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A PHYSICAL APPROACH TO LAGRANGIAN EQUATIONS WITH BUNDLE STRUCTURE FOR MINKOWSKI 3-SPACE

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Abstract. Minkowski 3-space is important because it is the spatial structure in which physical events occur. Therefore, in this study, we examined the time-dependent Lagrangian energy equations in Minkowski 3-space. To facilitate the examination of the necessary mechanical structure and solutions with respect to time, we employed the jet bundle structure. This approach has made the creation of the necessary geometric structures more comprehensible in terms of coordinate basis. To interpret the obtained Lagrangian energy equations and to understand the significance of the time parameter, an example is also provided in our study.

1. Introduction

We worked on the Lagrangian energy structure previously studied in Euclidean space by establishing the necessary mechanical structure in Minkowski space. This energy structure had not been obtained before in 3-dimensional Minkowski space.The structure of Minkowski metric is different from Euclid metric. Also for this difference, it can be seen no studies in this subject. Mathematicians, working in Minkowski space, believe that there is natural phenomenon with Minkowski geometry for explaining physical phenomena occuring in 3-dimensional Euclid space. To obtain the time-dependent energy structure, we constructed the jet bundle structure in 3-dimensional Minkowski space and based our study on the coordinate system of this bundle structure.
The jet bundles can be classfied into two manifold:
1) Total complex manifold,
2) Phase manifold.
The inclusion of time-dependent differential coordinates in the jet bundle structure makes it more suitable for the creation of time-dependent mechanical systems. The consraint, real, complex and Para-complex structures on the time-dependent Lagrangian systems can be researched in [2] and [5]. Then found that in the paper

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[1] Lagrangian and Hamiltonian mechanical systems were instructed on the vector bundle structures and jet bundle forms.Lagrangian equations are solved with real bundles by [2] , [3], [6], [5]. As shown in the studies presented in the references, none of them have been conducted in Minkowski space. Therefore, we chose to carry out our work in Minkowski space as a distinctive approach.
Let TQ be the tangent bundle of an m-dimensional configuration manifold Q. Given a Lagrangian energy function $L : TQ \to R$, the vector field X satisfying the energy equation

$$i_{X_L} w_L = dE_L \tag{1}$$

is unique. Here, w is a 2-form on the bundle T^*Q , and E_L represents the energy associated with L.[6], [5] If the families of curves that are solutions to the above energy equation on TQ are integral curves of the vector field X , then the vector field X is called a semi-spray.
The triple (TQ, w_L, L) is called Lagrangian system on the tangent bundle TQ. [6], [5]
Let, $L{:}R{\times}TQ{=}J\left(R,Q\right){\to}R$ and $TQ{=}\left\{t\right\}{\times}TQ$ be Lagrangian function. The coordinate system on TQ is $\{q_i, v_i\}$.
The Poincare cartan 1-form on the T^*Q associated with L Lagrangian energy function is

$$\alpha_L{=}d_J L{+}Ldv_i \ = \frac{\partial L}{\partial v_i}\ dq_i{+}Ldv_i$$

The Poincare cartan 2-form associated with L Lagrangian energy function is

$$\Omega_L{=}dd_j L{+}dL \wedge Ldv_i$$

If the paths of semisprays verify

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

then this is called as Euler-Lagrange equation.
[1], proved that equation (2) is not changed; but, this study investigates the possible shapes of Euler-Lagrange system (2) on jet bundles.

Bundles on Minkowski 3-Space

A bundle is a triple (E, π, M) where E and M are manifolds and $\pi{:}E \to M$ is a surjective submersion. E is called the total space, π, the projection and M the base space. This bundle denoted by π or E. The first jet manifold of π is the set $\left\{J^1_p\phi{:}p \in M, \phi \in \Gamma_p\left(\pi\right)\right\}$ and denoted by J^1E. Here, ϕ is a map $\phi{:}M \to E$ and is called a section of π. If it is satisfies the condition $\pi \circ \phi{=}id_M$, then the set of all sections of π will be denoted $\Gamma\left(\pi\right)$.
Let (E, π, M) a bundle and let (U, u) be an adapted coordinate system on E, where $u{=}\ (x, y, z, u_\alpha)$. The induced coordinate system (U^1, u^1) on J^1E is defined by

