_{g}-\varepsilon_{1}\frac{\zeta_{\mathcal{T}}}{\zeta})\left\langle{}^{2\mathcal{GD}}E,\mu^{*}\right\rangle \tag{2.57}$$

$$\left\langle{}^{2\mathcal{GD}}E,U^{*}\right\rangle_{\mathcal{T}} = -\varepsilon_{1}(\tau_{g}-\varepsilon_{1}\frac{\zeta_{\mathcal{T}}}{\zeta})\left\langle{}^{2\mathcal{GD}}E,U^{*}\right\rangle \tag{2.58}$$

From Eqs.(2.57) and (2.58), geometric phase around $\mathbf{T}^{*}$ for 2GDF in the tangential direction is $\varepsilon_{1}(\tau_{g}-\varepsilon_{1}\frac{\zeta_{\mathcal{T}}}{\zeta})$.

3. Magnetic curves for Darboux frames

$\mathbf{V}-$magnetic curves for null Darboux frame in the $\mathcal{T}-lines$ direction

Let α be a distinguished curve on timelike surface with null Darboux frame. α is called **V**-magnetic curve if it satisfied null Darboux Lorentz force equation

$$\mathbf{V}_{T}={}^{\mathcal{ND}}\,\Phi^{(V)}(\mathbf{V})=\mathcal{A}_{5}\times\mathbf{V} \tag{3.1}$$

Here, $\mathcal{A}_{5}$ is magnetic vector field. It can be written by

$${}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T})=\imath_{1}\mathbf{T}+\imath_{2}\mathbf{V}+\imath_{3}\mathbf{N} \tag{3.2}$$

From Eqs.(1.1), (3.1) and (3.2), it can be derived

$$\left\langle{}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}),\mathbf{T}\right\rangle = \imath_{2}=0 \tag{3.3}$$

$$\left\langle{}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}),\mathbf{V}\right\rangle = \imath_{1}=\kappa_{g}^{*} \tag{3.4}$$

$$\left\langle{}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T}),\mathbf{N}\right\rangle = \imath_{3}=\varsigma_{4} \tag{3.5}$$

ς_{4} is a function. If Eqs.(3.3), (3.4) and (3.5) are substituted Eq.(3.2),

$${}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{T})=\kappa_{g}^{*}\mathbf{T}+\varsigma_{4}\mathbf{N} \tag{3.6}$$

obtained. Also,

$${}^{\mathcal{ND}}\Phi^{(V)}(\mathbf{N})=\tau_{g}^{*}\mathbf{T}-\varsigma_{4}\mathbf{V} \tag{3.7}$$

$$\mathcal{A}_{5}=-\tau_{g}^{*}\mathbf{T}-\varsigma_{4}\mathbf{V}+\kappa_{g}^{*}\mathbf{N}$$

.

$\widetilde{\mu}-$magnetic curves for 1GDF on lightlike surface in the $\mathcal{T}-lines$ direction

α is called $\widetilde{\mu}$-magnetic curve for 1GDF on lightlike surface if it satisfied the Lorentz force equation

$$\widetilde{\mu}_{\mathcal{T}}={}^{1\mathcal{GD}}\,\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mu})=\widetilde{\mathbf{X}}_{1}\times\widetilde{\mu} \tag{3.8}$$

Furthermore, it can be written by

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}})=j_{1}\widetilde{\mathbf{T}}+j_{2}\widetilde{\mu}+j_{3}\widetilde{\mathbf{U}} \tag{3.9}$$

Via Eqs.(1.3), (3.8) and (3.9), it can be derived

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{T}}\right\rangle = j_1 = 0 \tag{3.10}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{U}}\right\rangle = \mathfrak{z}_1^2,\ j_2 = \varepsilon_1\mathfrak{z}_1^2 \tag{3.11}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}),\widetilde{\mu}\right\rangle = \varepsilon_1 j_3 \ \Rightarrow j_3 = \varepsilon_1\zeta\kappa_n \tag{3.12}$$

If Eqs.(3.10), (3.11) and (3.12) are rewritten in (3.9),

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_1^2\widetilde{\mu} + \varepsilon_1(\frac{\kappa_g}{\zeta} + \varepsilon_1\frac{\varpi\mathcal{T}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1\frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{U}}$$

As similar,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -\mathfrak{z}_1^2\widetilde{\mathbf{T}} - \varepsilon_1(\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1\frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mathbf{U}}$$

is derived. The magnetic field vector

$\widetilde{\mathbf{X}}_1 = -\mathfrak{z}_1^2\widetilde{\mu} + (\frac{\kappa_g}{\zeta} + \varepsilon_1\frac{\varpi\mathcal{T}}{\zeta} - \frac{\varpi\tau_g}{\zeta} + \varepsilon_1\frac{\varpi^2\kappa_n}{2\zeta})\widetilde{\mathbf{U}} + (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1\frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mathbf{T}}$

is obtained.

