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Araştırma Makalesi / Research Article

On the Graphical Analysis of a Fuzzy Problem

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Abstract

The fuzzy Laplace transform method is very useful to solve fuzzy differential equations and this method is an important method in practice. This paper is on a second-order fuzzy problem. In this study, we research the fuzzy problem with negative fuzzy coefficient using the method of fuzzy Laplace transform. Since we use generalized Hukuhara differentiability, solutions are investigated under the four different situations. A numerical example is given. Graphics of the solutions are drawn for alpha level sets. Conclusions are presented at the end of the paper.

Keywords: Second-order fuzzy problem, fuzzy number, generalized Hukuhara differentiability.

Bir Fuzzy Problemin Grafiksel Analizi Üzerine

Öz

Fuzzy Laplace dönüşüm metodu fuzzy diferansiyel denklemleri çözmek için çok kullanışlıdır ve bu metod pratikte önemli bir metoddur. Bu çalışma ikinci mertebeden bir fuzzy problem üzerinedir. Bu çalışmada, fuzzy Laplace dönüşüm metodunu kullanarak negatif fuzzy katsayılı bir fuzzy problemini araştırıyoruz. Genelleştirilmiş Hukuhara diferansiyellenebilirliği kullandığımız için, çözümler dört farklı durum altında incelenmiştir. Sayısal bir örnek verilmiştir. Çözümlerin grafikleri alfa seviye setleri için çizilmiştir. Çalışmanın sonunda sonuçlar sunulmuştur.

Anahtar Kelimeler: İkinci-mertebe fuzzy problem, fuzzy sayısı, genelleştirilmiş Hukuhara diferansiyellenebilirlik.

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1. Introduction

The fuzzy differential equations is an important topic. Fuzzy differential equations are used to model dynamical systems under uncertainty, which is an efficient way. So, the fuzzy differential equations have been growing rapidly. Many authors work on numerical solutions and theoretical solutions of fuzzy differential equations (Akın at al., 2016; Allahviranloo at al., 2007; Bayeğ at al., 2022; Bede at al., 2007; Gültekin Çitil, 2019; Gültekin Çitil, 2020; Ivaz at al., 2013; Jafaria at al., 2021; Mallak at al., 2022; Patel and Desai, 2017; Saqib at al., 2021).

Allahviranloo and Ahmadi (2010) proposed the fuzzy Laplace transform method for solving first order fuzzy differential equations. Salahshour and Haghi (2010) used the method of fuzzy Laplace transform in the study of fuzzy heat equations under the strong generalized Hukuhara differentiability. Using the fuzzy Laplace transform method, the fuzzy harmonic oscillator equation was solved by Salgado et al. (2019). The fuzzy Laplace transform method has been studied in many articles (Belhallaj at al., 2023; Eljaoui and Melliani, 2023; Gültekin Çitil, 2020; Salahshour and Allahviranloo, 2013; Salgado at al., 2021; Samuel and Tahir, 2021).

The aim of this work is to analyze the solutions of fuzzy problem with negative fuzzy coefficients using the fuzzy Laplace transform method.

This paper is organized as follows:

Section 2 is reserved for materials and methods. In Section 3, we first introduce the problem. Then, we investigate the solutions of the problem via the method of the fuzzy Laplace transform. In Section 4, we give numerical example for illustration. In the last section, we present our conclusions.

2. Materials and Methods

Definition 1. $\hat{v} \in \mathbb{R}_F$, where $\mathbb{R}_F$ is all the fuzzy sets.

$$[\hat{v}]^{\alpha} = \left[\underline{\hat{v}}_{\alpha}, \overline{\hat{v}}_{\alpha}\right] = \{x \in \mathbb{R} | \hat{v}(x) \geq \alpha\}$$

is the α-level set of $\hat{v}$, $0 < \alpha \leq 1$ (Khastan and Nieto, 2010).

Definition 2. $\left[\underline{\hat{v}}_{\alpha}, \overline{\hat{v}}_{\alpha}\right]$ satisfy the following conditions:

a) $\underline{\hat{v}}_{\alpha}$ is right-continuous for $\alpha = 0$ and non-decreasing bounded left-continuous on $(0,1]$,

b) $\overline{\hat{v}}_{\alpha}$ is right-continuous for $\alpha = 0$ and non-increasing bounded left-continuous on $(0,1]$,

c) $\underline{\hat{v}}_{\alpha} \leq \overline{\hat{v}}_{\alpha}$, $0 \leq \alpha \leq 1$ (Khastan and Nieto, 2010).

