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Gönderim Sınıfı Grubunda Denk Çapraz Homomorfizmler

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ÖZET

Bir yüzeyin gönderim sınıfı grubu, yön koruyan kendisine giden diffeomorfizmalarının izotopi sınıflarını tanımlayan bir grup olup, matematiğin pek çok alanında, özellikle topoloji, cebir ve geometride önemli bir rol oynar. Topolojide, gönderim sınıfı grupları 3-manifoldların ve lif demetlerin incelenmesinde önemlidir; cebir ve geometri alanlarında ise otomorfizm teorisi, modül uzayları ve yüzeyler üzerindeki kompleks yapılar ile yakın bir ilişkiye sahiptir. Gönderim sınıfı grupları konusunda ilginç bir bakış açısı, gönderim sınıfı gruplarının kohomoloji sınıflarının incelenmesini içerir. Yönlendirilebilir yüzeylerin gönderim sınıfı gruplarının kohomoloji sınıfları, yüzey demetlerinin karakteristik sınıfları olarak düşünülebilir. Earle, Morita, Furuta ve Trapp tarafından verilen, yönlendirilebilir yüzeylerin gönderim sınıfı gruplarının kohomoloji sınıfının çeşitli inşaları vardır. Bu inşalar çok farklı görünmektedir. Bu nedenle, çeşitli yazarlar bu yapıları karşılaştırarak aralarındaki ilişkileri daha iyi anlamak için çaba sarf ettiler. Furuta tarafından önerilen ve Trapp tarafından sunulan gönderim sınıfı gruplarının kohomoloji sınıflarını veren çapraz homomorfizmler dolanım sayıları ile ilişkilidir. Bu çalışmada, bu iki farklı yapı arasındaki ilişkiyi gösteriyoruz.

Equivalent Crossed Homomorphisms on The Mapping Class Group

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ABSTRACT

The mapping class group of a surface, which describes the isotopy classes of orientation-preserving self-diffeomorphisms, plays an important role in many areas of mathematics, particularly in topology, algebra and geometry. In topology, mapping class groups are essential for studying 3-manifolds and fiber bundles, while in algebra and geometry, they are closely related to the theory of automorphisms, moduli spaces, and complex structures on surfaces. An interesting perspective on mapping class groups involves the study of their cohomology classes. Cohomology classes of the mapping class groups of orientable surfaces can be considered as characteristic classes of surface bundles. There are several constructions of the cohomology class of the mapping class groups of orientable surfaces given by Earle, Morita, Furuta, and Trapp. These constructions seem very different. Therefore, various authors have made efforts to better understand the relationships between these constructions by comparing them. The crossed homomorphisms which yield the cohomology classes of the mapping class groups, as proposed by Furuta and presented by Trapp, are related to winding numbers. In this study, we show the relation between these two different constructions.

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INTRODUCTION

Let $\Sigma_{g,1}$ be a compact connected oriented smooth surface of genus g with 1 boundary component and $\Sigma_{g,*}$ be a surface obtained by attaching a disc to the boundary $\partial\Sigma_{g,1}$ with a fixed point $*$. The mapping class groups of $\Sigma_{g,1}$ and $\Sigma_{g,*}$ are the groups of all isotopy classes of orientation preserving self-diffeomorphisms, with the former fixing the boundary pointwise and the latter fixing the marked point $*$, respectively. Let $MCG(\Sigma_{g,1})$ and $MCG(\Sigma_{g,*})$ denote the mapping class groups of $\Sigma_{g,1}$ and $\Sigma_{g,*}$, respectively [1].

Various crossed homomorphisms from $MCG(\Sigma_{g,1})$ to $H_1(\Sigma_{g,1}, \mathbb{Z})$ were constructed to get a generator of the first cohomology class of the mapping class group. Earle [2] first constructed a crossed homomorphism $MCG(\Sigma_{g,*}) \to \frac{1}{2g-2} H_1(\Sigma_g; \mathbb{Z})$ for $g \geq 2$ and a crossed homomorphism $MCG(\Sigma_{g,*}) \longrightarrow H_1(\Sigma_g;\ \mathbb{Z})$ can be obtained after the multiplication by $(2g-2)$.