$$U^1{=}\left\{J^1_p\phi{:}\phi(p) \in U\right\}$$

$$u^1{=}\ (x, y, z, u_\alpha, u^i_\alpha)$$

where $x(J^1_p\phi) = x(p)$,$y(J^1_p\phi) = y(p)$,$z(J^1_p\phi) = z(p)$, $u_\alpha(J^1_p\phi) = u_\alpha(\phi m)$ and are known as derivative coordinates.[1]
In this study, we consider the bundle structure $\left(E^3_1, \pi, R\right)$. The coordinates of the manifold $E^3_1$are (x, y, z), the coordinate of the manifold R is (t). Also, the coordinates of the manifold $J^1E^3_1$ are $(t, x, y, z, x^\cdot, y^\cdot, z^\cdot)$.

Here derivative coordinates denoted by

$$\dot{x} = \frac{dx}{dt}$$

$$\dot{y} = \frac{dy}{dt}$$

$$\dot{z} = \frac{dz}{dt}$$

Lagrangian Mechanical Systems of Minkowski Space with Bundle Structure

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = -dx^2 + dy^2 + dz^2$$

where (x, y, z) is a rectangular coordinate system of E^3. Let g be the Minkowski metric, and let $v \in E_1^3$ be,

- $g(v, v) > 0$ or $g(v) = 0$, v is spacelike vector
- $g(v, v) < 0$, v is timelike vector
- $g(v, v) = 0$ and $v \neq 0$, v is null(lightlike)

A similar analysis can be performed within an α curve on E_1^3 .
τ is the set of all time-like vectors in E_1^3. For $\forall u \in \tau$; the set

$$C(\overrightarrow{u}) = \{\overrightarrow{x} \in \tau : \langle \overrightarrow{u}, \overrightarrow{x} \rangle < 0\} = \left\{\overrightarrow{x} \in E_1^3 : g(x-u, x-u) < 0\right\}$$

defined as timecone.

Theorem 1. The time-like vectors $\overrightarrow{\mathrm{x}}$ and $\overrightarrow{\mathrm{y}}$ in Minkowski 3-space E_1^3 are in the same timecone,

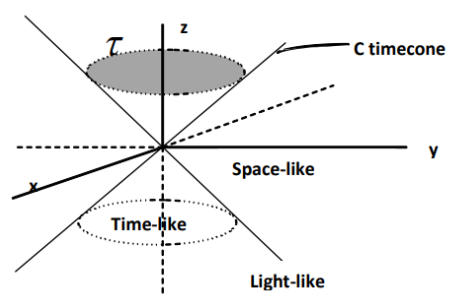

FIGURE 1

$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -\|\overrightarrow{x}\| \|\overrightarrow{y}\| \cosh\theta$ and here θ is the *Lorentz timelike angle* between $\overrightarrow{x}$ and $\overrightarrow{y}$ vectors.

Definition 1.1. A (1,1)- type tensor field J that satisfies the $J^2=0$ condition is approximately called a tangent structure. Here, $J : T\left(J^1E_1^3\right) \longrightarrow T\left(J^1E_1^3\right)$ *is*

$$J\left(\frac{\partial}{\partial x}\right)=-\frac{\partial}{\partial x^{\cdot}},\quad J\left(\frac{\partial}{\partial y}\right)=\frac{\partial}{\partial y^{\cdot}},\quad J\left(\frac{\partial}{\partial z}\right)=\frac{\partial}{\partial z^{\cdot}}=J\left(\frac{\partial}{\partial z^{\cdot}}\right)$$

$$J\left(\frac{\partial}{\partial x^{\cdot}}\right)=J\left(\frac{\partial}{\partial y^{\cdot}}\right)=J\left(\frac{\partial}{\partial z^{\cdot}}\right)=0$$

(3) $$J\left(\frac{\partial}{\partial t}\right)=-\dot{x}\frac{\partial}{\partial x^{\cdot}}+\dot{y}\frac{\partial}{\partial y^{\cdot}}+\dot{z}\frac{\partial}{\partial z^{\cdot}}$$