Definition 3. Let $\hat{a}, \hat{b} \in \mathbb{R}_F$. The generalized Hukuhara difference of $\hat{a}$ and $\hat{b}$ is the set $\hat{c} \in \mathbb{R}_F$ which $\hat{a} \ominus_g \hat{b} = \hat{c}$ if and only if $\hat{a} = \hat{b} + \hat{c}$ (Allahviranloo and Gholami, 2012).

Definition 4. Let $\hat{h}: [a_1, a_2] \longrightarrow \mathbb{R}_F$ *and* $t_0 \in [a_1, a_2]$.

(a) If there exists $\hat{h}'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently small,

$\exists \hat{h}(t_0 + h) \ominus \hat{h}(t_0), \exists \hat{h}(t_0) \ominus \hat{h}(t_0 - h)$

and the limits

$$\lim_{h \to 0^+} \frac{\hat{h}(t_0+h) \ominus \hat{h}(t_0)}{h} = \lim_{h \to 0^+} \frac{\hat{h}(t_0) \ominus \hat{h}(t_0-h)}{h} = \hat{h}'(t_0),$$

$\hat{h}$ is (1)-differentiable at t_0.

(b) If there exists $\hat{h}'(t_0) \in \mathbb{R}_F$ such that for all h > 0 sufficiently small,

$\exists \hat{h}(t_0) \ominus \hat{h}(t_0 + h), \exists \hat{h}(t_0 - h) \ominus \hat{h}(t_0)$

and the limits

$$\lim_{h \to 0^+} \frac{\hat{h}(t_0) \ominus \hat{h}(t_0+h)}{-h} = \lim_{h \to 0^+} \frac{\hat{h}(t_0-h) \ominus \hat{h}(t_0)}{-h} = \hat{h}'(t_0),$$

$\hat{h}$ is (2)-differentiable at t_0 (Khastan and Nieto, 2010).

Theorem 1. Let $\hat{h}: [a_1, a_2] \longrightarrow \mathbb{R}_F$ and for each $\alpha \in [0,1]$,

$$\left[\hat{h}(t)\right]^\alpha = \left[\underline{\hat{h}}_\alpha(t), \overline{\hat{h}}_\alpha(t)\right].$$

(a). If $\hat{h}$ is (1)-differentiable, $\underline{\hat{h}}_\alpha(t)$, $\overline{\hat{h}}_\alpha(t)$ are differentiable functions and

$$\left[\hat{h}'(t)\right]^\alpha = \left[\underline{\hat{h}}_\alpha'(t), \overline{\hat{h}}_\alpha'(t)\right].$$

(b). If $\hat{h}$ is (2)-differentiable, then $\underline{\hat{h}}_\alpha(t)$, $\overline{\hat{h}}_\alpha(t)$ are differentiable functions and

$\left[\hat{h}'(t)\right]^\alpha = \left[\overline{\hat{h}}_\alpha'(t), \underline{\hat{h}}_\alpha'(t)\right]$ (Khastan at al., 2009).

Definition 5. Let $\hat{g}: [a_1, a_2] \longrightarrow \mathbb{R}_F$.

$$\hat{G}(s) = \hat{L}(\hat{g}(x)) = \int_0^\infty e^{-sx} \hat{g}(x) dx = \left[\lim_{\rho \to \infty} \int_0^\rho e^{-sx} \underline{\hat{g}}(x) dx \ , \lim_{\rho \to \infty} \int_0^\rho e^{-sx} \overline{\hat{g}}(x) dx \ \right],$$

$$\hat{G}(s, \alpha) = \hat{L}([\hat{g}(x)]^\alpha) = \left[\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right), L\left(\overline{\hat{g}}_\alpha(x)\right)\right],$$

$$\hat{L}\left(\underline{\hat{g}}_\alpha(x)\right) = \int_0^\infty e^{-sx} \underline{\hat{g}}_\alpha(x) dx = \lim_{\rho \to \infty} \int_0^\rho e^{-sx} \underline{\hat{g}}_\alpha(x) dx,$$

$$\hat{L}\left(\overline{\hat{g}}_\alpha(x)\right) = \int_0^\infty e^{-sx} \overline{\hat{g}}_\alpha(x) dx = \lim_{\rho \to \infty} \int_0^\rho e^{-sx} \overline{\hat{g}}_\alpha(x) dx$$

is the fuzzy Laplace transform of $\hat{g}$ (Patel and Desai, 2017).

