Journal of Universal Mathematics
To Memory Assoc. Prof. Dr. Zeynep Akdemirci Şanlı
Vol.7, pp.57-66 (2024)
ISSN-2618-5660
DOI: 10.33773/jum.1562659

BINORMAL CURVES

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Abstract. In this paper, a new class of curves is called binormal curves are introduced. Here,we calculate the Frenet vector fields of binormal curves and using them give the curvature and torsion of such curves. Also, we provide some examples of binormal curves in Euclidean space R^3.

1. Introduction

Curves are the most important tools for elementary differential geometry. In the study of fundamental theory and characterization of space curves, the calculation of the curvature function, the torsion function and Frenet frame are very interesting and important problem in three dimensional space. So there are many articles on curves in the literature. In [1], the authors investigate the quaternionic Bertrand curves in Euclidean $3-$space. In the papers [2, 8], the curve theory in Galilean space is investigated. Further studies about curves and its applications are found in [3, 4, 5, 9]. In addition to these, the motion of parallel curves in Euclidean $3-$space is given in [7]. Gözütok, Çoban and Sağıroğlu [6] are study the classical differential geometry of curves with respect to conformable fractional derivative. In [7], Aldossary and Gazwani defined the notion of parallel curve based on binormal vector and calculated the Frenet frame of such a curve. Then, in [11] Sağıroğlu and Köse defined the notion of normal curve based on normal vector and gave some properties. Inspried by [11], in this paper we introduce a new class of curves which is called binormal curves. Here we study the Frenet frame of such curves and then calculate the curvature and torsion of binormal curves in Euclidean space R^3. Also, we provide some examples of binormal curves. The definitions of curvature, torsion and Frenet frame of a curve is given in [10].

2. Preliminaries

Let $\alpha : I \to R^3$ be a unit speed curve , so $\|\alpha'(s)\| = 1$ for each s in I. Then

Date: **Received:** 2024-10-07; **Accepted:** 2024-11-05.
2000 *Mathematics Subject Classification.* 53A04; 14HXX.
Key words and phrases. Curvature, Torsion, Frenet Frame, Curve.

$T = \alpha'$ is called the unit tangent vector field on α. Since T has constant lenght 1, its derivative $T' = \alpha''$ measures the way the curve is turning in R^3. Differention of $T.T = 1$ gives T' always orthogonal to T. The lenght of T' gives a numerical measurement of the turning of α. The function $\kappa(s) = \|T'(s)\|$ for all s in I is called the curvature function of α. The unit vector field $N = \frac{T'}{\kappa}$ on α is called the principal normal vector field on α. The vector field $B = T \times N$ is called the binormal vector field of α. The vector fields T, N, B are called the Frenet frame field of α. Also, it is clear that the torsion function τ of α satisfies $B' = -\tau N$ such that the torsion function τ measures twisting of α.

Theorem 2.1. [10] *If $\alpha : I \to R^3$ is a unit speed curve with curvature $\kappa > 0$ and torsion τ, then*

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N. \end{aligned} \tag{2.1}$$

Theorem 2.2. [10] *Let α be a regular curve in R^3. Then*

$$\begin{gathered} T = \frac{\alpha'}{\|\alpha'\|}, B = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, N = B \times T \\ \kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\alpha' \times \alpha''.\alpha'''}{\|\alpha' \times \alpha''\|^2}. \end{gathered} \tag{2.2}$$

3. The Curvature Function, The Torsion Function and Frenet Frame of a Binormal Curve

Definition 3.1. Let α be a unit speed curve in R^3 and $t(s), n(s), b(s)$ be its Franet frame in a point $\alpha(s)$. Including k to represent a real number constant, binormal curve of the curve α is defined as

$$\alpha_b(s) = \alpha(s) + kb(s). \tag{3.1}$$

Now let us consider the binormal curve of a curve α having the equation (3.1). Let be s_3 be the arc lenght function of this curve. So, the unit tangent vector of the curve $\alpha_b(s)$ has the form;

$$t_b(s_3) = \frac{d\alpha_b}{ds}(s_3).\frac{ds}{ds_3} = (t(s) + kb'(s))(s_3).\frac{ds}{ds_3} = (t(s) - k\tau(s)n(s))(s_3).\frac{ds}{ds_3}. \tag{3.2}$$