J can be calculated as a tensor field from (3), as

(4) $$J=\left(-dx-\dot{x}\,dt\right)\times\frac{\partial}{\partial x^{\cdot}}+\left(dy+\dot{y}\,dt\right)\times\frac{\partial}{\partial y^{\cdot}}+\left(dz+\dot{z}\,dt\right)\times\frac{\partial}{\partial z^{\cdot}}$$

A semi-spray is a vector field over E_1^3 and defined as below;

(5) $$\varepsilon=\frac{\partial}{\partial t}-\dot{x}\frac{\partial}{\partial x}+\dot{y}\frac{\partial}{\partial y}+\dot{z}\frac{\partial}{\partial z}-\varepsilon_1\frac{\partial}{\partial x^{\cdot}}+\varepsilon_2\frac{\partial}{\partial y^{\cdot}}+\varepsilon_3\frac{\partial}{\partial z^{\cdot}}$$

By calculate $J(\varepsilon)$, then equation (6) are found

(6) $$V=J\varepsilon=-2\dot{x}\frac{\partial}{\partial x^{\cdot}}+2\dot{y}\frac{\partial}{\partial y^{\cdot}}+2\dot{z}\frac{\partial}{\partial z^{\cdot}}$$

which is called "*Liouville vector field*"
Moreover, "Poincare-Cartan 1-form" is written as:

$$\alpha L=d_JL+Ldt$$

(7) $$\alpha L=-\dot{x}\frac{\partial L}{\partial x^{\cdot}}dt+\dot{y}\frac{\partial L}{\partial y^{\cdot}}dt+\dot{z}\frac{\partial L}{\partial z^{\cdot}}dt-\frac{\partial L}{\partial x^{\cdot}}dx+\frac{\partial L}{\partial y^{\cdot}}dy+\frac{\partial L}{\partial z^{\cdot}}dz+Ld$$

Then we can write differential operator d,

(8) $$d=\frac{\partial}{\partial t}dt-\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz-\frac{\partial}{\partial x^{\cdot}}d\dot{x}+\frac{\partial}{\partial y^{\cdot}}d\dot{y}+\frac{\partial}{\partial z^{\cdot}}d\dot{z}$$

By using the differentiation d (8), then Poincare-Cartan 2-form is obtained.

$$\Omega_L=dd_JL+dL\wedge dt$$

$$\begin{aligned}
\Omega_L=(dx\wedge dt)&\left(\frac{\partial^2L}{\partial t\partial\dot{x}}+\dot{x}\frac{\partial^2L}{\partial x\partial\dot{x}}-\dot{y}\frac{\partial^2L}{\partial x\partial\dot{y}}-\dot{z}\frac{\partial^2L}{\partial x\partial\dot{z}}-\frac{\partial L}{\partial x}\right)\\
+(dy\wedge dt)&\left(-\frac{\partial^2L}{\partial t\partial\dot{y}}-\dot{x}\frac{\partial^2L}{\partial y\partial\dot{x}}+\dot{y}\frac{\partial^2L}{\partial y\partial\dot{y}}+\dot{z}\frac{\partial^2L}{\partial y\partial\dot{z}}-\frac{\partial L}{\partial y}\right)\\
+(dz\wedge dt)&\left(-\frac{\partial^2L}{\partial t\partial\dot{z}}-\dot{x}\frac{\partial^2L}{\partial z\partial\dot{x}}+\dot{y}\frac{\partial^2L}{\partial z\partial\dot{y}}+\dot{z}\frac{\partial^2L}{\partial z\partial\dot{z}}-\frac{\partial L}{\partial z}\right)\\
+\left(d\dot{x}\wedge dt\right)&\left(\dot{x}\frac{\partial^2L}{\partial\dot{x}^2}-\dot{y}\frac{\partial^2L}{\partial\dot{x}\partial\dot{y}}-\dot{z}\frac{\partial^2L}{\partial\dot{x}\partial\dot{z}}\right)\\
+\left(d\dot{y}\wedge dt\right)&\left(-\dot{x}\frac{\partial^2L}{\partial\dot{x}\partial\dot{y}}+\dot{y}\frac{\partial^2L}{\partial\dot{y}^2}+\dot{z}\frac{\partial^2L}{\partial\dot{y}\partial\dot{z}}+2\frac{\partial L}{\partial\dot{y}}\right)\\
+\left(d\dot{z}\wedge dt\right)&\left(-\dot{x}\frac{\partial^2L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2L}{\partial\dot{z}^2}+2\frac{\partial L}{\partial\dot{z}_3}\right)
\end{aligned}$$