$\widetilde{\mathbf{U}}-$magnetic curves for 1GDF on lightlike surface in the $\mathcal{T}-lines$ direction

α is called $\widetilde{\mathbf{U}}-$magnetic curve for 1GDF on lightlike surface if it satisfied the Lorentz force equation for 1GDF

$$\widetilde{\mathbf{U}}_{\mathcal{T}} = {}^{1\mathcal{GD}}\ \Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{X}}_2\times\widetilde{\mathbf{U}} \tag{3.13}$$

where $\widetilde{\mathbf{X}}_2$ is magnetic field vector Let α be a spacelike curve 1GDF on lightlike surface. We get,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}) = v_1\widetilde{\mathbf{T}} + v_2\widetilde{\mu} + v_3\widetilde{\mathbf{U}} \tag{3.14}$$

Via Eqs.(3.13) and (3.14), it can be obtained

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{T}}\right\rangle = v_1 = 0 \tag{3.15}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}),\widetilde{\mathbf{U}}\right\rangle = \varepsilon_1 v_2 \Rightarrow v_2 = \varepsilon_1\zeta\kappa_n \tag{3.16}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}),\widetilde{\mu}\right\rangle = \varepsilon_1 v_3 \Rightarrow v_3 = \varepsilon_1\mathfrak{z}_1^3 \tag{3.17}$$

If Eqs.(3.15), (3.16) and (3.17) are substituted in Eq.(3.14), then

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_1^3\widetilde{\mathbf{U}} + \varepsilon_1\zeta\kappa_n\widetilde{\mu}$$

As similarly,

$${}^{1\mathcal{GD}}\Phi^{(\mathcal{T})\widetilde{\mathbf{U}}}(\widetilde{\mu}) = -\varepsilon_1\mathfrak{z}_1^3\widetilde{\mathbf{T}} + (\tau_g - \varepsilon_1\varpi\kappa_n - \varepsilon_1\frac{\zeta_{\mathcal{T}}}{\zeta})\widetilde{\mu}$$

4. Intrinsic directional derivatives for 1GDF on a lightlike surface and magnetic curves

${}^{1\mathcal{GD}}\nabla$ is called the gradient operator for 1GDF on lightlike surface. It can be written by

$$^{1\mathcal{GD}}\nabla = \widetilde{\mathbf{T}}\frac{\delta}{\delta\mathcal{T}} + \varepsilon_1\widetilde{\mu}\frac{\delta}{\delta\widetilde{U}} + \varepsilon_1\widetilde{\mathbf{U}}\frac{\delta}{\delta\widetilde{\mu}}$$

where $\widetilde{\mathcal{T}}$, $\widetilde{\mu}$ and $\widetilde{U}$ are the arc length coordinates on the $\widetilde{\mathcal{T}}$, $\widetilde{\mu}-lines$ and $\widetilde{U}-lines$ for 1GDF. $\frac{\delta}{\delta\widetilde{\mathcal{T}}}$, $\frac{\delta}{\delta\widetilde{\mu}}$ and $\frac{\delta}{\delta\widetilde{U}}$ are the intrinsic directional derivatives in $\widetilde{\mathcal{T}}-lines$, $\widetilde{\mu}-lines$ and $\widetilde{U}-lines$ directions. The divergence vector for 1GDF is

$$^{1\mathcal{GD}}\ div\widetilde{\mathbf{T}} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mathbf{T}}\right\rangle = \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{U}},\widetilde{\mu}\right\rangle + \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{\mu}},\widetilde{\mathbf{U}}\right\rangle$$

where

$$\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} = \left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{U}},\widetilde{\mu}\right\rangle,\ \varphi^{(\widetilde{\mathcal{T}}\widetilde{U})} = \left\langle \frac{\delta\widetilde{\mathbf{T}}}{\delta\widetilde{\mu}},\widetilde{\mathbf{U}}\right\rangle$$

