Journal of Engineering Technology and Applied Sciences, 2025 *e-ISSN:* 2548-0391
Received 6 October 2024, Accepted 1 June 2025 *Vol. 10, No. 2, 115-128*
Published online 30 August 2025, Research Article *doi: 10.30931/jetas.1562212*

Citation: Akkus, H., Ozkan, E., "On the k-Vieta-Pell and k-Vieta-Pell-Lucas Sequences". Journal of Engineering Technology and Applied Sciences 10 (2) 2025 : 115-128.

ON THE k-VIETA-PELL AND k-VIETA-PELL-LUCAS SEQUENCES

Hakan Akkuş[a*], Engin Özkan[b]

[a*]*Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Türkiye*
*hakan.akkus@ogr.ebyu.edu.tr (*corresponding author)*

[b]*Department of Mathematics, Faculty of Science, Marmara University, Istanbul, Türkiye*
engin.ozkan@marmara.edu.tr

Abstract

The aim of this paper is to define the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences, and some terms of these sequences are given. Then, we find the relations between the terms of the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences. Also, we give the summation formulas, generating functions, etc. We also derive the Binet formulas using two different approaches. The first is in the known classical way and the second is with the help of the sequence's generating functions. Moreover, we calculate the special identities of these sequences like Catalan and Melham. Finally, we examine the relations between the k-Vieta-Pell sequence and various other sequences, including Fibonacci, Pell, and Chebyshev polynomials of the first kind. Similarly, we analyze the k-Vieta-Pell-Lucas sequence in relation to Lucas, Pell-Lucas numbers, Chebyshev polynomials of the second kind, and other sequences. In addition, for special k values, these sequences are associated with the sequences in OEIS.

Keywords: Vieta polynomials, Generating function, Pell number, Cassini Identity, Binet formula

1. Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. The golden ratio that we can reach with the Fibonacci sequence is revealed by the proportions of the sensory organs on the human face. For example, the area from under our ears and nose to our chin contains the golden ratio. To give another example, the ratio of the base to the height of the Egyptian Pyramids gives the golden ratio. These sequences have intrigued scientists for a long time. The Fibonacci sequence has applications in diverse fields such as Cryptology [1], Phylotaxis [2], Biomathematics [3], Chemistry [4], Engineering [5], etc. Many generalizations of the Fibonacci sequence have been given. The known examples of such sequences are the k-Jacobsthal-Lucas

[6], k-Pell [7-8], k-Fibonacci [8], Perrin [20], Horadam [22], sequences, Gaussian Fibonacci numbers [22], Bronze Leonardo [23] and k-Leonardo [24] sequences, etc.

For $n \in \mathbb{N}$, the Fibonacci numbers F_n, Bronze Fibonacci BF_n, Lucas numbers L_n, and Bronze Lucas numbers BL_n defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n,\ BF_{n+2} = 3BF_{n+1} + BF_n,\ L_{n+2} = L_{n+1} + L_n,\ BL_{n+2} = 3BL_{n+1} + BL_n$$

with the initial conditions $F_0 = 0$, $F_1 = 1$, $BF_0 = 0$, $BF_1 = 1$, $L_0 = 2$, $L_1 = 1$, and $BL_0 = 2$, $BL_1 = 3$.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ L_n = \alpha^n + \beta^n,\ BF_n = \frac{\lambda^n - \psi^n}{\lambda - \psi},\ BL_n = \lambda^n + \psi^n$$

where:

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \lambda = \frac{3+\sqrt{13}}{2},\ \psi = \frac{3+\sqrt{13}}{2}$$

The characteristic equation for these terms are $r^2 - r - 1 = 0$, and $r^2 - 3r - 1 = 0$. Additionaly, clarify the significance of α and λ as the known golden and bronze ratios, respectively. Metallic ratios appear frequently in such sequences.

The recurrence relations for the Pell numbers p_n, Pell-Lucas numbers q_n, Balancing number B_n, Balancing-Lucas number C_n, Mersenne numbers M_n, and Mersenne-Lucas numbers N_n are presented. For improved readability, it might be beneficial to align the relations as follows:

$$p_{n+2} = 2p_{n+1} + p_n,\ q_{n+2} = 2q_{n+1} + q_n,\ B_{n+2} = 6B_{n+1} - B_n,$$
$$C_{n+2} = 6C_{n+1} - C_n,\ M_{n+2} = 3M_{n+1} - 2M_n,\ N_{n+2} = 3N_{n+1} - 2N_n$$

with the initial conditions:

$$p_0 = 0,\ p_1 = 1,\ q_0 = 2,\ q_1 = 2,\ B_0 = 0,\ B_1 = 1,$$
$$C_0 = 2,\ C_1 = 6,\ M_0 = 0,\ M_1 = 1,\ N_0 = 2,\ N_1 = 5.$$

The Binet formulas for these sequences are given as follows. Ensure consistency by defining the parameters used in each formula:

$$p_n = \frac{\varphi^n - \omega^n}{\varphi - \omega},\ q_n = \varphi^n + \omega^n,\ B_n = \frac{\gamma^n - \delta^n}{\gamma - \delta},\ C_n = \gamma^n + \delta^n,\ M_n = \frac{\mu^n - \sigma^n}{\mu - \sigma},\ N_n = \mu^n + \sigma^n.$$

where:

$$\varphi = 1 + \sqrt{2},\ \omega = 1 - \sqrt{2},\ \gamma = 3 + 2\sqrt{2},\ \delta = 3 - 2\sqrt{2},\ \mu = 2,\ \sigma = 1$$

The characteristic equation for these terms are $r^2 - 2r - 1 = 0$, $r^2 - 6r - 1 = 0$, and $r^2 - 3r + 2 = 0$. Additionaly, clarify the significance of φ the known silver ratio. The silver ratio $\varphi = 2.414213562\ldots$

In [11], Horadam worked on Vieta polynomials. In addition, Shannon and Horadam studied the relationship between the Vieta, Morgan Voyce, and Jacobsthal polynomials [12]. Also, Neville defined a new sequence of triangles with Vieta polynomials, and she found the properties of this sequence [13]. In [14], for $n \in \mathbb{N}$, Mason and Handscomb defined the Chebyshev polynomials of the first kind U_n and Chebyshev polynomials of the second kind T_n by the recurrence relations, respectively,

$$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) \text{ and } T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x),$$

with the initial conditions $U_0(x) = 1$, $U_1(x) = 2x$ and $T_0(x) = 1$, $T_1 = x$.

In addition, Binet formulas $U_n(x)$ and $T_n(x)$ gave by relations, respectively,

$$U_n(x) = \frac{\tau^{n+1} - \rho^{n+1}}{\tau - \rho} \text{ and } T_n(x) = \frac{\tau^n + \rho^n}{2}$$

where $\tau = x + \sqrt{x^2 - 1}$ and $\rho = x - \sqrt{x^2 - 1}$ are the roots of the characteristic equation $r^2 - 2xr + 1 = 0$.

With the help of the recurrence relation of the Fibonacci sequence, k-sequences were introduced, and these sequences have an important place in number theory [15]. In [16], Falcon and Plaza introduced the k-Fibonacci sequence and obtained many properties related to this sequence. In addition, Falcon defined the k-Lucas sequences [17]. Moreover, Falcon applied the Hankel transform to the k-Fibonacci sequence and obtained the terms of Fibonacci sequences differently [18]. Furthermore, Shannon et al defined the partial recurrence Fibonacci link and found many of its properties [19].

The initial conditions for the k-Oresme sequence [21] $O_{k,n}$ and k-Oresme-Lucas sequence [21] $P_{k,n}$ are specified as:

$$O_{k,0} = 0,\ O_{k,1}(x) = \frac{1}{k},\ P_{k,0}(x) = 2,\ P_{k,1} = 1$$
$$O_{k,n+2} = O_{k,n+1} - \frac{1}{k^2} O_{k,n}(x),\ \ P_{k,n+2} = P_{k,n+1} - \frac{1}{k^2} P_{k,n},$$

The Binet formulas are given as:

$$O_{k,n} = \frac{\vartheta^n - \theta^n}{(\vartheta - \theta)k} \text{ and } P_{k,n} = \vartheta^n + \theta^n$$

where:

$$\vartheta = \frac{k + \sqrt{k^2 - 4}}{2k},\ \theta = \frac{k - \sqrt{k^2 - 4}}{2k}$$

These values are the roots of the characteristic equation:

$$r^2 - r + \frac{1}{k^2} = 0.$$

As seen above, many generalizations of Fibonacci and Lucas sequences have been given so far. In this study, we give new generalizations inspired by the k-Fibonacci sequence and Vieta polynomials. We call these sequences the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences and denote them as $V\mathcal{P}_{k,n}$, and $V\mathbb{Q}_{k,n}$, respectively.