Theorem 2. Let $\hat{h}(x)$ and $\hat{h}'(x)$ be primitive of $\hat{h}'(x)$ and $\hat{h}''(x)$ on $[0, \infty)$, respectively. Also, let $\hat{h}''(x)$ be an integrable fuzzy function.

(a). If the functions $\hat{h}$, $\hat{h}'$ are (1)-differentiable,

$$\hat{L}\left(\hat{h}''(x)\right) = s^2 \hat{L}\left(\hat{h}(x)\right) \ominus s\hat{h}(0) \ominus \hat{h}'(0).$$

(b). If the functions $\hat{h}$, $\hat{h}'$ are (2)-differentiable,

$$\hat{L}\left(\hat{h}''(x)\right) = s^2\hat{L}\big(h(x)\big) \ominus s\hat{h}(0) - \hat{h}'(0).$$

(c). If $\hat{h}$ is (1)-differentiable, $\hat{h}'$ is (2)-differentiable,

$$\hat{L}\left(\hat{h}''(x)\right) = \ominus (-s^2)\hat{L}\left(\hat{h}(x)\right) - s\hat{h}(0) - \hat{h}'(0).$$

(d). If $\hat{h}$ is (2)-differentiable, $\hat{h}'$ is (1)-differentiable,

$$\hat{L}\left(\hat{h}''(x)\right) = \ominus (-s^2)\hat{L}\left(\hat{h}(x)\right) - s\hat{h}(0) \ominus \hat{h}'(0)$$ (Patel and Desai, 2017).

3. Findings and Discussion

We research the second-order fuzzy initial value problem

$$\begin{cases} \hat{u}'' = -[\hat{\eta}]^\alpha \hat{u} \\ \hat{u}(0) = [\hat{\gamma}]^\alpha \\ \hat{u}'(0) = \left[\hat{\zeta}\right]^\alpha \end{cases} \tag{1}$$

where

$$[\hat{\eta}]^\alpha = \left[\underline{\hat{\eta}}_\alpha, \overline{\hat{\eta}}_\alpha\right], [\hat{\gamma}]^\alpha = \left[\underline{\hat{\gamma}}_\alpha, \overline{\hat{\gamma}}_\alpha\right], \quad \left[\hat{\zeta}\right]^\alpha = \left[\underline{\hat{\zeta}}_\alpha, \overline{\hat{\zeta}}_\alpha\right]$$

are positive symmetric triangular fuzzy numbers, $[\hat{u}(t)]^\alpha = \left[\underline{\hat{u}}_\alpha(t), \overline{\hat{u}}_\alpha(t)\right]$, $\hat{u}$ is positive fuzzy function and $\hat{L}\big(\hat{u}(t)\big) = \hat{U}(s)$.

3.1. The solution (1,1)

Since $\hat{u}$ and $\hat{u}'$ are (1)-differentiable, we have

$$\underline{\hat{U}}_\alpha(s) = \frac{1}{s^2}\left(\underline{\hat{u}}_\alpha'(0) - \overline{\hat{\eta}}_\alpha \overline{\hat{U}}_\alpha(s)\right) + \frac{1}{s}\underline{\hat{u}}_\alpha(0), \tag{2}$$

$$\overline{\hat{U}}_\alpha(s) = \frac{1}{s^2}\left(\overline{\hat{u}}_\alpha'(0) - \underline{\hat{\eta}}_\alpha \underline{\hat{U}}_\alpha(s)\right) + \frac{1}{s}\overline{\hat{u}}_\alpha(0). \tag{3}$$