Morita [3] proved the following isomorphisms:

$$H^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_g;\ \mathbb{Z})\right) \cong\ H^1\left(MCG(\Sigma_{g,*});\ H^1(\Sigma_g;\ \mathbb{Z})\right) \cong\ \mathbb{Z}.$$

Moreover, Morita [3] gave a combinatorial construction of a crossed homomorphism representing a generator of $H^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_g;\ \mathbb{Z})\right)$. Kuno [4] compared Earle's and Morita's constructions. Recently, Chen [5] and Maruyama [6] constructed new crossed homomorphisms.

Trapp and Furuta also constructed crossed homomorphisms representing a generator of $H^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_g;\ \mathbb{Z})\right)$ using winding numbers. In this paper, we provide a survey of different constructions given by Trapp [7] and proposed by Furuta [8]. We also show that both constructions are equivalent by using difference cocycles after we present an overview to provide the necessary background information.

MATERIALS AND METHODS

This section is devoted to the basic definitions which will be needed in the next section. We define crossed homomorphisms, the winding number of a smooth curve, and the difference cocycle. We refer the reader to [9] for any unexplained terminology on the homology and cohomology of surfaces. In this paper, we are interested in homologies and cohomologies with $\mathbb{Z}$ coefficients, so we will no longer emphize the coefficients.

Crossed Homomorphism

Let us recall the definition of the first cohomology group $H^1(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1}))$ of $MCG(\Sigma_{g,1})$ with coefficients in $H^1(\Sigma_{g,1})$ [10].

There is an action of $MCG(\Sigma_{g,1})$ on $H_1(\Sigma_{g,1})$ via the symplectic representation ρ, which takes an element of $MCG(\Sigma_{g,1})$ to an automorphism of $H_1(\Sigma_{g,1})$ preserving the intersection form. If we identify $H^1(\Sigma_{g,1})$ with $\mathrm{Hom}(H_1(\Sigma_{g,1}), \mathbb{Z})$, the action of $MCG(\Sigma_{g,1})$ on $H^1(\Sigma_{g,1})$ is defined to be $\phi_1 f(x) = f((\phi_1)_*^{-1}(x)) = f(\rho(\phi_1^{-1}(x))$, where $\phi_1 \in MCG(\Sigma_{g,1})$, $f \in H^1(\Sigma_{g,1})$, $x \in H_1(\Sigma_{g,1})$, and $(\phi_1)_*$ is the induced homomorphism from ϕ_1.

A *crossed homomorphism* ψ is a function $\psi\colon MCG(\Sigma_{g,1}) \longrightarrow H^1(\Sigma_{g,1})$ such that $\psi(\mu_1\mu_2) = \psi(\mu_1) + \mu_1\, \psi(\mu_2)$ for all $\mu_1,\ \mu_2 \in \mathrm{MCG}(\Sigma_{g,1})$. Let $Z^1(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1}))$ denote the set of all crossed homomorphisms $\psi\colon \mathrm{MCG}(\Sigma_{g,1}) \longrightarrow H^1(\Sigma_{g,1})$.

Let us define a function $\psi_m: MCG(\Sigma_{g,1}) \longrightarrow H^1(\Sigma_{g,1})$ such that $\psi_m(\mu_1) = \mu_1\, m - m$, where m is a fixed element of $H^1(\Sigma_{g,1})$. This defined function ψ_m is called a *principal crossed homomorphism*. Let us denote the set of all such principal crossed homomorphisms by $B^1(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1}))$.

The first cohomology group of $MCG(\Sigma_{g,1})$ with coefficients in $H^1(\Sigma_{g,1})$ is defined as the quotient

$$H^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1})\right) := \frac{Z^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1})\right)}{B^1\left(MCG(\Sigma_{g,1});\ H^1(\Sigma_{g,1})\right)}$$

Similarly, we can define $H^1(MCG(\Sigma_{g,1});\ H_1(\Sigma_{g,1}))$.