$$
\begin{aligned}
&+(dx\wedge dy)\left(-\frac{\partial^2 L}{\partial x\partial \dot{y}}+\frac{\partial^2 L}{\partial y\partial \dot{x}}\right)+(dx\wedge dz)\left(-\frac{\partial^2 L}{\partial x\partial \dot{z}}+\frac{\partial^2 L}{\partial z\partial \dot{x}}\right)\\
&+(dy\wedge dz)\left(-\frac{\partial^2 L}{\partial y\partial \dot{z}}+\frac{\partial^2 L}{\partial z\partial \dot{y}}\right)+\left(dx\wedge d\dot{x}\right)\left(-\frac{\partial^2 L}{\partial \dot{x}^2}\right)\\
&+\left(dx\wedge d\dot{y}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}\right)+\left(dx\wedge d\dot{z}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\right)\\
&+\left(dy\wedge d\dot{x}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}\right)+\left(dy\wedge d\dot{y}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}^2}\right)\\
&+\left(dy\wedge d\dot{z}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}\right)+\left(dz\wedge d\dot{x}\right)\left(\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\right)\\
&+\left(dz\wedge d\dot{y}\right)\left(-\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}\right)+\left(dz\wedge d\dot{z}\right)\left(-\frac{\partial^2 L}{\partial \dot{z}^2}\right)
\end{aligned}
$$

(9)

Definition 1.2. Solutions of the Euler-Lagrange equation can be found by assuming