$$^{1\mathcal{GD}}\ div\widetilde{\mu} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mu}\right\rangle = -\widetilde{\kappa}_g + \varepsilon_1\left\langle \frac{\delta\widetilde{\mu}}{\delta\widetilde{\mu}},\widetilde{\mathbf{U}}\right\rangle \tag{4.1}$$

$$^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}} = \left\langle {}^{1\mathcal{GD}}\nabla, \widetilde{\mathbf{U}}\right\rangle = -\widetilde{\kappa}_n + \varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{U}}}{\delta\widetilde{U}},\widetilde{\mu}\right\rangle \tag{4.2}$$

$$\gamma^{(\widetilde{\mu})} = \left\langle {}^{1\mathcal{GD}}\nabla\times\widetilde{\mu}, \widetilde{\mu}\right\rangle = \varepsilon_1\left\langle \frac{\delta\widetilde{\mu}}{\delta\widetilde{\mu}},\widetilde{\mathbf{T}}\right\rangle$$

$$\gamma^{(\widetilde{U})} = \left\langle {}^{1\mathcal{GD}}\nabla\times\widetilde{\mathbf{U}}, \widetilde{\mathbf{U}}\right\rangle = -\varepsilon_1\left\langle \frac{\delta\widetilde{\mathbf{U}}}{\delta\widetilde{U}},\widetilde{\mathbf{T}}\right\rangle$$

$\gamma^{(T)}$, $\gamma^{(\widetilde{\mu})}$, $\gamma^{(\widetilde{U})}$ are total moments of the $\widetilde{\mathbf{T}}$, $\widetilde{\mu}$, $\widetilde{\mathbf{U}}$ fields of spacelike curve for 1GDF on lightlike surface. Intrinsic directional derivatives in $\widetilde{\mu}$ and $\widetilde{\mathbf{U}}$ lines directions for 1GDF on lightlike surface are obtained

$$\frac{\delta}{\delta\widetilde{\mu}}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} = \begin{bmatrix}0 & \varepsilon_1\varphi^{(T\widetilde{U})} & -\gamma^{(\widetilde{\mu})}\\ \varepsilon_1\gamma^{(\widetilde{\mu})} & \widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu} & 0\\ -\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})} & 0 & -(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} \tag{4.3}$$

$$\frac{\delta}{\delta\widetilde{U}}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} = \begin{bmatrix}0 & \gamma^{(\widetilde{U})} & \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\\ -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} & -(\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}}) & 0\\ -\varepsilon_1\gamma^{(\widetilde{U})} & 0 & (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{T}}\\ \widetilde{\mu}\\ \widetilde{\mathbf{U}}\end{bmatrix} \tag{4.4}$$

Geometric phase in $\widetilde{\mu}-lines$ direction for 1GDF on lightlike surface

In the general form, the change of the electric field $\frac{\delta({}^{1\mathcal{GD}}\mathbf{E}_1)}{\delta\widetilde{\mu}}$ for 1GDF on lightlike surface in $\widetilde{\mu}-lines$ direction is

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{\mu}} = l_1\widetilde{\mathbf{T}} + l_2\widetilde{\mu} + l_3\widetilde{\mathbf{U}}. \tag{4.5}$$

Assume that

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}} \right\rangle = 0 \tag{4.6}$$

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\,\mathbf{E} \right\rangle = const. \tag{4.7}$$

Here,

$$l_1 = -\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle. \tag{4.8}$$

$$l_2 = \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle, \quad l_3 = -\mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \tag{4.9}$$

When Eqs.(4.6), (4.7) are written in Eq.(4.5),

$$\begin{aligned} \frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} &= (-\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle)\widetilde{\mathbf{T}} \\ &\quad + \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle\widetilde{\mu} - \mathfrak{z}_2^1\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle\widetilde{\mathbf{U}} \end{aligned} \tag{4.10}$$

where $\mathfrak{z}_2^1$ is a parameter. $\mathfrak{z}_2^1({}^{1\mathcal{GD}}\mathbf{E}\times\widetilde{\mathbf{T}})$ is the rotation around $\widetilde{\mathbf{T}}$ for 1GDF on lightlike surface in $\widetilde{\mu}-lines$ direction.