If we take the dot product with t of both sides;

$$t_b.t = \frac{ds}{ds_3} = cos\theta \tag{3.3}$$

is obtained, where θ is the angle between the vectors t_b and t. If it is multiplied both sides of this equation by the vector b, we get the equation

$$t_b.b = 0. \tag{3.4}$$

This shows that the vectors t_b and b are orthogonal. Multiplying both sides by the vector n

$$t_b.n = -k\tau(s)\frac{ds}{ds_3} \tag{3.5}$$

is found. If the norm of both sides of the expression $t_b(s_3)$ is taken, we get

$$\|t_b(s_3)\| = \left\|(t(s) - k\tau(s)n(s))\frac{ds}{ds_3}\right\| = \|(t(s) - k\tau(s)n(s))\|\frac{ds}{ds_3}$$
$$= \frac{ds}{ds_3}\left[[(t(s) - k\tau(s)n(s))\right]^{\frac{1}{2}} = \frac{ds}{ds_3}\left[1 + k^2\tau^2(s)\right]^{\frac{1}{2}} \quad (3.6)$$

Then since $\|t_b(s_3)\| = 1$,

$$\frac{ds_3}{ds} = \sqrt{1 + k^2\tau^2(s)} \quad (3.7)$$

and from here

$$\frac{ds}{ds_3} = \frac{1}{\sqrt{1 + k^2\tau^2(s)}} \quad (3.8)$$

is obtained.
If it is taken as $t_b = n$, from the derivative of the vector $t_b(s_3)$ with respect to s_3, then in according to Frenet formulas

$$\kappa_b n_b = (-\kappa(s)t(s) + \tau(s)b(s)) . \frac{ds}{ds_3} \quad (3.9)$$

is found. If we substitute the expression for $\frac{ds}{ds_3}$ in here, we get

$$\kappa_b n_b = \frac{-\kappa(s)t(s) + \tau(s)b(s)}{\sqrt{1 + k^2\tau^2(s)}}. \quad (3.10)$$

If we take the dot product of this vector with itself,
(3.11)
$$\kappa_b n_b . \kappa_b n_b = \kappa_b^2 = \frac{1}{1 + k^2\tau^2(s)}\left(-\kappa(s)t(s) + \tau(s)b(s)\right) . \left(-\kappa(s)t(s) + \tau(s)b(s)\right)$$
$$= \frac{\kappa^2(s) + \tau^2(s)}{1 + k^2\tau^2(s)}$$

is obtained. Principal normal vector and binormal vector of binormal curve of the curve α are

$$n_b = \frac{-\kappa(s)t(s) + \tau(s)b(s)}{\kappa_b\sqrt{1 + k^2\tau^2(s)}} \quad (3.12)$$

and

$$b_b = t_b \times n_b = \left(\frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}}\right) \times \left(\frac{-\kappa(s)t(s) + \tau(s)b(s)}{\kappa_b\sqrt{1 + k^2\tau^2(s)}}\right)$$
$$= \frac{1}{\kappa_b\left(1 + k^2\tau^2(s)\right)}\left(-k\tau^2(s)t(s) - \tau(s)n(s) - k\kappa(s)\tau(s)b(s)\right) \quad (3.13)$$

respectively.
Now let us determine the Frenet frame of the curve $\alpha_b(s)$ in terms of the Frenet frame of $\alpha(s)$ in the general case. We know that

$$\frac{d\alpha_b(s)}{ds} = \frac{d\alpha(s)}{ds} + k\frac{db(s)}{ds} = t(s) + k(-\tau(s)n(s)) = t(s) - k\tau(s)n(s). \quad (3.14)$$

If we take the norm of this equation;

$$\left\|\frac{d\alpha_b(s)}{ds}\right\| = \sqrt{1 + k^2\tau^2(s)} = K(s) \quad (3.15)$$

is obtained. Then, the unit tangent vector $t_b(s)$ is as;

$$t_b(s) = \frac{\frac{d\alpha_b(s)}{ds}}{\left\|\frac{d\alpha_b(s)}{ds}\right\|} = \frac{t(s) - k\tau(s)n(s)}{\sqrt{1+k^2\tau^2(s)}}. \tag{3.16}$$