$$
i_\varepsilon \Omega_L=\Omega_L(\varepsilon)=0
$$

$$
\begin{aligned}
i_\varepsilon \Omega_L&=\Omega_L(\varepsilon)\\
=&-\Big(\frac{\partial^2 L}{\partial t\partial \dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{x}}-\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}}-\frac{\partial L}{\partial x}\\
&-\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}}+\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{x}}-\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}}+\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{x}}\\
&+\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}^2}+\varepsilon_2\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}+\varepsilon_3\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\Big)dx\\
&-\Big(-\frac{\partial^2 L}{\partial t\partial \dot{y}}-\dot{x}\frac{\partial^2 L}{\partial y\partial \dot{x}}+\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial y\partial \dot{z}}+\frac{\partial L}{\partial y}\\
&-\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{y}}+\dot{x}\frac{\partial^2 L}{\partial y\partial \dot{x}}-\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial y\partial \dot{z}}\\
&-\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\varepsilon_2\frac{\partial^2 L}{\partial \dot{y}^2}-\varepsilon_3\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}_3}\Big)dy\\
&-\Big(-\frac{\partial^2 L}{\partial t\partial \dot{z}}-\dot{x}\frac{\partial^2 L}{\partial z\partial \dot{x}}+\dot{y}\frac{\partial^2 L}{\partial z\partial \dot{y}}+\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{z}}+\frac{\partial L}{\partial z}\\
&-\dot{x}\frac{\partial^2 L}{\partial x\partial \dot{z}}+\dot{x}\frac{\partial^2 L}{\partial z\partial \dot{x}}-\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{z}}+\dot{y}\frac{\partial^2 L}{\partial z\partial \dot{y}}\\
&-\varepsilon_1\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}-\varepsilon_2\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}-\varepsilon_3\frac{\partial^2 L}{\partial \dot{z}^2}\Big)dz\\
&-\Big(\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2}-\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\\
&-\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2}-\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}-\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}}\Big)d\dot{x}\\
&-\Big(-\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}}+\dot{y}\frac{\partial^2 L}{\partial \dot{y}^2}+\dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}}+2\frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
&+\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}^2}+\dot{z}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}})d\dot{y}\\
&-(-\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2}+2\frac{\partial L}{\partial\dot{z}}\\
&+\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2})d\dot{z}\\
&+(-\dot{x}\frac{\partial^2 L}{\partial t\partial\dot{x}}-\dot{x}^2\frac{\partial^2 L}{\partial x\partial\dot{x}}+\dot{x}\dot{y}\frac{\partial^2 L}{\partial x\partial\dot{y}}+\dot{x}\dot{z}\frac{\partial^2 L}{\partial x\partial\dot{z}}+\dot{x}\frac{\partial L}{\partial x}\\
&-\dot{y}\frac{\partial^2 L}{\partial t\partial\dot{y}}-\dot{x}\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{y}^2\frac{\partial^2 L}{\partial y\partial\dot{y}}+\dot{z}\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{z}}+\dot{y}\frac{\partial L}{\partial y}\\
&-\dot{z}\frac{\partial^2 L}{\partial t\partial\dot{z}}-\dot{x}\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{x}}+\dot{y}\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{y}}+\dot{z}^2\frac{\partial^2 L}{\partial z\partial\dot{z}}+\dot{z}\frac{\partial L}{\partial z}\\
&-\varepsilon_1\dot{x}\frac{\partial^2 L}{\partial\dot{x}^2}+\varepsilon_1\dot{y}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\varepsilon_1\dot{z}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}\\
&-\varepsilon_2\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{y}}+\varepsilon_2\dot{y}\frac{\partial^2 L}{\partial\dot{y}^2}+\varepsilon_2\dot{z}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}\\
&-\varepsilon_3\dot{x}\frac{\partial^2 L}{\partial\dot{x}\partial\dot{z}}+\varepsilon_3\dot{y}\frac{\partial^2 L}{\partial\dot{y}\partial\dot{z}}+\varepsilon_3\dot{z}\frac{\partial^2 L}{\partial\dot{z}^2}+2\varepsilon_2\frac{\partial L}{\partial\dot{y}}+2\varepsilon_3\frac{\partial L}{\partial\dot{z}})dt
\end{aligned}
$$

(10)

By equalizing equation (10) to zero, then (11) are obtained.

$$
\begin{aligned}
I\quad:\quad 0=&-\frac{\partial^2 L}{\partial t\partial\dot{x}}+\frac{\partial}{\partial x}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(\dot{y}\frac{\partial^2 L}{\partial x\partial\dot{y}}-\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{z}\frac{\partial^2 L}{\partial x\partial\dot{z}}-\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{x}})\\
&-\frac{\partial}{\partial\dot{x}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
II\quad:\quad 0=&\frac{\partial^2 L}{\partial t\partial\dot{y}}-\frac{\partial}{\partial y}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(-\dot{x}\frac{\partial^2 L}{\partial y\partial\dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial\dot{y}}-\dot{z}\frac{\partial^2 L}{\partial y\partial\dot{z}}+\dot{z}\frac{\partial^2 L}{\partial z\partial\dot{y}}\\
&+\frac{\partial}{\partial\dot{y}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
III\quad:\quad 0=&\frac{\partial^2 L}{\partial t\partial\dot{z}}-\frac{\partial}{\partial z_3}(-\dot{x}\frac{\partial L}{\partial\dot{x}}+\dot{y}\frac{\partial L}{\partial\dot{y}}+\dot{z}\frac{\partial L}{\partial\dot{z}}+L)\\
&+(-\dot{x}\frac{\partial^2 L}{\partial z\partial\dot{x}}+\dot{x}\frac{\partial^2 L}{\partial x\partial\dot{z}}-\dot{y}\frac{\partial^2 L}{\partial z\partial\dot{y}}+\dot{y}\frac{\partial^2 L}{\partial y\partial\dot{z}})\\
&+\frac{\partial}{\partial\dot{z}}(\varepsilon_1\frac{\partial L}{\partial\dot{x}}+\varepsilon_2\frac{\partial L}{\partial\dot{y}}+\varepsilon_3\frac{\partial L}{\partial\dot{z}})
\end{aligned}
$$