From this, we obtain $\underline{\hat{U}}_\alpha(s)$ and $\overline{U}_\alpha(s)$ following as

$$\underline{\hat{U}}_\alpha(s) = \frac{1}{s^4 - \underline{\hat{\eta}}_\alpha \overline{\hat{\eta}}_\alpha}\left(s\left(s^2\underline{\hat{\gamma}}_\alpha + s\underline{\hat{\zeta}}_\alpha - \overline{\hat{\eta}}_\alpha \overline{\hat{\gamma}}_\alpha\right) - \overline{\hat{\eta}}_\alpha \overline{\hat{\zeta}}_\alpha\right)$$

$$\overline{U}_\alpha(s) = \frac{1}{s^4 - \underline{\hat{\eta}}_\alpha \overline{\hat{\eta}}_\alpha}\left(s\left(s^2\overline{\hat{\gamma}}_\alpha + s\overline{\hat{\zeta}}_\alpha - \underline{\hat{\eta}}_\alpha \underline{\hat{\gamma}}_\alpha\right) - \underline{\hat{\eta}}_\alpha \underline{\hat{\zeta}}_\alpha\right)$$

Then, $(1,1)$-solution of the problem (1) is obtained as

$$\underline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(\frac{\underline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}} - \frac{\sqrt[4]{\left(\overline{\hat{\eta}}_\alpha\right)}\,\overline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\right)^3}}\right)\left(sin\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + sinh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right)$$

$$+\frac{1}{2}\left(\underline{\hat{\gamma}}_\alpha - \sqrt{\left(\frac{\overline{\hat{\eta}}_\alpha}{\underline{\hat{\eta}}_\alpha}\right)}\,\overline{\hat{\gamma}}_\alpha\right)\left(cos\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + +cosh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right),$$

$$\overline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(\frac{\overline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}} - \frac{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\right)}\,\underline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\overline{\hat{\eta}}_\alpha\right)^3}}\right)\left(sin\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + sinh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right)$$

$$+\frac{1}{2}\left(\overline{\hat{\gamma}}_\alpha - \sqrt{\left(\frac{\underline{\hat{\eta}}_\alpha}{\overline{\hat{\eta}}_\alpha}\right)}\,\underline{\hat{\gamma}}_\alpha\right)\left(cos\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + cosh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right).$$

3.2. The solution (1,2)

Since $\hat{u}$ and $\hat{u}'$ are (1)-differentiable and (2)-differentiable, respectively, we can write the equations,

$$\underline{\hat{U}}_\alpha(s)\left(s^2 + \underline{\hat{\eta}}_\alpha\right) = s\underline{\hat{u}}_\alpha(0) + \underline{\hat{u}}_\alpha'(0),$$

$$\overline{\hat{U}}_\alpha(s)\left(s^2 + \overline{\hat{\eta}}_\alpha\right) = s\overline{\hat{u}}_\alpha(0) + \overline{\hat{u}}_\alpha'(0).$$

Then, we have

$$\underline{\hat{U}}_\alpha(s) = \frac{1}{s^2+\underline{\hat{\eta}}_\alpha}\left(s\underline{\hat{\gamma}}_\alpha + \underline{\hat{\zeta}}_\alpha\right),$$

$$\overline{\hat{U}}_\alpha(s) = \frac{1}{s^2+\overline{\hat{\eta}}_\alpha}\left(s\overline{\hat{\gamma}}_\alpha + \overline{\hat{\zeta}}_\alpha\right).$$

From this, (1,2)-solution of the problem (1) is obtained as

$$\underline{\hat{u}}_\alpha(t) = \underline{\hat{\gamma}}_\alpha cos\left(\sqrt{\underline{\hat{\eta}}_\alpha}t\right) + \frac{\underline{\hat{\zeta}}_\alpha}{\sqrt{\underline{\hat{\eta}}_\alpha}} sin\left(\sqrt{\underline{\hat{\eta}}_\alpha}t\right),$$

$$\overline{\hat{u}}_\alpha(t) = \overline{\hat{\gamma}}_\alpha cos\left(\sqrt{\overline{\hat{\eta}}_\alpha}t\right) + \frac{\overline{\hat{\zeta}}_\alpha}{\sqrt{\overline{\hat{\eta}}_\alpha}} sin\left(\sqrt{\overline{\hat{\eta}}_\alpha}t\right).$$

3.3. The solution (2,1)

Using Theorem 2, similar to (1,2)-solution, (2,1)-solution of the problem (1) is

$$\underline{\hat{u}}_\alpha(t) = \underline{\hat{\gamma}}_\alpha cos\left(\sqrt{\underline{\hat{\eta}}_\alpha}t\right) + \frac{\overline{\hat{\zeta}}_\alpha}{\sqrt{\underline{\hat{\eta}}_\alpha}} sin\left(\sqrt{\underline{\hat{\eta}}_\alpha}t\right),$$

$$\overline{\hat{u}}_\alpha(t) = \overline{\hat{\gamma}}_\alpha cos\left(\sqrt{\overline{\hat{\eta}}_\alpha}t\right) + \frac{\underline{\hat{\zeta}}_\alpha}{\sqrt{\overline{\hat{\eta}}_\alpha}} sin\left(\sqrt{\overline{\hat{\eta}}_\alpha}t\right).$$