When $\mathfrak{z}_2^1 = 0$, Eq.(4.10) is derived as the following:

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{\mu}} = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{\mu}} = (-\varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle + \gamma^{(\widetilde{\mu})}\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle)\widetilde{\mathbf{T}} \tag{4.11}$$

Lorentz force equation ${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}$ of ${}^{1\mathcal{GD}}\mathbf{E}$ in $\widetilde{\mu}-lines$ direction is described

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}({}^{1\mathcal{GD}}\mathbf{E}) = {}^{1\mathcal{GD}}\,\mathbf{E}_{\widetilde{\mu}} = \widetilde{\mathbf{Y}}_1\times{}^{1\mathcal{GD}}\mathbf{E} \tag{4.12}$$

With Eqs.(4.11) and (4.12), Lorentz force equations of 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction are derived :

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mathbf{T}}) = \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\widetilde{\mu} - \gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}} \tag{4.13}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mu}) = \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mathbf{T}} + \varepsilon_1\mathfrak{z}_2^1\widetilde{\mu} \tag{4.14}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})}(\widetilde{\mathbf{U}}) = -\varphi^{(\widetilde{\mathcal{T}}\widetilde{U})}\widetilde{\mathbf{T}} - \varepsilon_1\mathfrak{z}_2^1\widetilde{\mathbf{U}} \tag{4.15}$$

Here

$$\widetilde{\mathbf{Y}}_1 = \mathfrak{z}_2^1\widetilde{\mathbf{T}} - \gamma^{(\widetilde{\mathcal{T}}\widetilde{\mu})})\widetilde{\mu} - \varepsilon_{\mathbf{1}}\varphi^{(\widetilde{\mu}}\widetilde{\mathbf{U}}$$

the magnetic vector field satisfies Eqs.(4.13), (4.14) and (4.15).

$\widetilde{\mu}-$magnetic curves in the $\widetilde{\mu}-lines$ direction for 1GDF on lightlike surface

The curve is called $\widetilde{\mu}$-magnetic curve 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction if it satisfied Lorentz force equation

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mu}) = \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mu})\widetilde{\mu} = \widetilde{\mathbf{Y}}_2\times\widetilde{\mu}$$

Consider

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = l_1\widetilde{\mathbf{T}}+l_2\widetilde{\mu} + l_3\widetilde{\mathbf{U}} \tag{4.16}$$

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}}\right\rangle = l_1 = 0 \tag{4.17}$$

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}}\right\rangle = \mathfrak{z}_2^2 \Rightarrow l_2 = \varepsilon_1\mathfrak{z}_2^2 \tag{4.18}$$

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mu}\right\rangle = \varepsilon_1 l_3 \Rightarrow l_3 = -\gamma^{(\widetilde{\mu})} \tag{4.19}$$

$\mathfrak{z}_2^2$ is parameter. If Eqs.(4.17), (4.18) and (4.19) are substituted in Eq.(4.16)

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_2^2\widetilde{\mu} - \gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}}$$

As similarly,

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mathbf{U}} - \mathfrak{z}_2^2\widetilde{\mathbf{T}}$$

Thus,

$$\widetilde{\mathbf{Y}}_2 = -\varepsilon_1(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mathbf{T}} - \mathfrak{z}_2^2\widetilde{\mu} + \varepsilon_1\gamma^{(\widetilde{\mu})}\widetilde{\mathbf{U}}$$

$\widetilde{\mathbf{U}}-$magnetic curves 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction

The curve is called $\widetilde{\mathbf{U}}-$magnetic curve if it satisfied the Lorentz force equation 1GDF on lightlike surface in the $\widetilde{\mu}-lines$ direction

$$\widetilde{\mathbf{U}}_{\widetilde{\mu}} = {}^{1\mathcal{GD}}\ \Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{Y}}_3\times\widetilde{\mathbf{U}} \tag{4.20}$$

It can be written by

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = h_1\widetilde{\mathbf{T}}+h_2\widetilde{\mu} + h_3\widetilde{\mathbf{U}} \tag{4.21}$$

Using Eqs.(4.20) and (4.21), it can be obtained

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}}\right\rangle = h_1 = 0 \tag{4.22}$$

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}}\right\rangle = \varepsilon_1 h_2 \Rightarrow h_2 = \varepsilon_1\varphi^{(\widetilde{T}\widetilde{U})} \tag{4.23}$$

$$\left\langle{}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mu}\right\rangle = \varepsilon_1 h_3 \Rightarrow h_3 = \varepsilon_1\mathfrak{z}_2^3 \tag{4.24}$$

$\mathfrak{z}_2^3$ is a parameter. Via Eqs.(4.22), (4.23) and (4.24), Eq.(4.21) is rewritten by

$$\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_2^3\widetilde{\mathbf{U}} + \varepsilon_1\varphi^{(\widetilde{T}\widetilde{U})}\widetilde{\mu}$$