Winding Number

In this subsection, we provide the definition of the winding number [11]. Intuitively, the winding number of a smooth oriented closed curve is the number of rotations made by its tangent vector with respect to a nonvanishing vector field X, as the curve is traversed once in the positive direction.

Assume that $\Sigma_{g,1}$ is given some Riemannian structure. Let $T^1\Sigma_{g,1}$ be the unit tangent bundle of $\Sigma_{g,1}$ and $prj: T^1\Sigma_{g,1} \to \Sigma_{g,1}$ be the natural projection defined by $prj(t, v) = t$ for each unit vector $v \in T_t\Sigma_{g,1}$ and $t \in \Sigma_{g,1}$, where $T_t\Sigma_{g,1}$ is the tangent space over the point t. Let $\beta: S^1 \to \Sigma_{g,1}$ be a smooth oriented closed curve with $\beta(S^1) = \gamma$ based at the point t and continuously varying non-zero tangents exist at all points of γ. The continuous map β induces a pullback over S^1 from $T^1\Sigma_{g,1}$ to $\Sigma_{g,1}$. That is, if $prj^\beta: S^1 \times S^1 \to S^1$ is the first projection map, then there exists a map F taking the fiber of $S^1 \times S^1$ over each point $a \in S^1$ isomorphically onto the fiber of $T^1\Sigma_{g,1}$ over $\beta(a)$ such that the following diagram is commutative:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\ F\ } & T^1\Sigma_{g,1} \\ {\scriptstyle prj^\beta}\downarrow & & \downarrow{\scriptstyle prj} \\ S^1 & \xrightarrow[\ \beta\ ]{} & \Sigma_{g,1} \end{array} \tag{1}$$

Here, the total space is $S^1 \times S^1$, as $\beta(S^1) = \gamma$ is an orientation preserving curve.

Since the surface $\Sigma_{g,1}$ has a nonempty boundary, a nonvanishing vector field X on $\Sigma_{g,1}$ exists. If a nonvanishing vector field X is given, there is a section $X^\beta: S^1 \to S^1 \times S^1$ such that $F \circ X^\beta = \tilde{X} \circ \beta$, where $\tilde{X}(t) := X(t)/\|X(t)\|_t$ for $t \in \Sigma_{g,1}$ and $\|X(t)\|_t$ denotes the norm of $X(t)$ on $T_t\Sigma_{g,1}$. This section is defined so that $X^\beta(a) = (a, \tilde{X}(\beta(a))$ for every $a \in S^1$.

Now by considering the tangent map $d\beta: TS^1 \to T\Sigma_{g,1}$, we can define $d_0\beta: S^1 \to T^1\Sigma_{g,1}$ as follows:

$$d_0\beta(a) = \frac{d\beta(a, 1)}{\|d\beta(a, 1)\|_{\beta(a)}}.$$

This defined map $d_0\beta$ pulls back to a unique section $Y^\beta: S^1 \to S^1 \times S^1$ which satisfies the equality $F \circ Y^\beta = d_0\beta$.

Assume that $X^\beta(a) = Y^\beta(a) = a_0$ for a point $a \in S^1$. Then X^β and Y^β represent elements of

$\pi_1(S^1 \times S^1, a_0)$. The projection map prj^β induces the homomorphism $prj^\beta_*: \pi_1(S^1 \times S^1, a_0) \to \pi_1(S^1, a)$. Clearly, $prj^\beta_*(X^\beta) = prj^\beta_*(Y^\beta)$. Therefore, we see that $Y^\beta\left(X^\beta\right)^{-1}$is an element of the kernel of prj^β_*. Let A_0 denote the fiber over $a \in S^1$, and $i^\beta: A_0 \to S^1 \times S^1$ be the inclusion map. By the exactness of the following sequence

$$0 \to \pi_1(A_0, a_0) \xrightarrow{i^\beta_*} \pi_1(S^1 \times S^1, a_0) \xrightarrow{prj^\beta_*} \pi_1(S^1, a), \tag{2}$$

we get

$$i^\beta_*\left(w^\beta\right) = Y^\beta\left(X^\beta\right)^{-1}$$

for some unique $w^\beta \in \pi_1(A_0, a_0)$. Indeed, the exactness of the sequence (2) implies that i^β_* is one to one and there exists an element w^β of $\pi_1(A_0, a_0)$ such that the image of w^β is $Y^\beta\left(X^\beta\right)^{-1}$, which is an element of the kernel of prj^β_*. Because of the injectivity of i^β_*, w^β is unique. A choice of the orientation of $T_t\Sigma_{g,1}$ for $t \in \Sigma_{g,1}$ gives us an orientation of A_0. Hence, we can regard w^β as an integer, which is defined to be the winding number $wind(\gamma, X)$ of γ with respect to X.