2. k-Vieta-Pell and k-Vieta-Pell-Lucas sequences

For $k \in \mathbb{R}$ and $n \in \mathbb{N}$, the sequences k-Vieta-Pell $V\mathcal{P}_{k,n}$ and k-Vieta-Pell-Lucas $V\mathbb{Q}_{k,n}$ are defined by the recurrence relations:

$$V\mathcal{P}_{k,n+2} = 2kV\mathcal{P}_{k,n+1} - V\mathcal{P}_{k,n}, \quad V\mathcal{P}_{k,0} = 0, V\mathcal{P}_{k,1} = 1, \tag{1}$$
$$V\mathbb{Q}_{k,n+2} = 2kV\mathbb{Q}_{k,n+1} - V\mathbb{Q}_{k,n}, \quad V\mathbb{Q}_{k,0} = 2, V\mathbb{Q}_{k,1} = 2k. \tag{2}$$

Then, let's give some information about the equations of these sequences.

The characteristic equation for both sequences is provided as:

$$r^2 - 2kr + 1 = 0 \tag{3}$$

with roots:

$$r_1 = k + \sqrt{k^2 - 1}, \quad r_2 = k - \sqrt{k^2 - 1}.$$

The relationship between these roots is given below;

$$r_1 + r_2 = 2k, r_1 - r_2 = 2\sqrt{k^2 - 1}, r_1^2 + r_2^2 = 4k^2 - 2 \text{ and } r_1 r_2 = 1.$$

The $V\mathcal{P}_{k,n}$ and $V\mathbb{Q}_{k,n}$ values for the first four n natural numbers are given below;
$V\mathcal{P}_{k,0} = 0, V\mathcal{P}_{k,1} = 1, V\mathcal{P}_{k,2} = 2k, V\mathcal{P}_{k,3} = 4k^2 - 1, V\mathcal{P}_{k,4} = 8k^3 - 4k,$
and
$V\mathbb{Q}_{k,0} = 2, V\mathbb{Q}_{k,1} = 2k, V\mathbb{Q}_{k,2} = 4k^2 - 2, V\mathbb{Q}_{k,3} = 8k^3 - 6k, V\mathbb{Q}_{k,4} = 16k^4 - 16k^2 + 2.$
Also, the terms of the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences can be found with the help of the following relations. Let $n \in \mathbb{N}^+$:

$$V\mathbb{Q}_{k,n} = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \frac{n}{n-i} (-1)^i (2k)^{n-2i} \text{ and } V\mathcal{P}_{k,n} = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (2k)^{n-1-2i} (-1)^i .$$

In the following theorem, the Binet formulas of the k-Vieta-Pell sequence $V\mathcal{P}_{k,n}$, and k-Vieta-Pell-Lucas sequence $V\mathbb{Q}_{k,n}$ are expressed.

Theorem 2.1. Let $n \in \mathbb{N}$. We obtain
i. $V\mathcal{P}_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$, **ii.** $V\mathbb{Q}_{k,n} = r_1^n + r_2^n$.

Proof. The Binet form of a sequence is as follows

$$V\mathcal{P}_{k,n} = x{r_1}^n + y{r_2}^n.$$

The scalars x and y can be obtained by substituting the initial conditions. It is obtained by solving the given system of equations. For $n = 0$, $V\mathcal{P}_{k,0} = 0$ and for $n = 1$, $V\mathcal{P}_{k,1} = 1$. Thus $x = \frac{1}{2\sqrt{k^2-4}}$ and $y = \frac{-1}{2\sqrt{k^2-4}}$ are obtained. From here

$$V\mathcal{P}_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

The proof of the other is shown similarly. □

Next, we examine the relationships between the roots of the characteristic equation of these sequences and these sequences.

Theorem 2.2. Let $i \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We have

i. ${r_1}^{2i} = \frac{V\mathcal{P}_{k,2i}}{k} r_1\sqrt{k^2-1} + \frac{V\mathbb{Q}_{k,2i-1}}{2k}$, **ii.** ${r_2}^{2i} = -\frac{V\mathcal{P}_{k,2i}}{k} r_2\sqrt{k^2-1} + \frac{V\mathbb{Q}_{k,2i-1}}{2k}$,
iii. ${r_1}^{2i+1} = \frac{V\mathcal{P}_{k,2i}}{k}\sqrt{k^2-1} + r_1\frac{V\mathbb{Q}_{k,2i+1}}{2k}$, **iv.** ${r_2}^{2i+1} = -\frac{V\mathcal{P}_{k,2i}}{k}\sqrt{k^2-1} + r_2\frac{V\mathbb{Q}_{k,2i+1}}{2k}$,
v. $2\sqrt{k^2-1}\,V\mathcal{P}_{k,i} + V\mathbb{Q}_{k,i} = 2r_1^i$, **vi.** $2\sqrt{k^2-1}\,V\mathcal{P}_{k,i} - V\mathbb{Q}_{k,i} = -2r_2^i$.

Proof. i. If the Binet formula is used, we obtain

$$\frac{V\mathcal{P}_{k,2i}}{k} r_1\sqrt{k^2-1} + \frac{V\mathbb{Q}_{k,2i-1}}{2k} = r_1\sqrt{k^2-1}\frac{{r_1}^{2i}-{r_2}^{2i}}{(r_1-r_2)k} + \frac{{r_1}^{2i-1}+{r_2}^{2i-1}}{2k}$$
$$= \frac{{r_1}^{2i+1}-r_1{r_2}^{2i}+{r_1}^{2i-1}+{r_2}^{2i-1}}{2k} = \frac{{r_1}^{2i}\left(r_1+\frac{1}{r_1}\right)+{r_2}^{2i}(-r_1+\frac{1}{r_2})}{2k} = {r_1}^{2i}$$

The proofs of the others are shown similarly. □

Theorem 2.3. Let $s = r_1$ or $s = r_2$ and $x, y, z, t \in \mathbb{N}$. We obtain
i. $s^x = sV\mathcal{P}_{k,x} - V\mathcal{P}_{k,x-1}$, **ii.** $V\mathcal{P}_{k,x(y-z)} = s^{xz}V\mathcal{P}_{k,xy} - s^{xy}V\mathcal{P}_{k,xz}$
iii. $s^{2x} = s^xV\mathbb{Q}_{k,x} - 1$, **iv.** $s^x = s^yV\mathcal{P}_{k,x-y+1} - s^{y-1}V\mathcal{P}_{k,x-y}$,
v. $s^{xt} = \frac{s^xV\mathcal{P}_{k,xt}}{V\mathcal{P}_{k,x}} - \frac{V\mathcal{P}_{k,x(t-1)}}{V\mathcal{P}_{k,x}}$, **vi.** $-1 + 2ks + s^{2(2^{x+1}+1)} = s^{2(2^x+1)}V\mathbb{Q}_{k,2^{x+1}}$.

Proof. i. For $s = r_1$, we have

$$sV\mathcal{P}_{k,x} - V\mathcal{P}_{k,x-1} = r_1\left(\frac{r_1^x-r_2^x}{r_1-r_2}\right) - \left(\frac{r_1^{x-1}-r_2^{x-1}}{r_1-r_2}\right) = \frac{r_1^{x-1}(r_1^2-1)-r_2^{x-1}(r_1r_2-1)}{r_1-r_2} = {r_1}^x.$$

For $s = r_2$, we get

$$sV\mathcal{P}_{k,x} - V\mathcal{P}_{k,x-1} = r_2\left(\frac{r_1^x-r_2^x}{r_1-r_2}\right) - \left(\frac{r_1^{x-1}-r_2^{x-1}}{r_1-r_2}\right) = \frac{r_1^x\left(r_2-\frac{1}{r_1}\right)+r_2^x(\frac{1}{r_2}-r_2)}{r_1-r_2} = {r_2}^x.$$

The proofs of the others are shown similarly. □

In the following theorems, we give special relations between the k-Vieta-Pell $V\mathcal{P}_{k,n}$ and k-Vieta-Pell-Lucas $V\mathbb{Q}_{k,n}$ sequences.