3.4. The solution (2,2)

Similar to (1,1)-solution, (2,2)-solution is

$$\underline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(\frac{\overline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}} - \frac{\sqrt[4]{\left(\overline{\hat{\eta}}_\alpha\right)}\underline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\right)^3}}\right)\left(sin\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + sinh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right)$$

$$+\frac{1}{2}\left(\underline{\hat{\gamma}}_\alpha - \sqrt{\left(\frac{\overline{\hat{\eta}}_\alpha}{\underline{\hat{\eta}}_\alpha}\right)}\overline{\hat{\gamma}}_\alpha\right)\left(cos\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + cosh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right),$$

$$\overline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(\frac{\underline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}} - \frac{\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\right)}\overline{\hat{\zeta}}_\alpha}{\sqrt[4]{\left(\overline{\hat{\eta}}_\alpha\right)^3}}\right)\left(sin\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + sinh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right)$$

$$+\frac{1}{2}\left(\overline{\hat{\gamma}}_\alpha - \sqrt{\left(\frac{\underline{\hat{\eta}}_\alpha}{\overline{\hat{\eta}}_\alpha}\right)}\underline{\hat{\gamma}}_\alpha\right)\left(cos\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right) + cosh\left(\sqrt[4]{\left(\underline{\hat{\eta}}_\alpha\overline{\hat{\eta}}_\alpha\right)}\,t\right)\right).$$

4. Numerical Example

Consider the problem

$$\begin{cases}\hat{u}'' = -\left[\hat{1}\right]^\alpha\hat{u}, \\ \hat{u}(0) = \left[\hat{2}\right]^\alpha \\ \hat{u}'(0) = \left[\hat{3}\right]^\alpha\end{cases}, \tag{4}$$

where

$$\left[\hat{1}\right]^\alpha = [\alpha, 2-\alpha], \left[\hat{2}\right]^\alpha = [1+\alpha, 3-\alpha], \left[\hat{3}\right]^\alpha = [2+\alpha, 4-\alpha].$$

(1,1)-solution of the problem (4) is

$$\underline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(-\sqrt{\frac{2-\alpha}{\alpha}}(3-\alpha) + 1 + \alpha\right)\left(cosh(\sqrt[4]{\alpha_1}t) + cos(\sqrt[4]{\alpha_1}t)\right)$$

$$-\frac{1}{2}\left(\frac{\sqrt[4]{2-\alpha}(4-\alpha)}{\sqrt[4]{\alpha^3}} - \frac{2+\alpha}{\sqrt[4]{\alpha_1}}\right)\left(sin(\sqrt[4]{\alpha_1}t) + sinh(\sqrt[4]{\alpha_1}t)\right),$$

$$\overline{\hat{u}}_\alpha(t) = \frac{1}{2}\left(3 - \alpha - \sqrt{\frac{\alpha}{2-\alpha}}(1+\alpha)\right)\left(cos(\sqrt[4]{\alpha_1}t) + cosh(\sqrt[4]{\alpha_1}t)\right)$$

$$-\frac{1}{2}\left(\frac{\sqrt[4]{\alpha}(2+\alpha)}{\sqrt[4]{(2-\alpha)^3}} - \frac{4-\alpha}{\sqrt[4]{\alpha_1}}\right)\left(sin(\sqrt[4]{\alpha_1}t) + sinh(\sqrt[4]{\alpha_1}t)\right),$$

(1,2)-solution of the problem (4) is

$$\underline{\hat{u}}_{\alpha}(t) = sin\left(\sqrt{\alpha}t\right)\left(\frac{2}{\sqrt{\alpha}}+\sqrt{\alpha}\right)+cos\left(\sqrt{\alpha}t\right)(1+\alpha),$$

$$\overline{\hat{u}}_{\alpha}(t) = sin\left(\sqrt{2-\alpha}t\right)\left(\frac{4-\alpha}{\sqrt{2-\alpha}}\right)+cos\left(\sqrt{2-\alpha}t\right)(3-\alpha),$$