Difference Cocycle

In this subsection, the difference cocycle introduced by Chillingworth [11] is constructed. To see that we get the same integer in the image of the difference cocycle for the different smooth representatives of a homology class, we present an example.

For a Riemannian metric on $\Sigma_{g,1}$ and two nonvanishing vector fields X_1 and X_2 on $\Sigma_{g,1}$, one can define sections $\widetilde{X_1}$ and $\widetilde{X_2}$ from $\Sigma_{g,1}$ to $T^1\Sigma_{g,1}$ by $\widetilde{X_1}(t) \coloneqq X_1(t)/\|X_1(t)\|_t$ and $\widetilde{X_2}(t) \coloneqq X_2(t)/\|X_2(t)\|_t$ for $t \in \Sigma_{g,1}$. Let $\gamma = \beta(S^1)$ be an oriented closed curve and let us consider the commutative diagram (1) exhibited in the construction of the winding number. Suppose that $X_1(t) = X_2(t)$ for some $t \in \Sigma_{g,1}$. The composition maps $\widetilde{X_1}\beta$ and $\widetilde{X_2}\beta$ pull back to unique sections $X^\beta_1, X^\beta_2: S^1 \to S^1 \times S^1$, respectively. From the exact sequence (2), we obtain $i^\beta_*\left(u^\beta\right) = X^\beta_1\left(X^\beta_2\right)^{-1}$ for a unique $u^\beta \in \pi_1(A_0, a_0) \cong \mathbb{Z}$ and u^β can be identified with an integer which is the total number of times that X_1 rotates relative to the X_2 as γ traversed once.

Definition [11] Let $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$ be smooth simple closed curves on $\Sigma_{g,1}$, based at t, and their homotopy classes generate $\pi_1(\Sigma_{g,1}, t)$. Then $\{[\gamma_1], [\gamma_2], \ldots, [\gamma_{2g}]\}$ form a basis of $H_1(\Sigma_{g,1})$. A difference cocycle is a homomorphism

$$d(X_1, X_2): H_1\left(\Sigma_{g,1}\right) \longrightarrow \mathbb{Z}$$

defined by $d(X_1, X_2)[\gamma_i] = u^{\beta_i}$, where $\gamma_i = \beta_i(S^1)$.

Given an oriented closed curve $\gamma = \beta(S^1)$, we have $u^\beta = d(X_1, X_2)[\gamma]$. Because γ is homotopic to a product of γ_i or γ_i^{-1}.

Chillingworth [11] obtains that $d(X_1, X_2)[\gamma] = wind(\gamma, X_2) - wind(\gamma, X_1)$ for a smooth oriented closed curve γ.

Notice that the image of $[\gamma]$ under $d(X_1, X_2)$ does not change for different choices of representatives. To be clear, let us consider the following example.

Example

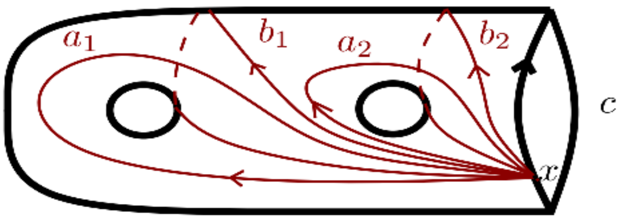

Figure 1
Orientable genus-2 surface with one boundary.