Moreover;

$$\begin{aligned} \frac{d^2\alpha_b(s)}{ds^2} &= \frac{dt(s)}{ds} - k\tau'(s)n(s) - k\tau(s)\frac{dn(s)}{ds} \\ &= k\kappa(s)\tau(s)t(s) + (\kappa(s) - k\tau'(s))n(s) - k\tau^2(s)b(s) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2} &= [t(s) - k\tau(s)n(s)] \times \left[k\kappa(s)\tau(s)t(s) + (\kappa(s) - k\tau'(s))n(s) - k\tau^2(s)b(s)\right] \\ &= (\kappa(s) - k\tau'(s))t(s) \times n(s) - k\tau^2(s)t(s) \times b(s) - k^2\kappa(s)\tau^2(s)n(s) \times t(s) + k^2\tau^3(s)n(s) \times b(s) \\ &= k^2\tau^3(s)t(s) + k\tau^2(s)n(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))b(s) \end{aligned} \tag{3.18}$$

are obtained. The norm of this expression is found as;

$$\left\|\frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2}\right\| = \sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2} = L(s) \tag{3.19}$$

The third derivative of the curve $\alpha_b(s)$ with respect to the s parameter is;

$$\begin{aligned} \frac{d^3\alpha_b(s)}{ds^3} &= k\kappa'(s)\tau(s)t(s) + k\kappa(s)\tau'(s)t(s) + k\kappa(s)\tau(s)\frac{dt(s)}{ds} + (\kappa'(s) - k\tau''(s))n(s) \\ &\quad + (\kappa(s) - k\tau'(s))\frac{dn(s)}{ds} - 2k\tau(s)\tau'(s)b(s) - k\tau^2(s)\frac{db(s)}{ds}. \\ &= (k\kappa'(s)\tau(s) + k\kappa(s)\tau'(s) - \kappa^2(s) + k\kappa(s)\tau'(s))t(s) + (k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s))n(s) \\ &\quad + (\kappa(s)\tau(s) - k\tau(s)\tau'(s) - 2k\tau(s)\tau'(s))b(s) \\ &= (k\kappa'(s)\tau(s) - \kappa^2(s) + 2k\kappa(s)\tau'(s))t(s) + (k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s))n(s) \\ &\quad + (\kappa(s)\tau(s) - 3k\tau(s)\tau'(s))b(s). \end{aligned} \tag{3.20}$$

Then we get

$$\begin{aligned} \left(\frac{d\alpha_b(s)}{ds} \times \frac{d^2\alpha_b(s)}{ds^2}\right).\frac{d^3\alpha_b(s)}{ds^3} &= k^2\tau^3(s).(k\kappa'(s)\tau(s) - \kappa^2(s) + 2k\kappa(s)\tau'(s)) \\ &\quad + k\tau^2(s).(k\kappa^2(s)\tau(s) + \kappa'(s) - k\tau''(s) + k\tau^3(s)) \\ &\quad + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau(s)).(\kappa(s)\tau(s) - 3k\tau(s)\tau'(s)) \\ &= k^3\kappa'(s)\tau^4(s) - k^2\kappa^2(s)\tau^3(s) + 2k^3\kappa(s)\tau'(s)\tau^3(s) + k^2\kappa^2(s)\tau^3(s) + k\kappa'(s)\tau^2(s) \\ &\quad - k^2\tau^2(s)\tau''(s) + k^2\tau^5(s) + \kappa^2(s)\tau(s) - 3k\kappa(s)\tau(s)\tau'(s) \\ &\quad - k\kappa(s)\tau(s)\tau'(s) + 3k^2\tau(s)(\tau'(s))^2 + k^2\kappa^2(s)\tau^3(s) - 3k^3\kappa(s)\tau^3(s)\tau'(s) \\ &= k^3\kappa'(s)\tau^4(s) + k^2\kappa^2(s)\tau^3(s) - k^3\kappa(s)\tau'(s)\tau^3(s) + k\kappa'(s)\tau^2(s) - k^2\tau^2(s)\tau''(s) \\ &\quad + k^2\tau^5(s) + \kappa^2(s)\tau(s) - 4k\kappa(s)\tau(s)\tau'(s) + 3k^2\tau(s)(\tau'(s))^2. \end{aligned} \tag{3.21}$$

In this case, the expressions of the curvature and torsion of the curve $\alpha_b(s)$ in terms of the curvature and torsion of the curve $\alpha(s)$ are;