$$
\begin{aligned}
IV \quad &: \quad 0 = -\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} + \frac{\partial L}{\partial \dot{x}} \\
&\quad + \dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \frac{\partial L}{\partial \dot{x}} \\
&\quad \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
V \quad &: \quad 0 = \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}} \\
&\quad - \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} - \dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \frac{\partial L}{\partial \dot{y}}
\end{aligned}
$$

$$
\begin{aligned}
VI \quad &: \quad 0 = \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} - \frac{\partial L}{\partial \dot{z}} \\
&\quad - \dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} - \dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} - \dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} - \frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

$$
\begin{aligned}
VII \quad &: \quad 0 = -\dot{x}\frac{\partial^2 L}{\partial t\partial \dot{x}} - \dot{x}^2\frac{\partial^2 L}{\partial x\partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial x\partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial x\partial \dot{z}} + \dot{x}\frac{\partial L}{\partial x} \\
&\quad - \dot{y}\frac{\partial^2 L}{\partial t\partial \dot{y}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{x}} + \dot{y}^2\frac{\partial^2 L}{\partial y\partial \dot{y}} + \dot{z}\dot{y}\frac{\partial^2 L}{\partial y\partial \dot{z}} + \dot{y}\frac{\partial L}{\partial y} \\
&\quad - \dot{z}\frac{\partial^2 L}{\partial t\partial \dot{z}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{x}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial z\partial \dot{y}} + \dot{z}^2\frac{\partial^2 L}{\partial z\partial \dot{z}} + \dot{z}\frac{\partial L}{\partial z} \\
&\quad - \varepsilon_1\dot{x}\frac{\partial^2 L}{\partial \dot{x}^2} + \varepsilon_1\dot{y}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \varepsilon_1\dot{z}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} \\
&\quad - \varepsilon_2\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} + \varepsilon_2\dot{y}\frac{\partial^2 L}{\partial \dot{y}^2} + \varepsilon_2\dot{z}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} \\
&\quad - \varepsilon_3\dot{x}\frac{\partial^2 L}{\partial \dot{x}\partial \dot{z}} + \varepsilon_3\dot{y}\frac{\partial^2 L}{\partial \dot{y}\partial \dot{z}} + \varepsilon_3\dot{z}\frac{\partial^2 L}{\partial \dot{z}^2} \\
&\quad + 2\varepsilon_2\frac{\partial L}{\partial \dot{y}} + 2\varepsilon_3\frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

(11)

This equation (11) represents a non-linear equations system. By assuming

$$\varepsilon_1 = -\dot{x}, \quad \varepsilon_2 = \dot{y}, \quad \varepsilon_3 = \dot{z} \tag{12}$$

In this approach, it must be a negative term, because Minkowski metric is negatie definitly. Then following equalities can be written as below;

$$\dot{x}(I) - \dot{y}(II) - \dot{z}(III) + \dot{x}(IV) - \dot{y}(V) - \dot{z}(VI) + (VII) = 0 \tag{13}$$

Solving (13) lead to the equation

$$
\begin{aligned}
VIII \quad : \quad 0 = & -\dot{x}\frac{\partial^2 L}{\partial t \partial \dot{x}} - \dot{y}\frac{\partial^2 L}{\partial t \partial \dot{y}} - \dot{z}\frac{\partial^2 L}{\partial t \partial \dot{z}} \\
& -\dot{x}^2\frac{\partial^2 L}{\partial x \partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial x \partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial x \partial \dot{z}} + \dot{x}\frac{\partial L}{\partial x} \\
& -\dot{x}\dot{y}\frac{\partial^2 L}{\partial y \partial \dot{x}} + \dot{y}^2\frac{\partial^2 L}{\partial y \partial \dot{y}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial y \partial \dot{z}} + \dot{y}\frac{\partial L}{\partial y} \\
& -\dot{x}\dot{z}\frac{\partial^2 L}{\partial z \partial \dot{x}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial z \partial \dot{y}} + \dot{z}^2\frac{\partial^2 L}{\partial z \partial \dot{z}} + \dot{z}\frac{\partial L}{\partial z} \\
& +\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} + \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} + \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +\dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} + \dot{y}^2\frac{\partial^2 L}{\partial \dot{y} \partial \dot{y}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial \dot{y} \partial \dot{z}} \\
& +\dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} + \dot{y}\dot{z}\frac{\partial^2 L}{\partial \dot{y} \partial \dot{z}} + \dot{z}^2\frac{\partial^2 L}{\partial \dot{z} \partial \dot{z}} \\
& -\dot{x}^2\frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} - \dot{x}\dot{y}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} - \dot{x}\dot{z}\frac{\partial^2 L}{\partial \dot{x} \partial \dot{z}} \\
& +2\dot{y}\frac{\partial L}{\partial \dot{y}} + 2\dot{z}\frac{\partial L}{\partial \dot{z}}
\end{aligned}
$$