Consider a cellular decomposition of $\Sigma_{2,1}$. Let x be a 0-cell on the boundary of $\Sigma_{2,1}$ and a_1, b_1, a_2, b_2, c be 1-cells as depicted in Figure 1. The unique 2-cell $\Sigma_{2,1} \backslash \{a_1, b_1, a_2, b_2, c\}$ will be denoted by A. Let $< x >$, $< a_1, b_1, a_2, b_2, c >$, and $< A >$ be the free abelian groups generated by $\{x\}$, $\{a_1, b_1, a_2, b_2, c\}$, and $\{A\}$, respectively. We have the following cellular chain complex:

$$0 \longrightarrow < A > \xrightarrow{\partial_2} < a_1, b_1, a_2, b_2, c > \xrightarrow{0} < x > \longrightarrow 0.$$

In the above chain complex ∂_2 denotes the boundary map and the image of it is $\partial_2(A) = c$.

For two nonvanishing vector fields X_1 and X_2 on $\Sigma_{2,1}$, $d(X_1, X_2)$ is defined on the free abelian group $< a_1, b_1, a_2, b_2, c >$ and its image is an integer. Now, we aim to show that it induces a homomorphism from $H_1(\Sigma_{2,1})$ to $\mathbb{Z}$. Therefore, we need to get $d(X_1, X_2)(c) = 0$, because the boundary of the 2-cell A is c.

By Lemma 5.7 in [11], we have $\text{wind}(c, X_1) = \text{wind}(c, X_2)$. By Lemma 4.1 in [11], $d(X_1, X_2)(c) = wind(c, X_2) - wind(c, X_1) = 0$ is obtained. Similarly, one can show that $d(X_1, X_2)(\gamma) = 0$ for any separating curve γ. By the universal property of quotient groups, we get a homomorphism $d(X_1, X_2)\colon H_1(\Sigma_{2,1}) \longrightarrow \mathbb{Z}$, as desired.

MAIN RESULT

This section is devoted to two crossed homomorphisms defined by Trapp and proposed by Furuta, and it is shown that these definitions are equivalent.

Trapp [7] defined a map

$$\tau_X\colon MCG(\Sigma_{g,1}) \longrightarrow H^1(\Sigma_{g,1})$$

such that $\tau_X(f)[\gamma] = wind(f\gamma, X) - wind(\gamma, X)$ for any $f \in MCG(\Sigma_{g,1})$, $[\gamma] \in H_1(\Sigma_{g,1})$ and a nonvanishing vector field X on $\Sigma_{g,1}$. Let $\rho\colon MCG(\Sigma_{g,1}) \longrightarrow Sp(2g;\, \mathbb{Z})$ be the symplectic representation. Trapp showed that τ_X is a crossed homomorphism by getting the following formula:

$$\tau_X(fh) = \tau_X(f)\rho(h) + \tau_X(h),$$

for any $f, h \in MCG(\Sigma_{g,1})$.

Lemma The cohomology class $\tau_X(f)$ is equal to the difference cocycle $d(X, f^{-1}X)$ for any $f \in MCG(\Sigma_{g,1})$ and a nonvanishing vector field X on $\Sigma_{g,1}$.

Proof. Let $[\gamma] \in H_1(\Sigma_{g,1})$. From the fact that $wind(f\gamma, fX) = wind(\gamma, X)$, we have $wind(f\gamma, X) = wind(\gamma, f^{-1}X)$ and this implies that $\tau_X(f)$ is equal to $d(X, f^{-1}X)$ as follows:

$$\tau_X(f)[\gamma] = wind(f\gamma, X) - wind(\gamma, X) = wind(\gamma, f^{-1}X) - wind(\gamma, X) = d(X, f^{-1}X)[\gamma].$$

We now outline Furuta's crossed homomorphism [8].