(2,1)-solution of the problem (4) is

$$\underline{\hat{u}}_{\alpha}(t) = sin\left(\sqrt{\alpha}t\right)\left(\frac{4}{\sqrt{\alpha}}-\sqrt{\alpha}\right)+cos\left(\sqrt{\alpha}t\right)(1+\alpha),$$

$$\overline{\hat{u}}_{\alpha}(t) = sin\left(\sqrt{2-\alpha}t\right)\left(\frac{2+\alpha}{\sqrt{2-\alpha}}\right)+cos\left(\sqrt{2-\alpha}t\right)(3-\alpha),$$

(2,2)-solution of the problem (4) is

$$\underline{\hat{u}}_{\alpha}(t) = \frac{1}{2}\left(-\sqrt{\frac{2-\alpha}{\alpha}}(3-\alpha)+1+\alpha\right)\left(cosh(\sqrt[4]{\alpha_1}t)+cos(\sqrt[4]{\alpha_1}t)\right)$$

$$-\frac{1}{2}\left(\frac{\sqrt[4]{2-\alpha}(2+\alpha)}{\sqrt[4]{\alpha^3}}-\frac{4-\alpha}{\sqrt[4]{\alpha_1}}\right)\left(sin(\sqrt[4]{\alpha_1}t)+sinh(\sqrt[4]{\alpha_1}t)\right),$$

$$\overline{\hat{u}}_{\alpha}(t) = \frac{1}{2}\left(3-\alpha-\sqrt{\frac{\alpha}{2-\alpha}}(1+\alpha)\right)\left(cos(\sqrt[4]{\alpha_1}t)+cosh(\sqrt[4]{\alpha_1}t)\right)$$

$$-\frac{1}{2}\left(\frac{\sqrt[4]{\alpha}(4-\alpha)}{\sqrt[4]{(2-\alpha)^3}}-\frac{2+\alpha}{\sqrt[4]{\alpha_1}}\right)\left(sin(\sqrt[4]{\alpha_1}t)+sinh(\sqrt[4]{\alpha_1}t)\right),$$

where $\alpha(2-\alpha)=\alpha_1$, $[\hat{u}(t)]^{\alpha}=\left[\underline{\hat{u}}_{\alpha}(t),\overline{\hat{u}}_{\alpha}(t)\right]$.

According to Definition 2 and since $\hat{u}$ is positive fuzzy function, $\hat{u}(t)$ is a valid fuzzy function for $t\in(0,1.03092)$ in Fig. 2 and for $(0,0.536283)$ in Fig. 3. But, $\hat{u}(t)$ is not a valid fuzzy function in Fig. 1 and Fig. 4.