Theorem 2.4. Let $x, y \in \mathbb{N}$, $x > y$ and $k \in \mathbb{R}^+, k > 1$. The following equations are satisfied.
i. $V\mathbb{Q}_{k,x}V\mathbb{Q}_{k,y} = V\mathbb{Q}_{k,x+y} + V\mathbb{Q}_{k,x-y}$, **ii.** $V\mathbb{Q}_{k,y} = V\mathcal{P}_{k,y+1} - V\mathcal{P}_{k,y-1}$,
iii. $2V\mathcal{P}_{k,x+y} = V\mathcal{P}_{k,x}V\mathbb{Q}_{k,y} + V\mathbb{Q}_{k,x}V\mathcal{P}_{k,y}$, **iv.** $V\mathcal{P}_{k,y}V\mathbb{Q}_{k,y} = V\mathcal{P}_{k,2y}$,
v. $2\sqrt{k^2-1}\,V\mathcal{P}_{k,y} = V\mathbb{Q}_{k,y+1} + V\mathbb{Q}_{k,y-1}$, **vi.** $V\mathbb{Q}_{k,y}^2 - (4k^2-16)V\mathcal{P}_{k,y}^2 = 4$,

Proof. ii. If the Binet formula is used, we obtain

$$V\mathcal{P}_{k,y+1} - V\mathcal{P}_{k,y-1} = \frac{r_1^{y+1}-r_2^{y+1}}{r_1-r_2} - \frac{r_1^{y-1}-r_2^{y-1}}{r_1-r_2} = \frac{r_1^y\left(r_1-\frac{1}{r_1}\right)+r_2^y\left(-r_2+\frac{1}{r_2}\right)}{r_1-r_2} = r_1^n + r_2^n = V\mathbb{Q}_{k,y}.$$

The proofs of the others are shown similarly. □

Theorem 2.5. Let $x, y \in \mathbb{N}$, $x > y$ and $k \in \mathbb{R}^+, k > 1$. We have
i. $2V\mathbb{Q}_{k,x-y} = V\mathbb{Q}_{k,y}V\mathbb{Q}_{k,x} - (4k^2-4)V\mathcal{P}_{k,y}V\mathcal{P}_{k,x}$,
ii. $2V\mathcal{P}_{k,x-y} = V\mathbb{Q}_{k,y}V\mathcal{P}_{k,x} - V\mathbb{Q}_{k,x}V\mathcal{P}_{k,y}$,
iii. $V\mathcal{P}_{k,x+y+1} = V\mathcal{P}_{k,x+1}V\mathcal{P}_{k,y+1} - V\mathcal{P}_{k,x}V\mathcal{P}_{k,y}$,
iv. $V\mathbb{Q}_{k,x+y+1} = V\mathbb{Q}_{k,y+1}V\mathcal{P}_{k,x+1} - V\mathbb{Q}_{k,y}V\mathcal{P}_{k,x}$.

Proof. vii. If the Binet formula is used, we obtain

$$V\mathcal{P}_{k,x+1}V\mathcal{P}_{k,y+1} - V\mathcal{P}_{k,x}V\mathcal{P}_{k,y} = \frac{r_1^{x+1}-r_2^{x+1}}{r_1-r_2}\frac{r_1^{y+1}-r_2^{y+1}}{r_1-r_2} - \frac{r_1^x-r_2^x}{r_1-r_2}\frac{r_1^y-r_2^y}{r_1-r_2}$$
$$= \frac{r_1^{x+y+1}(r_1-r_2)-r_2^{x+y+1}(r_1-r_2)}{(r_1-r_2)^2} = \frac{r_1^{x+y+1}-r_2^{x+y+1}}{r_1-r_2} = V\mathcal{P}_{k,x+y+1}.$$

The proofs of the others are shown similarly. □

Theorem 2.6. Let $x, y \in \mathbb{N}$, $k \in \mathbb{R}^+$and $k > 1$. We get
i. $V\mathbb{Q}_{k,x} + V\mathbb{Q}_{k,x+4y} = V\mathbb{Q}_{k,x+2y}V\mathbb{Q}_{k,2y}$, **ii.** $V\mathbb{Q}_{k,x+y} + V\mathbb{Q}_{k,x+3y} = V\mathbb{Q}_{k,x+2y}V\mathbb{Q}_{k,y}$,
iii. $V\mathcal{P}_{k,x+y} + V\mathcal{P}_{k,x+3y} = V\mathbb{Q}_{k,y}V\mathcal{P}_{k,x+2y}$, **iv.** $V\mathcal{P}_{k,x} + V\mathcal{P}_{k,x+4y} = V\mathcal{P}_{k,x+2y}V\mathbb{Q}_{k,2y}$,
v. $V\mathcal{P}_{k,x+3y} - V\mathcal{P}_{k,x+y} = V\mathbb{Q}_{k,x+2y}V\mathcal{P}_{k,y}$ **vi.** $V\mathbb{Q}_{k,y}V\mathcal{P}_{k,x} = V\mathcal{P}_{k,x-y} + V\mathcal{P}_{k,x+y}$,
vii. $V\mathbb{Q}_{k,y} + 2kV\mathcal{P}_{k,y} = 2V\mathcal{P}_{k,y+1}$, **viii.** $(4k^2 - 4)V\mathcal{P}_{k,y} + 2kV\mathbb{Q}_{k,y} = 2V\mathbb{Q}_{k,y+1}$,
ix. $V\mathbb{Q}_{k,-x} = V\mathbb{Q}_{k,x}$, **x.** $V\mathcal{P}_{k,-x} = -V\mathcal{P}_{k,x}$,
xi. $V\mathcal{P}_{k,3x} = V\mathcal{P}_{k,2x}V\mathbb{Q}_{k,x} - V\mathcal{P}_{k,x}$, **xii.** $V\mathcal{P}_{k,3x} = (4k^2 - 4)V\mathcal{P}_{k,x}^3 + 3V\mathcal{P}_{k,x}$,

Proof. iii. If the Binet formula is used, we get
$$V\mathbb{Q}_{k,y}V\mathcal{P}_{k,x+2y} = (r_1{}^y + r_2{}^y)\frac{r_1^{x+2y} - r_2^{x+2y}}{r_1 - r_2} = \frac{r_1^{x+3y} - r_1^y r_2^{x+2y} + r_1^{x+2y} r_2^y - r_2^{x+3y}}{r_1 - r_2}$$
$$= \frac{r_1^{x+3y} - r_2^{x+3y} + r_1^y r_2^y (r_1^{x+y} - r_2^{x+y})}{r_1 - r_2} = \frac{r_1^{x+3y} - r_2^{x+3y}}{r_1 - r_2} + \frac{r_1^{x+y} - r_2^{x+y}}{r_1 - r_2} = V\mathcal{P}_{k,x+y} + V\mathcal{P}_{k,x+3y}.$$
The proofs of the others are shown similarly. □

Theorem 2.8. Let $x, y \in \mathbb{N}$, $x > y$ and $k \in \mathbb{R}^+, k > 1$. We have
i. $V\mathbb{Q}_{k,x+3}V\mathbb{Q}_{k,x}^2 - V\mathbb{Q}_{k,x+1}^3 = V\mathbb{Q}_{k,x-3} - 3V\mathbb{Q}_{k,x+1} + 2V\mathbb{Q}_{k,x+3}$,
ii. $V\mathcal{P}_{k,x+3}V\mathcal{P}_{k,x}^2 - V\mathcal{P}_{k,x+1}^3 = \frac{1}{4k^2-4}(3V\mathcal{P}_{k,x+1} - 2V\mathcal{P}_{k,x+3} - V\mathcal{P}_{k,x-3})$,
iii. $(4k^2 - 4)V\mathcal{P}_{k,2y+3}V\mathcal{P}_{k,2x-3} = V\mathbb{Q}_{k,4x} + V\mathbb{Q}_{k,6}$,
iv. $V\mathcal{P}_{k,x+y}^2 V\mathbb{Q}_{k,x+y}^2 - V\mathcal{P}_{k,y}^2 V\mathbb{Q}_{k,y}^2 = \frac{1}{4k^2-4}(V\mathbb{Q}_{k,4x+4y} + 2V\mathbb{Q}_{k,2x+2y} - V\mathbb{Q}_{k,4y} + 4)$,
v. $V\mathbb{Q}_{k,n} = 2kV\mathcal{P}_{k,n+2} + (2 - 4k^2)V\mathcal{P}_{k,n+1}$,
vi. $V\mathcal{P}_{k,n} = \frac{1}{2k^2-2}(kV\mathbb{Q}_{k,n+2} + (-2k^2 + 1)V\mathbb{Q}_{k,n+1})$.

Theorem 2.9. Let $k \in \mathbb{R}^+, k > 1$ and $x, y, z \in \mathbb{N}$. The following equations are satisfied.
i. $4V\mathcal{P}_{k,x+y+z} = V\mathbb{Q}_{k,x}V\mathbb{Q}_{k,y}V\mathcal{P}_{k,z} + V\mathcal{P}_{k,x}V\mathbb{Q}_{k,y}V\mathbb{Q}_{k,z} + V\mathbb{Q}_{k,x}V\mathcal{P}_{k,y}V\mathbb{Q}_{k,z}$
$+(4k^2 - 4)V\mathcal{P}_{k,x}V\mathcal{P}_{k,y}V\mathcal{P}_{k,z}$,
ii. $4V\mathbb{Q}_{k,x+y+z} = V\mathbb{Q}_{k,x}V\mathbb{Q}_{k,y}V\mathbb{Q}_{k,z} + (4k^2 - 4)V\mathbb{Q}_{k,x}V\mathcal{P}_{k,y}V\mathcal{P}_{k,z}$
$+(4k^2 - 4)V\mathcal{P}_{k,x}V\mathbb{Q}_{k,y}V\mathcal{P}_{k,z} + (4k^2 - 4)V\mathcal{P}_{k,x}V\mathcal{P}_{k,y}V\mathbb{Q}_{k,z}$.