(3.22)
$$\kappa_b(s) = \frac{\|\alpha_b'(s) \times \alpha_b''(s)\|}{\|\alpha_b'(s)\|^3} = \frac{\sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2}}{(1 + k^2\tau^2(s))^{\frac{3}{2}}}$$

and

(3.23)
$$\tau_b(s) = \frac{(\alpha_b'(s) \times \alpha_b''(s)).\alpha_b'''(s)}{\|\alpha_b'(s) \times \alpha_b''(s)\|^2}$$
$$= \frac{k^3\kappa'\tau^4 + k^2\kappa^2\tau^3 - k^3\kappa\tau'\tau^3 + k\kappa'\tau^2 - k^2\tau^2\tau'' + k^2\tau^5 + \kappa^2\tau - 4k\kappa\tau\tau' + 3k^2\tau(\tau')^2}{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}$$

We also obtained the unit tangent vector of the curve $\alpha_b(s)$ as;

$$t_b(s) = \frac{\frac{d\alpha_b(s)}{ds}}{\left\|\frac{d\alpha_b(s)}{ds}\right\|} = \frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}} \tag{3.24}$$

The other Frenet frame elements of this curve;

$$b_b(s) = \frac{\alpha_b'(s) \times \alpha_b''(s)}{|\alpha_b'(s) \times \alpha_b''(s)\|} = \frac{k^2\tau^3 t + k\tau^2 n + (\kappa - k\tau' + k^2\kappa\tau^2)b}{\sqrt{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}} \tag{3.25}$$

and

(3.26)
$$n_b(s) = b_b(s) \times t_b(s) = \frac{1}{K(s).L(s)}[k^2\tau^3 t + k\tau^2 n + (\kappa - k\tau' + k^2\kappa\tau^2)b] \times [t - k\tau n]$$
$$= \frac{1}{K(s).L(s)}[-k^3\tau^4 t \times n + k\tau^2 n \times t + (\kappa - k\tau' + k^2\kappa^2\tau^2)b \times t + (-k\kappa\tau + k^2\tau\tau' - k^3\kappa\tau^3)b \times n]$$
$$= \frac{1}{K(s).L(s)}[(k\kappa\tau - k^2\tau\tau' + k^3\kappa\tau^3)t + (\kappa - k\tau' + k^2\kappa^2\tau^2)n + (-k^3\tau^4 - k\tau^2)b].$$

Theorem 3.2. *The expression of the Frenet frame of the binormal curve $\alpha_b(s)$ in terms of the Frenet frame of the curve $\alpha(s)$ is of the form;*

$$\begin{aligned} t_b(s) &= \frac{t(s) - k\tau(s)n(s)}{\sqrt{1 + k^2\tau^2(s)}} \\ b_b(s) &= \frac{k^2\tau^3(s)t(s) + k\tau^2(s)n(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))b(s)}{\sqrt{k^4\tau^6(s) + k^2\tau^4(s) + (\kappa(s) - k\tau'(s) + k^2\kappa(s)\tau^2(s))^2}} \\ n_b(s) &= \frac{1}{K(s).L(s)}[(k\kappa(s)\tau(s) - k^2\tau(s)\tau'(s) + k^3\kappa(s)\tau^3(s))t(s) \\ &\quad + (\kappa(s) - k\tau'(s) + k^2\kappa^2(s)\tau^2(s))n(s) + (-k^3\tau^4(s) - k\tau^2(s))b(s)]. \end{aligned} \tag{3.27}$$

Theorem 3.3. *The expression of the curvature and torsion functions of the binormal curve $\alpha_b(s)$ in terms of the curvature and torsion functions of the curve $\alpha(s)$*

is obtained as;

$$\kappa_b(s) = \frac{\sqrt{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}}{(1+k^2\tau^2)^{\frac{3}{2}}} \tag{3.28}$$

$$\tau_b(s) = \frac{k^3\kappa'\tau^4 + k^2\kappa^2\tau^3 - k^3\kappa\tau'\tau^3 + k\kappa'\tau^2 - k^2\tau^2\tau'' + k^2\tau^5 + \kappa^2\tau - 4k\kappa\tau\tau' + 3k^2\tau(\tau')^2}{k^4\tau^6 + k^2\tau^4 + (\kappa - k\tau' + k^2\kappa\tau^2)^2}$$