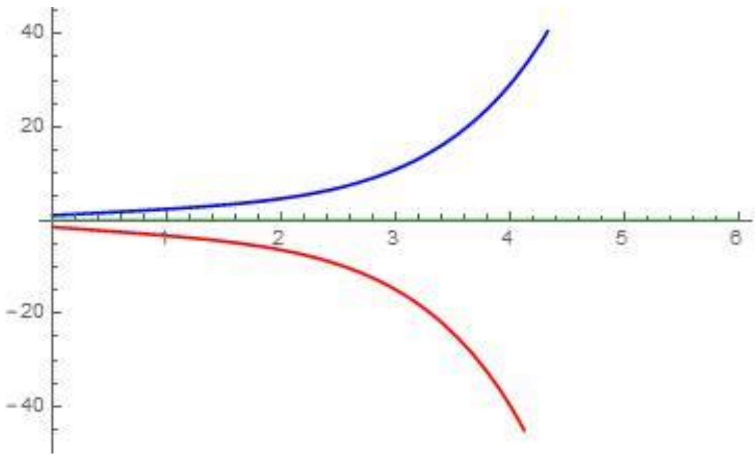

Figure 1. The solution (1,1), $\alpha = 0.7$

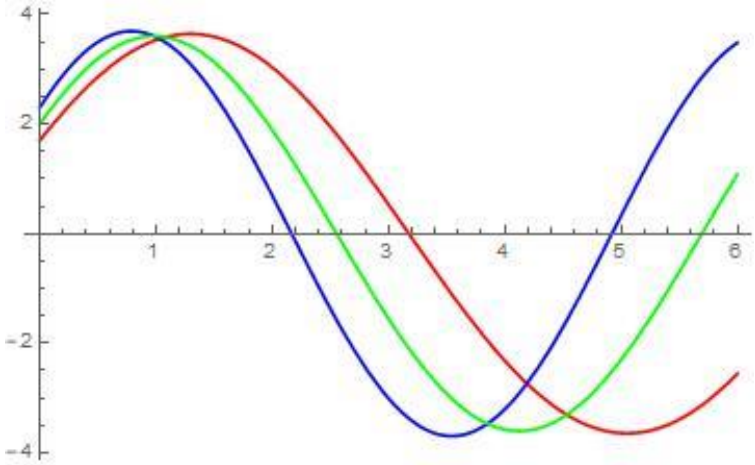

Figure 2. The solution (1,2), $\alpha = 0.7$

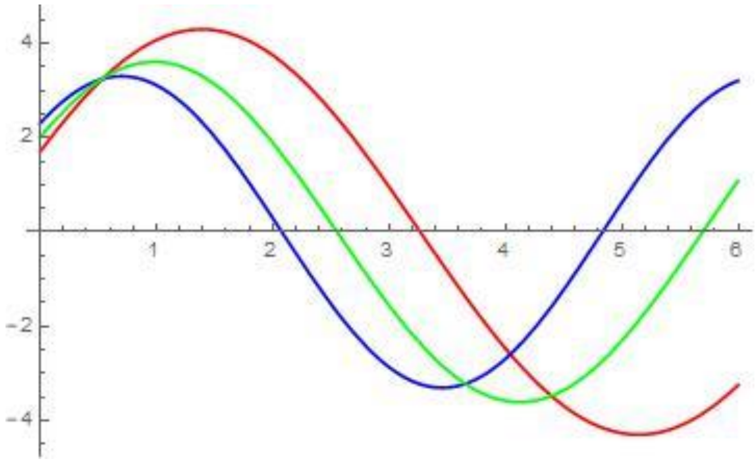

Figure 3. The solution (2,1), $\alpha = 0.7$

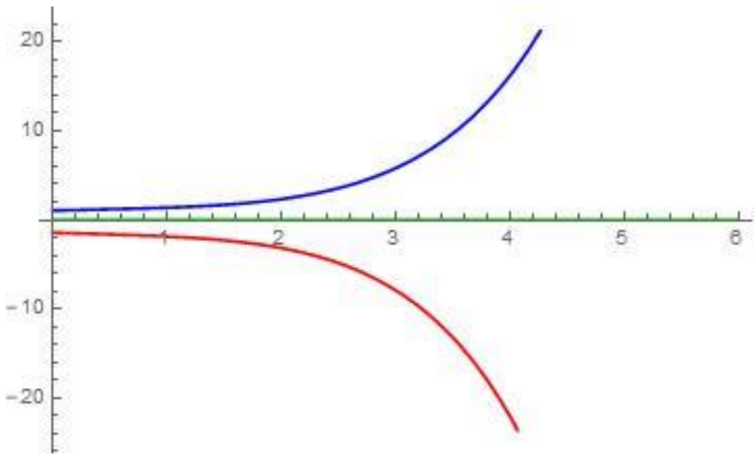

Figure 4. The solution (2,2), $\alpha = 0.7$

$$\text{red} \longrightarrow \underline{\hat{u}}_{\alpha}(t), \text{blue} \longrightarrow \overline{\hat{u}}_{\alpha}(t), \text{green} \longrightarrow \underline{\hat{u}}_{1}(t) = \overline{\hat{u}}_{1}(t)$$

5. Conclusions and Recommendations

In this study, a second order fuzzy problem was analyzed via the fuzzy Laplace transform method. We gave a numerical example. We have seen that (1,2) solution and (2,1) solution of the problem are valid fuzzy functions for all alpha-level sets in different intervals. But (1,1) solution and (2,2) solution of the problem are not valid fuzzy functions.

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

References

Akın Ö., Khaniyev T., Bayeğ S. and Türkşen B. (2016). Solving a second order fuzzy initial value problem using the heaviside function, *Turkish Journal of Mathematics and Computer Science*, 4, 16–25.

Allahviranloo T., Ahmady N. and Ahmady E. (2007). Numerical solution of fuzzy differential equations by predictor-corrector method, *Information Sciences*, 177(7), 1633-1647.