Theorem 2.10. Let $k \in \mathbb{R}^+, k > 1$ and $x \in \mathbb{N}$. We have
i. $V\mathcal{P}_{k,x}^2 + V\mathcal{P}_{k,x+1}^2 = \frac{V\mathbb{Q}_{k,2x} + V\mathbb{Q}_{k,2x+2} - 4}{4k^2-4}$,
ii. $V\mathbb{Q}_{k,x}^2 + V\mathbb{Q}_{k,x+1}^2 = V\mathbb{Q}_{k,2x} + V\mathbb{Q}_{k,2x+2} + 4$,
iii. $V\mathcal{P}_{k,x+1}^2 - V\mathcal{P}_{k,x-1}^2 = \frac{V\mathbb{Q}_{k,2x+2} - V\mathbb{Q}_{k,2x-2}}{4k^2-4}$,
iv. $V\mathbb{Q}_{k,x+1}^2 - V\mathbb{Q}_{k,x-1}^2 = V\mathbb{Q}_{k,2x+2} - V\mathbb{Q}_{k,2x-2}$,
v. $V\mathcal{P}_{k,x}V\mathcal{P}_{k,x+1} = \frac{V\mathbb{Q}_{k,2x+1} - 2k}{4k^2-4}$,
vi. $V\mathbb{Q}_{k,x}V\mathbb{Q}_{k,x+1} = \mathbb{Q}_{k,2x+1} + 2k$,
viii. $V\mathcal{P}_{k,n} = \frac{1}{2k^2-2}(V\mathbb{Q}_{k,n+1} - kV\mathbb{Q}_{k,n})$,
ix. $V\mathbb{Q}_{k,n} = 2V\mathcal{P}_{k,n+1} - 2kV\mathcal{P}_{k,n}$.

The proofs of Theorem 2.7.,-2.11., are shown using the Binet formulas in a similar way to Theorem 2.6.

Theorem 2.12. Let $x, y, z \in \mathbb{N}$, $z \neq x$, $k \in \mathbb{R}^+$ and $k > 1$. We obtain
i. $(4 - 4k^2)V\mathcal{P}_{k,z-x}^2 = V\mathbb{Q}_{k,y+z}^2 - V\mathbb{Q}_{k,x+y}V\mathbb{Q}_{k,z-x}V\mathbb{Q}_{k,y+z} + V\mathbb{Q}_{k,x+y}^2$,

ii. $V\mathcal{P}_{k,z-x}^2 = V\mathcal{P}_{k,y+z}^2 - V\mathbb{Q}_{k,x-z}V\mathcal{P}_{k,x+y}V\mathcal{P}_{k,y+z} + V\mathcal{P}_{k,x+y}^2$.

Proof. ii. If the Binet formulas are used, we have

$$V\mathcal{P}_{k,y+z}^2 - V\mathbb{Q}_{k,x-z}V\mathcal{P}_{k,x+y}V\mathcal{P}_{k,y+z} + V\mathcal{P}_{k,x+y}^2$$
$$= \left(\frac{{r_1}^{y+z}-{r_2}^{y+z}}{r_1-r_2}\right)^2 - ({r_1}^{x-z}+{r_2}^{x-z})\left(\frac{{r_1}^{x+y}-{r_2}^{x+y}}{r_1-r_2}\right)\left(\frac{{r_1}^{y+z}-{r_2}^{y+z}}{r_1-r_2}\right) + \left(\frac{{r_1}^{x+y}-{r_2}^{x+y}}{r_1-r_2}\right)^2$$
$$= \frac{{r_1}^{2x+2y}+{r_2}^{2x+2y}-2-{r_1}^{2x+2y}+{r_1}^{2z-2x}+1-{r_2}^{2y+2z}-{r_1}^{2y+2z}+1+{r_2}^{2z-2x}-{r_2}^{2x+2y}+{r_1}^{2y+2z}+{r_2}^{2y+2z}-2}{(r_1-r_2)^2}$$
$$= \left(\frac{{r_1}^{z-x}-{r_2}^{z-x}}{r_1-r_2}\right)^2 = V\mathcal{P}_{k,z-x}^2.$$

The proof of the other is shown similarly. □

In the following theorems, we calculate the specific identities of the k-Vieta-Pell $V\mathcal{P}_{k,n}$ and k-Vieta-Pell-Lucas $V\mathbb{Q}_{k,n}$ sequences.

Theorem 2.13. ***(Cassini Identity)*** Let $n \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We get
i. $V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n-1} - V\mathcal{P}_{k,n}^2 = -1$, **ii.** $V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n-1} - V\mathbb{Q}_{k,n}^2 = 4k^2 - 4$.

Proof. If the Binet formula is used, we get

i. $V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n-1} - V\mathcal{P}_{k,n}^2 = \frac{{r_1}^{n+1}-{r_2}^{n+1}}{r_1-r_2}\frac{{r_1}^{n-1}-{r_2}^{n-1}}{r_1-r_2} - \frac{{r_1}^{n}-{r_2}^{n}}{r_1-r_2}\frac{{r_1}^{n}-{r_2}^{n}}{r_1-r_2}$

$$= \frac{{r_1}^{2n}-{r_1}^{n+1}{r_2}^{n-1}-{r_2}^{n+1}{r_1}^{n-1}+{r_2}^{2n}}{(r_1-r_2)^2} - \frac{{r_1}^{2n}-2{r_1}^{n}{r_2}^{n}+{r_2}^{2n}}{(r_1-r_2)^2}$$
$$= \frac{(r_1r_2)^n\frac{-r_1}{r_2}}{(r_1-r_2)^2} + \frac{(r_1r_2)^n\frac{-r_2}{r_1}}{(r_1-r_2)^2} + \frac{2(r_1r_2)^n}{(r_1-r_2)^2}$$

If the properties of the characteristic equation of the k-Vieta-Pell sequence are used, we obtain

$$V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n-1} - V\mathcal{P}_{k,n}^2 = -1.$$

ii. $V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n-1} - V\mathbb{Q}_{k,n}^2 = ({r_1}^{n+1}+{r_2}^{n+1})({r_1}^{n-1}+{r_2}^{n-1}) - ({r_1}^{n}+{r_2}^{n})({r_1}^{n}+{r_2}^{n})$

$$= {r_1}^{2n} + {r_1}^{n+1}{r_2}^{n-1} + {r_2}^{n+1}{r_1}^{n-1} + {r_2}^{2n} - {r_1}^{2n} - 2{r_1}^{n}{r_2}^{n} - {r_2}^{2n}$$

If the properties of the characteristic equation of the k-Vieta-Pell-Lucas sequence are used, we obtain

$$V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n-1} - V\mathbb{Q}_{k,n}^2 = 4k^2 - 4.$$ □

Theorem 2.14. ***(Catalan Identity)*** Let $k \in \mathbb{R}^+, k > 1$ and $n, r \in \mathbb{N}$, $r \le n$. We obtain
i. $V\mathcal{P}_{k,n+r}V\mathcal{P}_{k,n-r} - V\mathcal{P}_{k,n}^2 = -V\mathcal{P}_{k,r}^2$, **ii.** $V\mathbb{Q}_{k,n+r}V\mathbb{Q}_{k,n-r} - V\mathbb{Q}_{k,n}^2 = (4k^2-4)V\mathcal{P}_{k,r}^2$.

Theorem 2.15. ***(D'ocagne Identity)*** Let $k \in \mathbb{R}^+, k > 1$ and $n, r \in \mathbb{N}$, $r \le n$. We have
i. $V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,r} - V\mathcal{P}_{k,n}V\mathcal{P}_{k,r+1} = -V\mathcal{P}_{k,n-r}$,
ii. $V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,r} - V\mathbb{Q}_{k,n}V\mathbb{Q}_{k,r+1} = (4k^2-4)V\mathcal{P}_{k,n-r}$.

Theorem 2.16. ***(Vajda Identity)*** Let $n, i, j \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We have
i. $V\mathcal{P}_{k,n+i}V\mathcal{P}_{k,n+j} - V\mathcal{P}_{k,n}V\mathcal{P}_{k,n+i+j} = V\mathcal{P}_{k,i}V\mathcal{P}_{k,j}$,
ii. $V\mathbb{Q}_{k,n+i}V\mathbb{Q}_{k,n+j} - V\mathbb{Q}_{k,n}V\mathbb{Q}_{k,n+i+j} = (4-4k^2)V\mathcal{P}_{k,i}V\mathcal{P}_{k,j}$.

Theorem 2.17. ***(Halton Identity)*** Let $n \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We get
i. $V\mathcal{P}_{k,n+i}V\mathcal{P}_{k,n-i} - V\mathcal{P}_{k,n+j}V\mathcal{P}_{k,n-j} = \frac{1}{4k^2-4}(V\mathbb{Q}_{k,2j} - V\mathbb{Q}_{k,2i})$
ii. $V\mathbb{Q}_{k,n+i}V\mathbb{Q}_{k,n-i} - V\mathbb{Q}_{k,n+j}V\mathbb{Q}_{k,n-j} = 2V\mathbb{Q}_{k,2n} + V\mathbb{Q}_{k,2i} + V\mathbb{Q}_{k,2j}$.