As similar,

$$^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mu}) = -\mathfrak{z}_2^3\widetilde{\mathbf{T}}+(\widetilde{\kappa}_g + {}^{1\mathcal{GD}}\ div\widetilde{\mu})\widetilde{\mu}.$$

$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}}$ in $\widetilde{\mathbf{U}}-lines$ direction for 1GDF on lightlike surface

$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}}$ for 1GDF in $\widetilde{\mathbf{U}}-lines$ direction in the general form on lightlike surface is given by

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{U}} = f_1\widetilde{\mathbf{T}} + f_2\widetilde{\mu} + f_3\widetilde{\mathbf{U}}. \tag{4.25}$$

Assume that

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{T}} \right\rangle = 0 \tag{4.26}$$

$$\left\langle {}^{1\mathcal{GD}}\mathbf{E}, {}^{1\mathcal{GD}}\ \mathbf{E} \right\rangle = const. \tag{4.27}$$

Via Eqs.(4.26) and (4.27), it can be obtained:

$$f_1 = -(\varepsilon_1 \varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle + \gamma^{(\widetilde{U})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle) \tag{4.28}$$

$$f_2 = \mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle, \quad f_3 = -\mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \tag{4.29}$$

where $\mathfrak{z}_3^1$ is a parameter. If Eqs.(4.26), (4.27) are written in Eq.(4.25), then

$$\begin{aligned} \frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} &= {}^{1\mathcal{GD}}\mathbf{E}_{\widetilde{U}} = -(\varepsilon_1 \varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle + \gamma^{(\widetilde{U})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle)\widetilde{\mathbf{T}} \\ &\quad + \mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle \widetilde{\mu} - \mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \widetilde{\mathbf{U}} \end{aligned} \tag{4.30}$$

$\mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle - \mathfrak{z}_3^1 \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle \widetilde{\mathbf{U}}$ gives the rotation around $\widetilde{\mathbf{T}}$ for 1GDF on lightlike surface in the in $\widetilde{U}-lines$ direction. When $\mathfrak{z}_3^1 = 0$, Eq.(4.30) is

$$\frac{\delta({}^{1\mathcal{GD}}\mathbf{E})}{\delta\widetilde{U}} = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{U}} = -(\varepsilon_1 \varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mathbf{U}} \right\rangle - \gamma^{(\widetilde{U})} \left\langle {}^{1\mathcal{GD}}\mathbf{E}, \widetilde{\mu} \right\rangle)\widetilde{\mathbf{T}} \tag{4.31}$$

Lorentz force equation ${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}$ of ${}^{1\mathcal{GD}}\mathbf{E}$ in $\widetilde{U}-lines$ direction is given by

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}({}^{1\mathcal{GD}}\mathbf{E}) = {}^{1\mathcal{GD}}\ \mathbf{E}_{\widetilde{U}} = \widetilde{\mathbf{Y}}_3 \times {}^{1\mathcal{GD}}\mathbf{E}_1 \tag{4.32}$$

With Eqs.(4.31) and (4.32), Lorentz force equations of 1GDF on lightlike surface are given by:

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{T}}) = \gamma^{(\widetilde{U})}\widetilde{\mu} + \varepsilon_1 \varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{U}} \tag{4.33}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mu}) = -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{T}} + \varepsilon_1 \mathfrak{z}_3^2 \widetilde{\mu} \tag{4.34}$$

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{\mu})\widetilde{U}}(\widetilde{\mathbf{U}}) = -\varepsilon_1 \gamma^{(\widetilde{U})}\widetilde{\mathbf{T}} - \varepsilon_1 \mathfrak{z}_3^2 \widetilde{\mathbf{U}} \tag{4.35}$$

Here

$$\widetilde{\mathbf{Y}}_3 = \mathfrak{z}_3^1 \widetilde{\mathbf{T}} - \varepsilon_1 \gamma^{(\widetilde{U})}\widetilde{\mu} - \varphi^{(\mathcal{T}\widetilde{\mu})}\widetilde{\mathbf{U}}$$

magnetic vector field satisfies Eqs.(4.33), (4.34) and (4.35).