Allahviranloo T. and Ahmadi M. B. (2010). Fuzzy Laplace transforms, *Soft Computing*, 14(3), 235–243.

Allahviranloo T. and Gholami S. (2012). Note on ‘‘Generalized Hukuhara differentiability of interval-valued functions and interval differential equations’’, *Journal of Fuzzy Set Valued Analysis*, 2012, 1-4.

Bayeğ S., Mert R., Akın Ö. and Khaniyev T. (2022). On a type-2 fuzzy approach to solution of second-order initial value problem, *Soft Computing*, 26, 1671-1683.

Bede B., Rudas I. J. and Bencsik A. L. (2007). First order linear fuzzy differential equations under generalized differentiability, *Information Sciences*, 177(7), 1648–1662.

Belhallaj Z., Melliani S., Elomari M. and Chadli L. S. (2023). Application of the intuitionistic fuzzy Laplace transform method for resolution of one dimensional wave equations, *International Journal of Difference Equations*, 18(1), 211-225.

Eljaoui E. and Melliani S. (2023). A study of some properties of fuzzy Laplace transform with their applications in solving the second-order fuzzy linear partial differential equations, *Advances in Fuzzy Systems*, 2023(7868762), 1-15.

Gültekin Çitil H. (2019). Comparisons of the exact and the approximate solutions of second-order fuzzy linear boundary value problems, *Miskolc Mathematical Notes*, 20(2) 823–837.

Gültekin Çitil H. (2020). Solving the fuzzy initial value problem with negative coefficient by using fuzzy Laplace transform, *Facta Universitatis, Series: Mathematics and Informatics*, 35(1), 201-215.

Gültekin Çitil H. (2020). The problem with fuzzy eigenvalue parameter in one of the boundary conditions, *An International Journal of Optimization and Control: Theories & Applications*, 10(2), 159-165.

Ivaz K., Khastan A. and Nieto J. J. (2013). A numerical method for fuzzy differential equations and hybrid fuzzy differential equations, *Abstract and Applied Analysis*, 2013(735128),1-10.

Jafari R., Yu W., Razvarz S. and. Gegov A. (2021). Numerical methods for solving fuzzy equations: A survey, *Fuzzy Sets and Systems*, 404, 1–22.

Khastan A. and Nieto J. J. (2010). A boundary value problem for second order fuzzy differential equations, *Nonlinear Analysis*, 72(9-10), 3583-3593.

Khastan A., Bahrami F. and Ivaz K. (2009). New results on multiple solutions for nth-order fuzzy differential equations under generalized differentiability, *Boundary Value Problems*, 2009(395714), 1-13.

Mallak S., Attili B. and Subuh M. (2022). Numerical treatment of hybrid fuzzy differential equations subject to trapezoidal and triangular fuzzy initial conditions using Picard's and the general linear method, *Computation*, 10(168), 1-19.

Patel K. R. and Desai N. B. (2017). Solution of fuzzy initial value problems by fuzzy Laplace transform, *Kalpa Publications in Computing*, 2, 25-37.

Salahshour S. and Allahviranloo T. (2013) Applications of fuzzy Laplace transforms, *Soft Computing*, 17(1), 145-158.

Salahshour S. and Haghi E. (2010). Solving fuzzy heat equation by fuzzy Laplace transforms, *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Applications, Communications in Computer and Information*, 81, 512-521.

Salgado S. A. B., Barros L. C., Esmi E. and Eduardo Sanchez D. (2019). Solution of a fuzzy differential equation with interactivity via Laplace transform, *Journal of Intelligent & Fuzzy Systems*, 37(2), 2495-2501.

Salgado S. A. B., Esmi E., Eduardo Sanchez D. and Barros L. C. (2021). Solving interactive fuzzy initial value problem via fuzzy Laplace transform, *Computational and Applied Mathematics*, 40, 1-14.

Samuel M. Y. and Tahir A. (2021). Solution of first order fuzzy partial differential equations by fuzzy Laplace transform method, *Bayero Journal of Pure and Applied Sciences*, 14(2), 37 – 51.

Saqib M., Akram M., Bashir S. and Allahviranloo T. (2021). A Runge-Kutta numerical method to approximate the solution of bipolar fuzzy initial value problems, *Computational and Applied Mathematics*, 40(151), 1-43.