(14)

We can write this equation in a general form. But for this writing, we can assume a notation for negative terms. We denote this notation as follows,

$$
\delta_i = \begin{cases} -1, & ,i = 1 \\ 1 & ,i = 2,3 \end{cases}
$$

Also we can write

$$
\begin{aligned}
& -\frac{\partial}{\partial t}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) - \dot{x}\frac{\partial L}{\partial x}\left(-\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) \\
& +\dot{y}\frac{\partial L}{\partial \dot{y}}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) + \dot{z}\frac{\partial L}{\partial \dot{z}}\left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) \\
& + \left(\dot{x}\frac{\partial L}{\partial x} + \dot{y}\frac{\partial L}{\partial y} + \dot{z}\frac{\partial L}{\partial z}\right) + \left(-\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) + \left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) \\
& + \left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right) = 0
\end{aligned}
$$

$$\dot{x}\frac{\partial L}{\partial \dot{x}}+\dot{y}\frac{\partial L}{\partial \dot{y}}+\dot{z}\frac{\partial L}{\partial \dot{z}}=M$$
$$-\dot{x}\frac{\partial L}{\partial \dot{x}}+\dot{y}\frac{\partial L}{\partial \dot{y}}+\dot{z}\frac{\partial L}{\partial \dot{z}}=N$$
$$\dot{x}\frac{\partial L}{\partial x}+\dot{y}\frac{\partial L}{\partial y}+\dot{z}\frac{\partial L}{\partial z}=P$$

$$(15)\quad -\frac{\partial}{\partial t}(M)-\dot{x}\frac{\partial L}{\partial \dot{x}}(N+P)+\dot{y}\frac{\partial}{\partial \dot{y}}(M+P)+\dot{z}\frac{\partial}{\partial \dot{z}}(M+P)+(N+M+P)=0$$

(15) is the Lagrange equation in Minkowski space . Following examples show an application of equation (14).

Example In this example, we will derive the Lagrangian energy equations for a particle moving within a time cone. We will assume that we use the same approach as outlined above for the solution method. First, let us define a helical curve within a time cone and examine the coordinate structure of this curve. In Figure-1, we show the helical curve within the time cone. Since the curve we are working with remains entirely within the time cone, it is also a timelike curve.

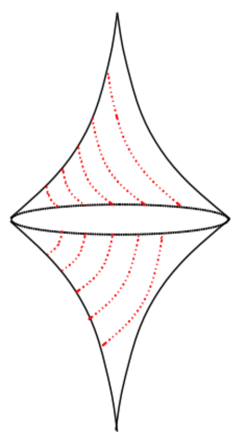

FIGURE 2

This helix is also referred to as a Minkowski helix and is represented as follows.

$$\alpha(\theta)=(r\sinh u\theta, r\cosh u\theta\sin\theta, r\cosh u\theta\cos\theta)$$

Here, $r=r(t)$ is the radius function depending on the time parameter t, θ is fixed angle.

For the reason mentioned above, the α curve is also a timelike curve, and thus $\left\langle \alpha', \alpha' \right\rangle < 0$.