Recall that $\Sigma_{g,1}$ is given a Riemannian structure. Let X be a nonvanishing vector field on $\Sigma_{g,1}$ and $f \in MCG(\Sigma_{g,1})$. Since X is nonvanishing, fX is also a nonvanishing vector field. Let S^1 denote the set of angles mod 2π. Let a fixed orientation be chosen on $\Sigma_{g,1}$. Furuta defined a map

$$\psi_f: \Sigma_{g,1} \longrightarrow S^1$$

such that $\psi_f(p) := \angle(X_p, (f\,X)_p)$ is the angle mod 2π from X_p to $(f\,X)_p$. Let the cohomology class $\psi_f^*([S^1]) \in H^1(\Sigma_{g,1})$ be denoted by $[\psi_f]$. Here $[S^1]$ indicates the generator of $H^1(S^1)$. The Poincaré dual of $[\psi_f]$ gives us a homology class $m_X(f) \in H_1(\Sigma_{g,1})$. In summary, Furuta obtained a map m_X: $\mathrm{MCG}(\Sigma_{g,1}) \longrightarrow H_1(\Sigma_{g,1})$ depending on X. In [8], it is proved that $m_X: MCG(\Sigma_{g,1}) \longrightarrow H_1(\Sigma_{g,1})$ is a crossed homomorphism. It is also shown that its cohomology class $[m_X]$ is independent of X and is a generator of $H^1(MCG(\Sigma_{g,1}); H_1(\Sigma_{g,1}))$. Moreover, it can be seen that the map $\Psi_X: MCG(\Sigma_{g,1}) \longrightarrow H^1(\Sigma_{g,1})$ defined by $\Psi_X(f) = [\psi_f]$ is also a crossed homomorphism.

Now our task is to give the relation between these two above crossed homomorphisms.

Theorem Let X be a nonvanishing vector field on $\Sigma_{g,1}$. Then we have $\tau_X(f)[\gamma] = \Psi_X(f^{-1})[\gamma] = [\psi_{f^{-1}}][\gamma]$ for any $f \in MCG(\Sigma_{g,1})$ and $[\gamma] \in H_1(\Sigma_{g,1})$.

Proof. Let $\beta: S^1 \to \Sigma_{g,1}$ be a smooth closed oriented curve with $\beta(S^1) = \gamma$ and continuously varying non-zero tangents exist at all points of γ. Consider the commutative diagram (1) which is depicted in the winding number subsection.

Let $\tilde{X}$ and $\widetilde{f^{-1}X}$ be the unit vector fields induced by X and $f^{-1}X$, respectively. There exist sections X^β and $(f^{-1}X)^\beta$ such that $F \circ X^\beta = \tilde{X} \circ \beta$ and $F \circ (f^{-1}X)^\beta = \widetilde{f^{-1}X} \circ \beta$. Suppose that $\tilde{X}$ and $\widetilde{f^{-1}X}$ rotate k and l-times, respectively, around the fiber on $T^1\Sigma_{g,1}$ restricted to γ. Then by the construction of the difference cocycle, X^β and $(f^{-1}X)^\beta$ are homotopic to maps sending θ to $(\theta, k\theta)$ and θ to $(\theta, l\theta)$, respectively. After the compositions of X^β and $(f^{-1}X)^\beta$ with the second projection map $pr_2: S^1 \times S^1 \to S^1$, we get $pr_2 \circ X^\beta: S^1 \to S^1$ taking θ to $k\theta$ and $pr_2 \circ (f^{-1}X)^\beta: S^1 \to S^1$ taking θ to $l\theta$. Therefore, the image of $[\gamma]$ under $d(X, f^{-1}X)$ can be regarded as the degree of the map $S^1 \to S^1$ sending θ to $(k-l)\theta$.

Now recall the map $\psi_{f^{-1}}: \Sigma_{g,1} \to S^1$ defined by $\psi_{f^{-1}}(p) = \angle(X)_p, (f^{-1}X)_p)$. From the restriction of the map $\psi_{f^{-1}}$ to γ we have a map denoted by $\psi_{f^{-1}} \circ \beta: S^1 \to S^1$. Since we assumed that $\tilde{X}$ and $\widetilde{f^{-1}X}$ rotate k and l-times, respectively, around the fiber on $T^1\Sigma_{g,1}$ restricted to γ, by the definition of $\psi_{f^{-1}}$ the composition map $\psi_{f^{-1}} \circ \beta$ is homotopic to a map sending θ to $(k-l)\theta$. Therefore, we have $d(X, f^{-1}X)[\gamma] = \deg(\psi_{f^{-1}} \circ \beta)$.