Example 3.4. Let the curve

$$\alpha(s) = (acos(\frac{s}{a}), asin(\frac{s}{a}), 0), a > 0 \tag{3.29}$$

be given. Then, let us calculate the Frenet apparatus of binormal curve. The definition of binormal curve was $\alpha_b(s) = \alpha(s) + kb(s)$. Let us compute the binormal vector $b(s)$ of the curve $\alpha(s)$. Since the curve $\alpha(s)$ is unit speed curve, we get

$$b(s) = t(s) \times n(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \\ -cos(\frac{s}{a}) & -sin(\frac{s}{a}) & 0 \end{vmatrix} = (0,0,1). \tag{3.30}$$

Then the equation of the curve $\alpha_b(s)$ is,

$$\alpha_b(s) = (acos(\frac{s}{a}), asin(\frac{s}{a}), 0) + k(0,0,1) = (acos(\frac{s}{a}), asin(\frac{s}{a}), k). \tag{3.31}$$

From here;

$$\alpha_b'(s) = (-sin(\frac{s}{a}), cos(\frac{s}{a}), 0) \tag{3.32}$$

and

$$\|\alpha_b'(s)\| = \left((-sin(\frac{s}{a}))^2 + (cos(\frac{s}{a}))^2\right)^{\frac{1}{2}} = 1 \tag{3.33}$$

are obtained. Therefore the binormal curve $\alpha_b(s)$ is the unit speed curve. If we take the necessary calculations;

$$\begin{aligned} \alpha_b''(s) &= (-\frac{1}{a}cos(\frac{s}{a}), -\frac{1}{a}sin(\frac{s}{a}), 0) \\ \alpha_b'''(s) &= (\frac{1}{a^2}sin(\frac{s}{a}), -\frac{1}{a^2}cos(\frac{s}{a}), 0) \\ \alpha_b'(s) \times \alpha_b''(s) &= \begin{vmatrix} U_1 & U_2 & U_3 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \\ -\frac{1}{a}cos(\frac{s}{a}) & -\frac{1}{a}sin(\frac{s}{a}) & 0 \end{vmatrix} = (0,0,\frac{1}{a}) \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \|\alpha_b'(s) \times \alpha_b''(s)\| &= \sqrt{0^2 + 0^2 + (\frac{1}{a})^2} = \frac{1}{a} \\ \alpha_b'(s) \times \alpha_b''(s).\alpha_b''' &= 0 \end{aligned} \tag{3.35}$$

are found. So the Frenet frame of the binormal curve is;

$$
\begin{aligned}
t_b(s) &= \frac{\alpha_b'(s)}{\|\alpha_b'(s)\|} = (-sin(\frac{s}{a}), cos(\frac{s}{a}), 0) \\
b_b(s) &= \frac{\alpha_b'(s)\times\alpha_b''(s)}{\|\alpha_b'(s)\times\alpha_b''(s)\|} = a(0,0,\frac{1}{a}) = (0,0,1) \\
n_b(s) = b_b(s)\times t_b(s) &= \begin{vmatrix} U_1 & U_2 & U_3 \\ 0 & 0 & 1 \\ -sin(\frac{s}{a}) & cos(\frac{s}{a}) & 0 \end{vmatrix} = (-cos(\frac{s}{a}), -sin(\frac{s}{a}), 0).
\end{aligned} \tag{3.36}
$$

The curvature and the torsion functions of this binormal curve are;

$$
\begin{aligned}
\kappa_b(s) &= \frac{\|\alpha_b'(s)\times\alpha_b''(s)\|}{\|\alpha_b'(s)\|^3} = \frac{\frac{1}{a}}{1} = \frac{1}{a} \\
\tau_b(s) &= \frac{\alpha_b'(s)\times\alpha_b''(s).\alpha_b'''(s)}{\|\alpha_b'(s)\times\alpha_b''(s)\|^2} = \frac{0}{(\frac{1}{a})^2} = 0.
\end{aligned} \tag{3.37}
$$

Hence the binormal curve of the curve $\alpha(s)$ is a circle.

Example 3.5. Let the helix $\alpha(s) = (cos(\frac{s}{\sqrt{2}}), sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}})$ be given. Then let us calculate the Frenet apparatus of the binormal curve. Since $\alpha(s)$ is a unit speed curve, we get

$$
b(s) = t(s)\times n(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}} \\ -cos(\frac{s}{\sqrt{2}}) & -sin(\frac{s}{\sqrt{2}}) & 0 \end{vmatrix} = (\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}). \tag{3.38}
$$

Then the equation of the binormal curve is,

$$
\alpha_b(s) = \left(cos(\frac{s}{\sqrt{2}}) + \frac{k}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}), sin(\frac{s}{\sqrt{2}}) - \frac{k}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} + \frac{k}{\sqrt{2}}\right) \tag{3.39}
$$