If $\overrightarrow{x}$, $\overrightarrow{y}$ is timelike vector,

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -\|\overrightarrow{x}\| \|\overrightarrow{y}\| \cosh\theta$$

The velocity vector for this curve is

$$\alpha'(\theta)=(ur\cosh u\theta, ur\sinh u\theta\sin\theta + r\cosh u\theta\cos\theta, ur\sinh u\theta\cos\theta - r\cosh u\theta\sin\theta)$$

If this curve is time-like, then it can be

(16) $$\left\langle \alpha^{\prime}(\theta), \alpha^{\prime}(\theta) \right\rangle = r^2 \left(\cosh^2 u\theta - u^2\right)$$

Thus, taking into account the condition provided above,

(17) $$-u < \cosh u\theta < u$$

On the other hand, for the helix, the jet bundle coordinate is

$$(t, r\sinh u\theta, r\cosh u\theta \sin\theta, r\cosh u\theta \cos\theta, \dot{r}\sinh u\theta, \dot{r}\cosh u\theta \sin\theta, \dot{r}\cosh u\theta \cos\theta)$$

(18)

When we rearrange equation (14) according to these bundle coordinates,

(19) $$-3\dot{r}\frac{\partial^2 L}{\partial t \partial r^{\cdot}} + 3\dot{r}^2 \frac{\partial^2 L}{\partial r \partial r^{\cdot}} + 3\dot{r}\frac{\partial L}{\partial r} + 5\dot{r}^2 \frac{\partial^2 L}{\partial r^{\cdot} \partial r^{\cdot}} + 4\dot{r}\frac{\partial L}{\partial r^{\cdot}} = 0$$

we obtain the equation.
Equation (19) is the Lagrangian energy equation for the Minkowski helix.
In this equation, it can be seen that the Lagrangian energy function depends on the parameter r .
Since $r = r(t)$, the energy function L also depends on time.
Also, we consider

$$\frac{dL}{dr^{\cdot}} = \lambda \Rightarrow L = \lambda \dot{r}$$

With calculation the equation (18), we get solution of Lagrange energy function;

(20) $$L = -\frac{4}{3}\lambda r$$

Furthermore, for radius function

$$-\frac{4}{3}\lambda r = \lambda \dot{r}$$

In the solution of the equation, the radius function

$$r = e^{-\frac{4}{3}t}$$

By using this value, the Lagrangian energy is,

(21) $$L = -\frac{4}{3}\lambda e^{-\frac{4}{3}t}$$

From this equation, it can be noticed that radius related by time. Really, with our accaptance, the main parameter is time. The progress of movement is related to time.

2. Conclusion

In this paper, unlike in Euclidean space, we have worked in 3-dimensional Minkowski space. Specifically, a jet bundle structure has been established in this space, and all proofs have been examined using the coordinate system of this bundle structure. The advantage of working with the bundle structure is that it allows us to directly construct the time-dependent mechanical system. Lagrangian energy equations are divided into time-dependent and time-independent categories. In applications and physical contexts, it is more practical to work with the time-dependent equation. Using or not using the time parameter in deriving these two equation structures changes and complicates the entire proof in the classical approach. Our method,

which involves using the jet bundle structure, makes the entire analysis more comprehensible by incorporating time within these bundle coordinates.
In our study, we demonstrated the ease of obtaining the energy function using bundle coordinates within the example structure we presented. As shown in the example, due to the differences in the metric structure of Minkowski space, the curves and vectors studied are categorized as timelike, spacelike, or null. Since we worked with a helix lying within a time cone, we dealt with timelike curves and vectors. However, as can be shown with other examples, the obtained equation (14) can also be applied to spacelike curves and vectors.
The system given in equation (11) is a nonlinear system of equations. Its solution is possible under special conditions. When proving this, we considered the general form of the Lagrangian energy structure based on our previous work in Euclidean space and complex space. Accordingly, by experimenting with all the special conditions that would allow us to obtain the Lagrangian energy equation in Minkowski space, we found that the conditions provided in equation (12) are the most suitable and developed the proof based on these. The negativity here is entirely due to the structure of the Minkowski metric.
As a result of energy equations (21), where time provide to in a big-far time interval or partial movement in all large velocity, Lagrange energy is in a develop state case.

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