Theorem 2.18. ***(Padilla Identity)*** Let $n \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We obtain
i. $V\mathcal{P}_{k,n+2}^3 + V\mathcal{P}_{k,n-1}^3 - 3V\mathcal{P}_{k,n}V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n+2}$

$= \frac{1}{4k^2-4}(V\mathcal{P}_{k,3n+6} - 3V\mathcal{P}_{k,3n+3} + V\mathcal{P}_{k,3n-3} + 3V\mathcal{P}_{k,2n+3} - 3V\mathcal{P}_{k,n+2} + 3V\mathcal{P}_{k,n+1})$
ii. $V\mathbb{Q}_{k,n+2}^3 + V\mathbb{Q}_{k,n-1}^3 - 3V\mathbb{Q}_{k,n}V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n+2} = V\mathbb{Q}_{k,3n+6} + 3V\mathbb{Q}_{k,n+2} + V\mathbb{Q}_{k,3n-3}$
$+3V\mathbb{Q}_{k,n-1} + (4k^2-4)(-V\mathcal{P}_{k,3n+3} - 3V\mathcal{P}_{k,n-1} + 3V\mathcal{P}_{k,n+1} + 3V\mathcal{P}_{k,2n+3})$.

Theorem 2.19. ***(Melham Identity)*** Let $n \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We get
i. $V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n+2}V\mathcal{P}_{k,n+6} - V\mathcal{P}_{k,n}^3 = \frac{1}{4k^2-4}(V\mathcal{P}_{k,3n+9} - V\mathcal{P}_{k,3n} - V\mathcal{P}_{k,n+7} - V\mathcal{P}_{k,n+5}$
$+3V\mathcal{P}_{k,n} - V\mathcal{P}_{k,n-3})$,
ii. $V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n+2}V\mathbb{Q}_{k,n+6} - V\mathbb{Q}_{k,n}^3 = V\mathbb{Q}_{k,3n+9} - V\mathbb{Q}_{k,3n} + V\mathbb{Q}_{k,n+7} + V\mathbb{Q}_{k,n+5}$
$-3V\mathbb{Q}_{k,n} + V\mathbb{Q}_{k,n-3}$.

Theorem 2.20. ***(Gelin-Cesaro Identity)*** Let $n \in \mathbb{N}$, $k \in \mathbb{R}^+$ and $k > 1$. We obtain
i. $V\mathcal{P}_{k,n+2}V\mathcal{P}_{k,n+1}V\mathcal{P}_{k,n-1}V\mathcal{P}_{k,n-2} - V\mathcal{P}_{k,n}^4$
$= \frac{-V\mathbb{Q}_{k,2n+4} - V\mathbb{Q}_{k,2n+2} + V\mathbb{Q}_{k,6} - V\mathbb{Q}_{k,2n-2} - V\mathbb{Q}_{k,2} - V\mathbb{Q}_{k,2n-4} + 4V\mathbb{Q}_{k,2n} - 5}{(4k^2-4)^2}$,
ii. $V\mathbb{Q}_{k,n+2}V\mathbb{Q}_{k,n+1}V\mathbb{Q}_{k,n-1}V\mathbb{Q}_{k,n-2} - V\mathbb{Q}_{k,n}^4 = V\mathbb{Q}_{k,2n+4} + V\mathbb{Q}_{k,2n+2} + V\mathbb{Q}_{k,6}$
$+V\mathbb{Q}_{k,2n-2} + V\mathbb{Q}_{k,2} + V\mathbb{Q}_{k,2n-4} - 4V\mathbb{Q}_{k,2n} - 5$.

The proofs of Theorems 2.14–2.20., can be shown in the same way as the proof of the Cassini identity, using Binet's formulas and properties of the characteristic equation of sequences.

In the following theorems, we obtain special sum formulas of the k-Vieta-Pell $V\mathcal{P}_{k,n}$ and k-Vieta-Pell-Lucas $V\mathbb{Q}_{k,n}$ sequences.

Theorem 2.21. For $n \in \mathbb{N}$, we have
i. $\sum_{t=0}^{n} V\mathcal{P}_{k,t} = \frac{1 + V\mathcal{P}_{k,n-1} + (1-2k)V\mathcal{P}_{k,n}}{2-2k}$, **ii.** $\sum_{t=0}^{n} V\mathbb{Q}_{k,t} = \frac{(1-2k)V\mathbb{Q}_{k,n} + V\mathbb{Q}_{k,n-1} - 2k + 2}{2-2k}$.

Proof. i. From the definition of the k-Vieta-Pell sequence, we have

$$V\mathcal{P}_{k,2} = 2kV\mathcal{P}_{k,1} - V\mathcal{P}_{k,0},$$
$$V\mathcal{P}_{k,3} = 2kV\mathcal{P}_{k,2} - V\mathcal{P}_{k,1},$$
$$\vdots$$
$$V\mathcal{P}_{k,n} = 2kV\mathcal{P}_{k,n-1} - V\mathcal{P}_{k,n-2}.$$

If the obtained equations are added side by side, we obtain

$$-1 + \sum_{t=0}^{n} V\mathcal{P}_{k,t} = 2k\sum_{t=1}^{n-1} V\mathcal{P}_{k,t} - \sum_{t=0}^{n-2} V\mathcal{P}_{k,t},$$
$$-1 + \sum_{t=0}^{n} V\mathcal{P}_{k,t} = \left((-V\mathcal{P}_{k,n} - V\mathcal{P}_{k,0})2k + 2k\sum_{t=0}^{n} V\mathcal{P}_{k,t}\right) - \left(-V\mathcal{P}_{k,n-1} - V\mathcal{P}_{k,n} + \sum_{t=0}^{n} V\mathcal{P}_{k,t}\right).$$

When the necessary adjustments are made, we obtain the following equation:

$$\sum_{t=0}^{n} V\mathcal{P}_{k,t} = \frac{(1-2k)V\mathcal{P}_{k,n} + V\mathcal{P}_{k,n-1} + 1}{2-2k}.$$

The proof of the other is shown similarly. □

Theorem 2.22. For $x, y, z \in \mathbb{N}$, we have
i. $\sum_{y=0}^{n} V\mathcal{P}_{k,xy} = \frac{1}{2-V\mathbb{Q}_{k,x}}(V\mathcal{P}_{k,x} + V\mathcal{P}_{k,nx} - V\mathcal{P}_{k,nx+x})$,
ii. $\sum_{y=0}^{n} V\mathbb{Q}_{k,xy} = \frac{1}{2-V\mathbb{Q}_{k,x}}(-V\mathbb{Q}_{k,nx+x} + V\mathbb{Q}_{k,nx} - V\mathbb{Q}_{k,x} + 2)$,
iii. $\sum_{y=0}^{n} V\mathcal{P}_{k,xy+z} = \begin{cases} \frac{-V\mathcal{P}_{k,nx+x+z} + V\mathcal{P}_{k,nx+z} + V\mathcal{P}_{k,x-z} + V\mathcal{P}_{k,z}}{2-V\mathbb{Q}_{k,x}}, & if\ z < x \\ \frac{-V\mathcal{P}_{k,nx+x+z} + V\mathcal{P}_{k,nx+z} + V\mathcal{P}_{k,z-x} + V\mathcal{P}_{k,z}}{2-V\mathbb{Q}_{k,x}}, & otherwise \end{cases}$,

iv. $\sum_{y=0}^{n} V\mathbb{Q}_{k,xy+z} = \begin{cases} \frac{-V\mathbb{Q}_{k,nx+x+z}+V\mathbb{Q}_{k,nx+z}-V\mathbb{Q}_{k,x-z}+V\mathbb{Q}_{k,z}}{2-V\mathbb{Q}_{k,x}}, \ if \ z < x \\ \frac{V\mathbb{Q}_{k,nx+z}-V\mathbb{Q}_{k,z-x}+V\mathbb{Q}_{k,z}-V\mathbb{Q}_{k,nx+x+z}}{2-V\mathbb{Q}_{k,x}}, \ otherwise \end{cases}.$

Proof. i. If the Binet formulas are used, we get

$\sum_{y=0}^{n} V\mathcal{P}_{k,xy} = \sum_{y=0}^{n} \frac{{r_1}^{xy}-{r_2}^{xy}}{r_1-r_2} = \frac{1}{r_1-r_2}(\sum_{y=0}^{n}({r_1}^{x})^{y} - \sum_{y=0}^{n}({r_2}^{x})^{y})$

$= \frac{1}{r_1-r_2}(\frac{{r_1}^{nx+x}-1}{{r_1}^{x}-1} - \frac{{r_2}^{nx+x}-1}{{r_2}^{x}-1}) = \frac{1}{2-V\mathbb{Q}_{k,x}}\frac{1}{r_1-r_2}({r_1}^{nx} - {r_2}^{nx} - {r_1}^{nx+x} + {r_2}^{nx+x} + {r_1}^{x} - {r_2}^{x})$

$= \frac{1}{2-V\mathbb{Q}_{k,x}}(V\mathcal{P}_{k,x} + V\mathcal{P}_{k,nx} - V\mathcal{P}_{k,nx+x}).$

The proofs of the others are shown similarly. □

Theorem 2.23. Let $k \in \mathbb{R}^{+}$. We obtain

i. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}V\mathcal{P}_{k,j} = (-1)^{n}V\mathcal{P}_{k,2n}$,

ii. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}V\mathbb{Q}_{k,j} = (-1)^{n}V\mathbb{Q}_{k,2n}$.