$\widetilde{\mu}-$magnetic curves for 1GDF on lightlike surface in the $\widetilde{\mathbf{U}}-lines$ direction

If the curve is called $\widetilde{\mu}$-magnetic curve in the $\widetilde{\mathbf{U}}-lines$ direction if it satisfied the Lorentz force equation

$$\begin{aligned} \frac{\delta\widetilde{\mu}}{\delta\widetilde{U}} &= {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mu}) = -\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{T}} - (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mu} \\ &= \widetilde{\mathbf{Z}}_1 \times \widetilde{\mu} \end{aligned} \tag{4.36}$$

Consider

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = m_1\widetilde{\mathbf{T}} + m_2\widetilde{\mu} + m_3\widetilde{\mathbf{U}}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = m_1 = 0 \tag{4.37}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \mathfrak{z}_3^2 \Rightarrow m_3 = \varphi^{(\mathcal{T}\widetilde{\mu})} \tag{4.38}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \varepsilon_1 m_2 \Rightarrow m_2 = \varepsilon_1\mathfrak{z}_3^2 \tag{4.39}$$

Via Eqs.(4.37), (4.38) and (4.39), it is obtained

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{T}}) = \varepsilon_1\mathfrak{z}_3^2\widetilde{\mu} + \varepsilon_1\varphi^{(\widetilde{\mathcal{T}}\widetilde{\mu})}\widetilde{\mathbf{U}} \tag{4.40}$$

As similarly,

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{\mu}}(\widetilde{\mathbf{U}}) = -\mathfrak{z}_3^2\widetilde{\mathbf{T}} + (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{U}} \tag{4.41}$$

Here, the magnetic vector field

$$\widetilde{\mathbf{Z}}_1 = -\varepsilon_1(\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{T}} - \mathfrak{z}_3^2\widetilde{\mu} + \varphi^{(\mathcal{T}\widetilde{\mu})}\widetilde{\mathbf{U}}$$

satisfies Eqs.(4.36), (4.40) and (4.41).

$\widetilde{\mathbf{U}}$−magnetic curves for 1GDF on lightlike surface in the $\widetilde{\mathbf{U}}-lines$ direction

If the curve is called $\widetilde{\mathbf{U}}$−magnetic curve for 1GDF on lightlike surface, if it satisfied the Lorentz force equation in the $\widetilde{U}-lines$ lines

$$\widetilde{\mathbf{U}}_{\widetilde{U}} = {}^{1\mathcal{GD}}\ \Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{U}}) = \widetilde{\mathbf{Z}}_2 \times \widetilde{\mathbf{U}} \tag{4.42}$$

It can be written by

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}) = n_1\widetilde{\mathbf{T}} + n_2\widetilde{\mu} + n_3\widetilde{\mathbf{U}} \tag{4.43}$$

From Eqs.(4.42) and (4.43), it can be obtained

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{T}} \right\rangle = n_1 = 0 \tag{4.44}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mu} \right\rangle = \varepsilon_1 n_3 \Rightarrow n_3 = \varepsilon_1\mathfrak{z}_3^3 \tag{4.45}$$

$$\left\langle {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}), \widetilde{\mathbf{U}} \right\rangle = \varepsilon_1 n_2 \Rightarrow n_2 = \gamma^{(\widetilde{U})} \tag{4.46}$$

Via Eqs.(4.44), (4.45) and (4.46), it is obtained

$${}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mathbf{T}}) = \gamma^{(\widetilde{U})}\widetilde{\mu} + \varepsilon_1\mathfrak{z}_3^3\widetilde{\mathbf{U}} \tag{4.47}$$

As similar,

$$ {}^{1\mathcal{GD}}\Phi^{(\widetilde{U})\widetilde{U}}(\widetilde{\mu}) = -\mathfrak{z}_3^3\widetilde{\mathbf{T}} - (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mu} \tag{4.48} $$

Here,

$$ \widetilde{\mathbf{Z}}_2 = (\widetilde{\kappa}_n + {}^{1\mathcal{GD}}\ div\widetilde{\mathbf{U}})\widetilde{\mathbf{T}} - \varepsilon_1\gamma^{(\widetilde{U})}\widetilde{\mu} + \mathfrak{z}_3^3\widetilde{\mathbf{U}} $$

satisfies Eqs.(4.42), (4.47) and (4.48).

5. Conclusion

In this manuscript, we studied variations of electric fields, geometric phases, Lorentz force equations and magnetic curves for Darboux frame of a spacelike curve on null surface, null Darboux frame on timelike surface, first and second kinds generalized Darboux frames on lightlike surface in the tangential direction. Finally, we presented geometric phases, Lorentz force equations and magnetic curves via anholonomic coordinates for the first generalized Darboux frame on lightlike surface.

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