Our next step is to show the equality $[\psi_{f^{-1}}][\gamma] = \deg(\psi_{f^{-1}} \circ \beta)$ in an explicit way. After identifying $H^1(\Sigma_{g,1})$ with $\mathrm{Hom}(H_1(\Sigma_{g,1}), \mathbb{Z})$, the cohomology class $[\psi_{f^{-1}}]$ is obtained from the induced map $(\psi_{f^{-1}})_*$. Indeed, the induced map $(\psi_{f^{-1}})_*: H_1(\Sigma_{g,1}) \to H_1(S^1)$ will give us a cohomology class after identifying $H_1(S^1)$ with $\mathbb{Z}$. Let us consider the following sequence:

$$H_1(S^1) \overset{\beta_*}{\to} H_1(\Sigma_{g,1}) \xrightarrow{(\psi_{f^{-1}})_*} H_1(S^1) \cong \mathbb{Z}.$$

The isomorphism in the above sequence comes from the cohomology class $[S^1] \in H^1(S^1)$. If $[a]$ is a generator of $H_1(S^1)$, we have

$$\left(\psi_{f^{-1}} \circ \beta\right)_*([a]) = \left(\psi_{f^{-1}}\right)_* \circ \beta_*([a]) = \left(\psi_{f^{-1}}\right)_*([\gamma]).$$

Consequently, we obtain

$$[\psi_{f^{-1}}][\gamma] = \left(\psi_{f^{-1}}\right)_*([\gamma]) = \left(\psi_{f^{-1}} \circ \beta\right)_*([a]) = \deg\left(\psi_{f^{-1}} \circ \beta\right) = d(X, f^{-1}X)[\gamma] = \tau_X(f)[\gamma],$$

for any smooth representative γ.

Corollary Let X be a nonvanishing vector field on $\Sigma_{g,1}$ and $f \in MCG(\Sigma_{g,1})$. If $n_X(f)$ is the Poincaré dual of $\tau_X(f)$ defined by Trapp, then $n_X: MCG(\Sigma_{g,1}) \to H_1(\Sigma_{g,1})$ depends only on X and it is also a crossed homomorphism. Moreover, the cohomology class $[n_X] \in H^1(MCG(\Sigma_{g,1}); H_1(\Sigma_{g,1}))$ is independent of X.

Proof. Recall that the Poincaré dual of $[\psi_{f^{-1}}]$ is $m_X(f^{-1}) \in H_1(\Sigma_{g,1})$ and the cohomology class $[m_X]$ is a generator of $H^1(MCG(\Sigma_{g,1}); H_1(\Sigma_{g,1}))$. In [8], it is also proved that m_X is a crossed homomorphism. By the previous theorem, we obtain $[\psi_{f^{-1}}][\gamma] = \tau_X(f)[\gamma]$. Therefore, the homology class $n_X(f)$ is equal to $m_X(f^{-1})$, which completes the proof.

DISCUSSION AND CONCLUSIONS

In this study, we obtain the equivalence of two crossed homomorphisms defined by Trapp and proposed by Furuta. Morita was aware of the equivalence, but to our knowledge, there is no explicit proof showing this equivalence. We fill this gap with this study. In the construction of difference cocycles, algebraic and geometric features are used. Moreover, crossed homomorphisms and winding numbers can be applied in other areas of mathematics such as Banach spaces. Therefore, we can consider this subject as an interdisciplinary subject in mathematics [12-14]. For future work, it can be constructed other crossed homomorphisms on the mapping class group of an orientable surface that contains more than one boundary component with the help of difference cocycles.

Ethical Statement

This study is derived from the Ph.D thesis titled "Generalized Chillingworth Classes on Subsurface Torelli Groups", submitted in August 2018 under the supervision of Prof. Dr. Mustafa KORKMAZ. It has also been developed and partially modified from the content of the paper titled "Crossed Homomorphisms on The Mapping Class Group", which was presented orally at the 6th Workshop of Association for Turkish Women in Maths but was not published in full text.

Author Contributions

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