Proof. i. The following equations are obtained with the help of the characteristic equation of the k-Vieta-Pell sequence:

$$r_1^2 = 2kr_1 - 1 \text{ and } r_2^2 = 2kr_2 - 1.$$

$\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}V\mathcal{P}_{k,j} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}(\frac{{r_1}^{j}-{r_2}^{j}}{r_1-r_2})$

$= \frac{1}{r_1-r_2}(\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}{r_1}^{j} - \sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}{r_2}^{j})$

$= \frac{1}{r_1-r_2}[(1-2kr_1)^{n} - (1-2kr_2)^{n}] = \frac{1}{r_1-r_2}[(-{r_1}^{2})^{n} - (-{r_2}^{2})^{n}] = (-1)^{n}V\mathcal{P}_{k,2n}$

The proof of the other is shown similarly. □

Theorem 2.24. For b, p, r, n natural number and $b > r$, we obtain

i. $\sum_{j=0}^{n}\binom{n}{j}(2k)^{j}(-1)^{j}V\mathcal{P}_{k,bn+r+j} = (-1)^{n}V\mathcal{P}_{k,bn+2n+r}$,

ii. $\sum_{j=0}^{n}\binom{n}{j}(2k)^{j}(-1)^{j}V\mathbb{Q}_{k,bn+r+j} = (-1)^{n}V\mathbb{Q}_{k,bn+2n+r}$,

iii. $\sum_{j=0}^{n}\frac{V\mathcal{P}_{k,bj+r}}{p^{j}} = \frac{1}{1-pV\mathbb{Q}_{k,b}+p^{2}}\frac{1}{p^{n}}(-pV\mathcal{P}_{k,bn+b+r} + V\mathcal{P}_{k,bn+r} - p^{n+1}V\mathcal{P}_{k,b-r} + p^{n+2}V\mathcal{P}_{k,r})$

iv. $\sum_{j=0}^{n}\frac{V\mathbb{Q}_{k,bj+r}}{p^{j}} = \frac{1}{1-pV\mathbb{Q}_{k,b}+p^{2}}\frac{1}{p^{n}}(p^{n+2}V\mathbb{Q}_{k,r} - p^{n+1}V\mathbb{Q}_{k,b-r} + V\mathbb{Q}_{k,bn+r} - pV\mathbb{Q}_{k,bn+b+r})$

Proof. If Binet formulas, definitions, and geometric series are used, we obtain

i. $\sum_{j=0}^{n}\binom{n}{j}(2k)^{j}(-1)^{j}V\mathcal{P}_{k,bn+r+j} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(2k)^{j}\frac{{r_1}^{bn+r+j}-{r_2}^{bn+r+j}}{r_1-r_2}$

$= \frac{1}{r_1-r_2}[{r_1}^{bn+r}(1-2kr_1)^{n} - {r_2}^{bn+r}(1-2kr_2)^{n}]$

$= \frac{(-1)^{n}}{r_1-r_2}({r_1}^{bn+2n+r} - {r_2}^{bn+2n+r}) = (-1)^{n}V\mathcal{P}_{k,bn+2n+r}.$

The proof of the other is shown similarly. □

Theorem 2.25. Let $k \in \mathbb{R}$ and $b, r, t, n \in \mathbb{N}$. We obtain

i. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}\frac{V\mathcal{P}_{k,bn+r+j}}{(2k)^{j}} = \frac{V\mathcal{P}_{k,bn-n+r}}{(2k)^{n}}$,

ii. $\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}\frac{V\mathbb{Q}_{k,bn+r+j}}{(2k)^{j}} = \frac{V\mathbb{Q}_{k,bn-n+r}}{(2k)^{n}}$,

iii. $\sum_{j=0}^{n}(\frac{t}{k})^{j}(tV\mathcal{P}_{k,j-r+2} - kV\mathcal{P}_{k,j-r+1}) = (\frac{t}{k})^{n}tV\mathcal{P}_{k,n-r+2} - kV\mathcal{P}_{k,-r+1}$,

iv. $\sum_{j=0}^{n}(\frac{t}{k})^{j}\left(tV\mathbb{Q}_{k,j-r+2}-kV\mathbb{Q}_{k,j-r+1}\right)=(\frac{t}{k})^{n}tV\mathbb{Q}_{k,n-r+2}-kV\mathbb{Q}_{k,-r+1}$.

Proof. With the help of characteristic equation, we have

$$r_1^2=2kr_1-1,\ r_2^2=2kr_2-1.$$

i. $\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{V\mathcal{P}_{k,bn+r+j}}{(2k)^{j}}=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\frac{{r_1}^{bn+r+j}-{r_2}^{bn+r+j}}{(2k)^{j}(r_1-r_2)}$

$=\frac{1}{r_1-r_2}\left[{r_1}^{bn+r}\left(1-\frac{r_1}{2k}\right)^{n}-{r_2}^{bn+r}\left(1-\frac{r_2}{k}\right)^{n}\right]=\frac{1}{r_1-r_2}[{r_1}^{bn+r}\left(\frac{1}{2kr_1}\right)^{n}-{r_2}^{bn+r}\left(\frac{1}{2kr_2}\right)^{n}]$

$=\frac{1}{(r_1-r_2)(2k)^{n}}({r_1}^{bn-n+r}-{r_2}^{bn-n+r})=\frac{V\mathcal{P}_{k,bn-n+r}}{(2k)^{n}}$.

The proof of the other is shown similarly. □

Theorem 2.26. Let $k\in\mathbb{R}$ and $b,r,t,n\in\mathbb{N}$. We obtain

i. $\sum_{j=0}^{n}(-1)^{j}(\frac{t}{k})^{j}\left(tV\mathcal{P}_{k,j-r+2}-kV\mathcal{P}_{k,j-r+1}\right)=\frac{1}{k^2+t+2tk^2}(-1)^{n+1}\left(\frac{t}{k}\right)^{n}$
$\left[t(k^2-t^2)V\mathcal{P}_{k,n-r+2}-2tkV\mathbb{Q}_{k,n-r+2}\right]+\frac{k}{k^2+t+2tk^2}[(t^2-k^2)V\mathcal{P}_{k,-r+1}+2tkV\mathbb{Q}_{k,-r+1}]$,

ii. $\sum_{j=0}^{n}(-1)^{j}(\frac{t}{k})^{j}\left(tV\mathbb{Q}_{k,j-r+2}-kV\mathbb{Q}_{k,j-r+1}\right)=\frac{k}{k^2+t+2tk^2}[(-1)^{n+1}\left(\frac{t}{k}\right)^{n+1}$
$(k^2-t^2)V\mathbb{Q}_{k,n-r+2}-2tk(4k^2-4)V\mathcal{P}_{k,n-r+2}+[(t^2-k^2)V\mathbb{Q}_{k,-r+1}$
$+2tk(4k^2-4)V\mathcal{P}_{k,-r+1}]$.

Proof. The proofs are shown in a similar to Theorem 2.25. □

In the following theorems, we obtain special generating functions of the k-Vieta-Pell $V\mathcal{P}_{k,n}$ and k-Vieta-Pell-Lucas $V\mathbb{Q}_{k,n}$ sequences.

Theorem 2.27. We obtain

i. $\wp(x)=\sum_{n=0}^{\infty}V\mathcal{P}_{k,n}x^{n}=\frac{x}{1-2kx+x^2}$, **ii.** $q(x)=\sum_{n=0}^{\infty}V\mathbb{Q}_{k,n}x^{n}=\frac{2-3kx}{1-2kx+x^2}$.

Proof. i. By the definition of the k-Vieta-Pell sequence, we get

$\wp(x)=\sum_{n=0}^{\infty}V\mathcal{P}_{k,n}x^{n}=x+\sum_{n=2}^{\infty}V\mathcal{P}_{k,n}x^{n}$
$=x+2k\sum_{n=2}^{\infty}V\mathcal{P}_{k,n-1}x^{n}-\sum_{n=2}^{\infty}V\mathcal{P}_{k,n-2}x^{n}=x+2kx\sum_{n=1}^{\infty}V\mathcal{P}_{k,n}x^{n}-x^2\sum_{n=0}^{\infty}V\mathcal{P}_{k,n}x^{n}$
$=\frac{x}{1-2kx+x^2}$.

The proof of the other is shown similarly. □

Theorem 2.28. For $V\mathcal{P}_{k,n}$, and $V\mathbb{Q}_{k,n}$ sequences, the Binet formulas can be obtained with the help of the generating functions.