Hence

$$
\alpha_b'(s) = \left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right) \tag{3.40}
$$

and

$$
\begin{aligned}
\|\alpha_b'(s)\| &= \left(\left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}} \\
&= \left(1 + \frac{k^2}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{4+k^2}}{2}
\end{aligned} \tag{3.41}
$$

are found. Therefore, the binormal curve is not unit speed curve. If we make necessary calculations on arbitrary speed curves;

(3.42)

$$\alpha_b''(s) = \left(-\frac{1}{2}cos(\frac{s}{\sqrt{2}}) - \frac{k}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}), 0\right)$$

$$\alpha_b'''(s) = \left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), 0\right)$$

$$\alpha_b'(s) \times \alpha_b''(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}) & \frac{1}{\sqrt{2}} \\ -\frac{1}{2}cos(\frac{s}{\sqrt{2}}) - \frac{k}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) & -\frac{1}{2}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) & 0 \end{vmatrix}$$

$$= \left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), \frac{2+k^2}{4\sqrt{2}}\right)$$

and

(3.43)

$$\|\alpha_b'(s) \times \alpha_b''(s)\|$$

$$= \sqrt{\left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}})\right)^2 + \left(-\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}})\right)^2 + \left(\frac{2+k^2}{4\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{k^4+6k^2+8}{32}}$$

$$\alpha_b'(s) \times \alpha_b''(s).\alpha_b''' = \frac{2+k^2}{16}$$

are obtained. Then Frenet frame of the binormal curve is;

(3.44)

$$t_b(s) = \frac{2}{\sqrt{4+k^2}}\left(-\frac{1}{\sqrt{2}}sin(\frac{s}{\sqrt{2}}) + \frac{k}{2}cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}cos(\frac{s}{\sqrt{2}}) + \frac{k}{2}sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)$$

$$b_b(s) = \sqrt{\frac{32}{k^4+6k^2+8}}\left(\frac{1}{2\sqrt{2}}sin(\frac{s}{\sqrt{2}}) - \frac{k}{4}cos(\frac{s}{\sqrt{2}}), -\frac{1}{2\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4}sin(\frac{s}{\sqrt{2}}), \frac{2+k^2}{4\sqrt{2}}\right)$$

$$\begin{aligned} n_b(s) = \sqrt{\frac{32}{k^4+6k^2+8}}.\frac{2}{\sqrt{4+k^2}}.(-\frac{1}{4}cos(\frac{s}{\sqrt{2}}) - \frac{k}{4\sqrt{2}}sin(\frac{s}{\sqrt{2}}) \\ -(\frac{2+k^2}{8})cos(\frac{s}{\sqrt{2}}) - (\frac{2k+k^3}{8\sqrt{2}}sin(\frac{s}{\sqrt{2}}), -\frac{2+k^2}{8}sin(\frac{s}{\sqrt{2}}) \\ +(\frac{2k+k^3}{8\sqrt{2}}cos(\frac{s}{\sqrt{2}}) - \frac{1}{4}sin(\frac{s}{\sqrt{2}}) + \frac{k}{4\sqrt{2}}cos(\frac{s}{\sqrt{2}}), 0). \end{aligned}$$

The curvature and the torsion function of the curve are;

$$\kappa_b(s) = \frac{\sqrt{2}.\sqrt{k^2+3}}{4+k^2} \tag{3.45}$$

and

$$\tau_b(s) = \frac{2}{4+k^2}. \tag{3.46}$$

Hence binormal curve is also a helix.

4. Conclusions

The definition of binormal curves is given by $\alpha_b(s) = \alpha(s) + kb(s)$ using the unit speed curve $\alpha(s)$. These curves are called parallel curves in the literature. The main goal of this paper is to investigate parallel curves using binormal vector and to study the associated geometry of these curves. We give a similar definition of parallel curves using the normal vector. The aim of this study is contribution to the literature on the theory of parallel curves based on binormal vector in three-dimensional space. In addition, the studies discussed here will later be expanded to surfaces and their geometric properties will be examined. Also, the instrinsic geometric formulas will be derived from the curvatures. This study was conducted at the Karadeniz Technical University in the Department of Mathematics and presented as a master's thesis.

5. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding
The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest
The author(s) declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The author(s) declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author(s) declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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