Proof. With the help of the roots of the characteristic equation of these sequences, the roots of the $1-2kx+x^2=0$ equation become $\frac{1}{r_1}$ and $\frac{1}{r_2}$. For $\mathcal{M}_{k,n}$, we obtain

$$\frac{x}{1-2kx+x^2}=\frac{1}{r_1-r_2}\frac{1}{1-r_1x}-\frac{1}{r_1-r_2}\frac{1}{1-r_2x}$$
$$=\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}{r_1}^{n}x^{n}-\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}{r_2}^{n}x^{n}=\frac{1}{r_1-r_2}\sum_{n=0}^{\infty}({r_2}^{n}-{r_2}^{n})x^{n}$$
$$=\sum_{n=0}^{\infty}V\mathcal{P}_{k,n}x^{n}.$$

Similarly, the Binet formula of the sequence $V\mathbb{Q}_{k,n}$ is found. □

Theorem 2.29. For $a,b\in\mathbb{N}$, and $b>a$, we obtain

i. $\sum_{i=0}^{\infty}V\mathcal{P}_{k,bn}x^{n}=\frac{xV\mathcal{P}_{k,b}}{1-xV\mathbb{Q}_{k,b}+x^2}$, **ii.** $\sum_{i=0}^{\infty}V\mathbb{Q}_{k,bn}x^{n}=\frac{2-xV\mathbb{Q}_{k,a}}{1-xV\mathbb{Q}_{k,a}+x^2}$,

iii. $\sum_{i=0}^{\infty} V\mathcal{P}_{k,an+b}x^n = \frac{V\mathcal{P}_{k,b}-xV\mathcal{P}_{k,b-a}}{1-xV\mathbb{Q}_{k,a}+x^2}$, **iv.** $\sum_{i=0}^{\infty} V\mathbb{Q}_{k,an+b}x^n = \frac{\sqrt{k^2-4}V\mathcal{P}_{k,b}-xV\mathcal{P}_{k,b-a}}{1-xV\mathbb{Q}_{k,a}+x^2}$,

v. $\sum_{n=0}^{\infty} \frac{V\mathcal{P}_{k,bn}}{n!}x^n = \frac{e^{r_1{}^b x}-e^{r_2{}^b x}}{r_1-r_2}$, **vi.** $\sum_{n=0}^{\infty} \frac{V\mathbb{Q}_{k,bn}}{n!}x^n = e^{r_1{}^b x}+e^{r_2{}^b x}$.

Proof. i. If the Binet formulas are used, we have

$$\sum_{i=0}^{\infty} V\mathcal{P}_{k,bn}x^n = \sum_{n=0}^{\infty} \frac{r_1{}^{bn}-r_2{}^{bn}}{r_1-r_2}x^n = \frac{1}{r_1-r_2}\sum_{n=0}^{\infty}(r_1{}^b x)^n - \frac{1}{r_1-r_2}\sum_{n=0}^{\infty}(r_2{}^b x)^n$$
$$= \frac{1}{r_1-r_2}\left(\frac{1}{1-r_1{}^b x} - \frac{1}{1-r_2{}^b x}\right) = \frac{xV\mathcal{P}_{k,b}}{1-xV\mathbb{Q}_{k,b}+x^2}.$$

The proofs of the others are shown similarly. □

3. Relations between special sequences

In this chapter, we examine the relations of the k-Vieta-Pell sequence with the Fibonacci, Pell, Chebyshev polynomials of the first kind, k-Oresme, Balancing, Mersenne, Oresme sequences and k-Vieta-Pell-Lucas sequence with the Lucas, Pell-Lucas numbers, Chebyshev polynomials of the second kind, k-Oresme-Lucas, Balancing-Lucas, Mersenne-Lucas, Oresme-Lucas sequences, respectively. In addition, for special k values, these sequences are associated with the sequences in OEIS.

Theorem 3.1. Let $k=\frac{3}{2}$, $k=\frac{7}{2}$ and $n\in\mathbb{N}$ values. Then, the following relations can be written between the k-Vieta-Pell sequence and Fibonacci sequence F_n, k-Vieta-Pell-Lucas sequence and Lucas sequence L_n, respectively;

i. $V\mathcal{P}_{\frac{3}{2},n} = F_{2n}$ and $V\mathcal{P}_{\frac{7}{2},n} = \frac{F_{4n}}{3}$, **ii.** $V\mathbb{Q}_{\frac{3}{2},n} = L_{2n}$ and $V\mathbb{Q}_{\frac{7}{2},n} = L_{4n}$.

Proof. i. The Binet formula of the k-Vieta-Pell sequence is

$$V\mathcal{P}_{k,n} = \frac{r_1^n-r_2^n}{r_1-r_2} = \frac{(k+\sqrt{k^2-1})^n-(k-\sqrt{k^2-1})^n}{2\sqrt{k^2-1}}.$$

For $k=\frac{3}{2}$ and $k=\frac{7}{2}$, the following relations can be written:

$$V\mathcal{P}_{\frac{3}{2},n} = \frac{(\frac{3+\sqrt{5}}{2})^n-(\frac{3-\sqrt{5}}{2})^n}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^{2n}-(\frac{1-\sqrt{5}}{2})^{2n}}{\sqrt{5}} \text{ and } V\mathcal{P}_{\frac{7}{2},n} = \frac{(\frac{7+3\sqrt{5}}{2})^n-(\frac{7-3\sqrt{5}}{2})^n}{3\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^{4n}-(\frac{1-\sqrt{5}}{2})^{4n}}{3\sqrt{5}}.$$

Thus, we obtain

$$V\mathcal{P}_{\frac{3}{2},n} = F_{2n} \text{ and } V\mathcal{P}_{\frac{7}{2},n} = \frac{F_{4n}}{3}.$$

ii. The Binet formula of the k-Vieta-Pell-Lucas sequence is

$$V\mathbb{Q}_{k,n} = r_1^n + r_2^n = (k+\sqrt{k^2-1})^n + (k-\sqrt{k^2-1})^n.$$

For $k=\frac{3}{2}$ and $k=\frac{7}{2}$, the following relations can be written:

$$V\mathbb{Q}_{\frac{3}{2},n} = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n = (\frac{1+\sqrt{5}}{2})^{2n} + (\frac{1-\sqrt{5}}{2})^{2n}$$

and

$$V\mathbb{Q}_{\frac{7}{2},n} = (\frac{7+3\sqrt{5}}{2})^n + (\frac{7-3\sqrt{5}}{2})^n = (\frac{1+\sqrt{5}}{2})^{4n} + (\frac{1-\sqrt{5}}{2})^{4n}.$$

Thus, we have

$$V\mathbb{Q}_{\frac{3}{2},n} = L_{2n} \text{ and } V\mathbb{Q}_{\frac{7}{2},n} = L_{4n}.$$ □

Theorem 3.2. Let $k = 3$ and $n \in \mathbb{N}$ values. Then, the following relations can be written between the k-Vieta-Pell and Pell sequence p_n, k-Vieta-Pell-Lucas sequence and Pell-Lucas sequence q_n, respectively;

i. $V\mathcal{P}_{3,n} = \frac{p_{2n}}{2}$, **ii.** $V\mathbb{Q}_{3,n} = q_{2n}$.

Proof. The proofs are shown in a similar to Theorem 3.1. □

Theorem 3.3. Let's take $\frac{k}{2}$ instead of k value in the Binet formulas of the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences. The following relations can be written between the k-Vieta-Pell and k-Oresme sequence $O_{k,n}$, k-Vieta-Pell-Lucas sequence and k-Oresme-Lucas sequence $P_{k,n}$, respectively;

i. $V\mathcal{P}_{\frac{k}{2},n} = k^n O_{k,n}$, **ii.** $V\mathbb{Q}_{\frac{k}{2},n} = k^n P_{k,n}$.

Proof. Let a value of $\frac{k}{2}$ instead of k.

i. The Binet formula of the k-Vieta-Pell sequence is

$$V\mathcal{P}_{\frac{k}{2},n} = \frac{(\frac{k}{2}+\sqrt{\frac{k^2}{4}-1})^n-(\frac{k}{2}-\sqrt{\frac{k^2}{4}-1})^n}{2\sqrt{\frac{k^2}{4}-1}} = \frac{(\frac{k+\sqrt{k^2-4}}{2})^n-\left(\frac{k-\sqrt{k^2-4}}{2}\right)^n}{\sqrt{k^2-4}}.$$

Thus, we have

$$V\mathcal{P}_{\frac{k}{2},n} = k^n O_{k,n}.$$

ii. The Binet formula of the k-Vieta-Pell-Lucas sequence is

$$V\mathbb{Q}_{\frac{k}{2},n} = r_1^n + r_2^n = (\frac{k}{2}+\sqrt{\frac{k^2}{4}-1})^n + (\frac{k}{2}-\sqrt{\frac{k^2}{4}-1})^n = (\frac{k+\sqrt{k^2-4}}{2})^n + (\frac{k-\sqrt{k^2-4}}{2})^n.$$

Thus, we obtain

$$V\mathbb{Q}_{\frac{k}{2},n} = k^n P_{k,n}.$$ □

Theorem 3.4. Let $n \in \mathbb{N}$. The following relations can be written between the k-Vieta-Pell sequence and Chebyshev polynomials of the first kind U_n, k-Vieta-Pell-Lucas sequence and the Chebyshev polynomials of the second kind T_n, respectively;

i. $V\mathcal{P}_{k,n} = U_{n-1}(k)$, **ii.** $V\mathbb{Q}_{k,n} = 2T_n(k)$.

Proof. The proofs are shown in a similar to Theorem 3.1. □

Theorem 3.5. Let $k = 17$ and $n \in \mathbb{N}$ values. Then, the following relations can be written between the k-Vieta-Pell sequence and Balancing sequence B_n, k-Vieta-Pell-Lucas sequence and Balancing-Lucas sequence C_n, respectively;

i. $V\mathcal{P}_{17,n} = \frac{1}{6}B_{2n}$, **ii.** $V\mathbb{Q}_{17,n} = C_{2n}$.

Proof. The proofs are shown in a similar to Theorem 3.1. □

Theorem 3.6. Let $k = \frac{65}{16}$ and $n \in \mathbb{N}$ values. Then, the following relations can be written between the k-Vieta-Pell sequence and Mersenne sequence M_n, k-Vieta-Pell-Lucas sequence and Mersenne-Lucas sequence N_n, respectively;

i. $V\mathcal{P}_{\frac{65}{16},n} = \frac{1}{63}\frac{1}{8^{n-1}}M_{6n}$, **ii.** $V\mathbb{Q}_{\frac{65}{16},n} = \frac{1}{8^n}N_{6n}$.

Proof. The proofs are shown in a similar to Theorem 3.1. □

Theorem 3.7. For the $k = \frac{5}{4}$ and $n \in \mathbb{N}$ values. Then, the following relations can be written between the k-Vieta-Pell sequence and Oresme sequence R_n, k-Vieta-Pell-Lucas sequence and Oresme-Lucas sequence H_n, respectively;

i. $\mathcal{M}_{\frac{5}{4},n} = \frac{2}{3n}(4^n - 1)R_n$, **ii.** $\mathcal{L}_{\frac{5}{4},n} = \frac{1}{2}(4^n + 1)H_n$.

Proof. The proofs are shown in a similar to Theorem 3.1. □

Theorem 3.8. Let $n \in \mathbb{N}$. The following relations are provided for some k values.

i. For $k = 2$, $V\mathcal{P}_{2,n} = A_n$ and $V\mathbb{Q}_{2,n} = B_n$, **ii.** For $k = 5$, $V\mathcal{P}_{5,n} = C_n$ and $V\mathbb{Q}_{5,n} = D_n$,

iii. For $k = 7$, $V\mathcal{P}_{7,n} = E_n$ and $V\mathbb{Q}_{7,n} = F_n$.

Here, A_n, B_n, C_n, D_n, E_n and F_n sequences are A001353, A003500, A004189, A087799, A007655 and A067902 sequences in OEIS, respectively.

Proof. The proofs are shown in a similar to Theorem 3.1. □

4. Conclusion

In this paper, we defined the k-Vieta-Pell and k-Vieta-Pell-Lucas sequences. Then, we obtained the many features of these sequences. Also, we found the relationships between the terms of these sequences. In addition, we calculate the special identities of these sequences. Moreover, we examine the relations of the k-Vieta-Pell sequence with the Fibonacci, Pell, Chebyshev polynomials of the first kind, k-Oresme, Balancing, Mersenne, Oresme sequences and k-Vieta-Pell-Lucas sequence with the Lucas, Pell-Lucas numbers, Chebyshev polynomials of the second kind, k-Oresme-Lucas, Balancing-Lucas, Mersenne-Lucas, Oresme-Lucas sequences, respectively. Finally, for special k values, these sequences are associated with the sequences in OEIS. If this study is examined, such features can be found in other sequences such as Jacobsthal and Oresme sequences.

Acknowledgments

We would like to thank the referees and the editor for their contributions to our article.

References

[1] Aydınyüz, S., Aşcı, M., ‘‘The Moore-Penrose Inverse of the Rectangular Fibonacci Matrix and Applications to the Cryptology’’, Advances and Applications in Discrete Mathematics 40(2) (2023) : 195-211.

[2] Turner, H.A., Humpage, M., Kerp, H., Hetherington, A.J., ‘‘Leaves and sporangia developed in rare non-Fibonacci spirals in early leafy plants’’, Science 380(6650) (2023) : 1188-1192.

[3] Avazzadeh, Z., Hassani, H., Agarwal, P., Mehrabi, S., Javad Ebadi, M., Hosseini Asl, M. K., ‘‘Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials’’, Mathematical Methods in the Applied Sciences 46(8) (2023) : 9332-9350.

[4] Otto, H.H., ‘‘Fibonacci Stoichiometry and Superb Performance of Nb16W5O55 and Related Super-Battery Materials’’, Journal of Applied Mathematics and Physics 10(6) (2022) : 1936-1950.

[5] de Oliveira, R.R., Alves, F.R.V., ‘’An investigation of the bivariate complex fibonacci polynomials supported in didactic engineering: an application of Theory of Didactics Situations (TSD)’’, Acta Scientiae 21(3) (2019) : 170-195.

[6] Akkuş, H., Üregen, R.N., Özkan, E., ‘‘A New Approach to $k-$Jacobsthal Lucas Sequences’’, Sakarya University Journal of Science 25(4) (2021) : 969-973.

[7] Tastan, M., Özkan, E., ‘‘Catalan transform of the k-Pell, k-Pell-Lucas and modified k-Pell sequence’’, Notes On Number Theory and Discrete Mathematics 27(1) (2021) : 198-207.

[8] Özkan, E., Akkus, H., ‘‘On k-Chebsyhev Sequence’’, Wseas Transactions On Mathematics 22 (2023) : 503-507.

[9] Akkuş, H., Deveci, Ö., Özkan, E., Shannon, A.G., ‘‘Discatenated and lacunary recurrences’’, Notes on Number Theory and Discrete Mathematics 30(1) (2024) : 8-19.

[10] Özkan, E., Akkuş, H., ‘‘Copper ratio obtained by generalizing the Fibonacci sequence’’, AIP Advances 14(7) (2024) : 1-11.

[11] Horadam, A.F., ‘‘Vieta polynomials’’, Fibonacci Quarterly 40(3) (2002) : 223-232.

[12] Shannon, A.G., Horadam, A.F. ‘‘Some relationships among Vieta, Morgan- Voyce and Jacobsthal polynomials’’, Applications of Fibonacci Numbers 8 (1999) : 307-323.

[13] Robbins, N., ‘‘Vieta's triangular array and a related family of polynomials’’, International Journal of Mathematics and Mathematical Sciences 14 (1991) : 239-244.

[14] Mason, J.C., Handscomb, D.C., ‘‘Chebyshev polynomials’’, CRC press, New York 2002.

[15] Griggs, J.R., Hanlon, P., Odlyzko, A.M., Waterman, M.S., ‘‘On the number of alignments of k sequences’’, Graphs and Combinatorics 6(2) (1990) : 133-146.

[16] Falcon, S., Plaza, A., ‘‘On the Fibonacci k-numbers’’, Chaos Solitons Fractals 32(5) (2007) : 1615–1624.

[17] Falcon, S., ‘‘On the k-Lucas numbers’’, International Journal of Contemporary Mathematical Sciences 6(21) (2011) : 1039-1050.

[18] Falcon, S., ‘‘Catalan Transform of the k-Fibonacci sequence’’, Communications of the Korean Mathematical Society 28(4) (2013) : 827-832.

[19] Shannon, A.G., Akkuş, H., Aküzüm, Y., Deveci, Ö., Özkan, E., ‘‘A partial recurrence Fibonacci link’’, Notes on Number Theory And Discrete Mathematics 30(3) (2024) : 530-537.

[20] Soykan, Y., Taşdemir, E., Okumuş, I., ‘‘A study on binomial transform of the generalized padovan Sequence’’, Journal of Science and Arts 22(1) (2022) : 63-90.

[21] Özkan, E., Akkuş, H., ‘‘A New Approach to k-Oresme and k-Oresme-Lucas Sequences’’, Symmetry 16 (2024) : 1-12.

[22] Koshy, T., ‘‘Fibonacci and Lucas Numbers with Applications’’, Volume 2. John Wiley & Sons (2019).

[23] Özkan, E., Akkuş, H., ‘‘Generalized Bronze Leonardo sequence’’, Notes on Number Theory and Discrete Mathematics 30(4) (2024) : 811-824.

[24] Akkuş, H., Özkan, E., ‘‘Generalization of the k-Leonardo Sequence and their Hyperbolic Quaternions’’, Mathematica Montisnigri 60 (2024) : 14-31.