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$\mathbf{I}^*$-CONVERGENCE OF FUNCTION SEQUENCES IN ASYMMETRIC METRIC SPACE

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Abstract. In this paper, by considering ideal which is special subfamily of power set of natural numbers $\mathbf{I}^*$-convergence of sequence of functions in asymmetric metric spaces is defined and some results about new concept are given. Obtained results is supported some examples to show differences by the classical ones.

1. Introduction

The definition of statistical convergence by using asymptotic density was first introduced by Fast [6] and Steinhaus [17] in the same year 1951, independently. Although, it looks a simple generalization of classical convergence, this definition gave a new perspective to the researchers.

In [8], Freedman A.R. and Sember J. J. introduced a general concept of density and studied the relationship between densities and strong convergence areas of different summability methods. In [2], It has been demonstrated that if a sequence is strongly $\mathfrak{p}$-Cesaro summable or $w_{\mathfrak{p}}$ convergent then the sequence must be statistically convergent for $0 < \mathfrak{p} < \infty$ Furthermore a bounded statistically convergent sequence must be $w_{\mathfrak{p}}$ convergent for any $\mathfrak{p}$, $0 < \mathfrak{p} < \infty$.

Di Maio G. and Kocinac L. D. R. introduced and examined statistical convergence in topological and uniform spaces in [3]. They demonstrated the applicability of this convergence to the theory of choice principles, function spaces, and hyperspaces.

Some years later in the paper [12], Ilkhan and Kara obtained some results about completeness, compactness and pre-compactness by using statistically Cauchy sequences in a quasi metric spaces.

Later, based on the idea of this definition, P. Kostyrko, T. Salat, W. Wilczynki [13] gave the concepts ideal convergence by characterizing the small sets of a space in different ways.

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In article [7], the Bolzano-Weierstrass theorem is generalized by using ideal convergence. The authors of the paper [7] provided instances of ideals possessing and lacking the Bolzano-Weierstrass property, and examined the BW property in relation to submeasures and its extendibility to a maximal **P**-ideal. Apart from these, The study [18] examined the completion of a linear n-normed space regarding ideal convergence by introducing the notion of uniform continuous n-norm.

B. K. Lahiri and P. Das [14] carried out studies on **I**-convergence and $\mathbf{I}^*$ convergence and obtained important results. These concepts were studied in arbitrary metric spaces or arbitrary topological spaces.

In [11], Argha Gosh discussed and examined the concepts of $\mathbf{I}^*(\alpha)$- convergence and $\mathbf{I}^*$-exhaustiveness of metric function sequences and explained the relationship between these two concepts.

In asymmetric metric spaces (or quasi metric spaces in some sources) which is a larger structure than metric spaces, some properties of quasi metric spaces were given by Otafudu O. O. in [15], Reilly et all. in [16], Doitchinov D. in [4] and Dutta R. in [5], where sequence and function sequence convergence and fixed point results were given. Then, Ghosh A. (in his paper [10]) investigated the convergence of sequences of functions in asymmetric metric spaces with the help of ideals.

In this paper, our aim to give new kind definitions of left (right) $\mathbf{I}^*(\alpha)$-convergence, left (right) $\mathbf{I}^*$- Alexandroff convergence, left (right) $\mathbf{I}^*$-uniformly convergence for function sequences in an asymmetric metric space and some relations between them will be investigated.

2. Preliminaries and new results

In this part, we will present several new definitions along with corresponding results related to them.Throughout the text, we are going to use $\mathbf{Y}^{\mathbf{X}}$ to indicate the set of all maps from the asymmetric metric spaces $(\mathbf{X}, \mathfrak{q})$ to $(\mathbf{Y}, \mathfrak{p})$

Definition 2.1. Let $\mathbf{X} \neq \emptyset$ be a set and $\mathfrak{q} : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ be a function. The function $\mathfrak{q}$ is defined as an asymmetric metric on $\mathbf{X}$ if it meets the following criteria: (i) $\mathfrak{q}(x, y) \geq 0$, for all $x, y \in \mathbf{X}$; (ii) $\mathfrak{q}(x, y) = 0$ if and only if $x = y$ and (iii) $\mathfrak{q}(x, z) \leq \mathfrak{q}(x, y) + \mathfrak{q}(y, z)$ holds for all $x, y, z \in \mathbf{X}$.

Then, the pair $(\mathbf{X}, \mathfrak{q})$ is referred to as an asymmetric metric space and in addition to this if $\mathfrak{q}$ possesses the property of symmetry, it is classified as a metric and $(\mathbf{X}, \mathfrak{q})$ is termed a metric space.

Definition 2.2. Let $(\mathbf{X}, \mathfrak{q})$ be an asymmetric metric space. A left(right) topology $\tau^-(\tau^+)$ induced by $\mathfrak{q}$ is generated by the collection of left(right) open balls

$$\mathbb{B}^-(x, r) := \{y \in \mathbf{X} : \mathfrak{q}(y, x) < r\} \, (\mathbb{B}^+(x, r) := \{y \in \mathbf{X} : \mathfrak{q}(x, y) < r\})$$

for all $x \in \mathbf{X}$ and positive reals $r > 0$, respectively.

A sequence $\tilde{x} = (x_n)$ is said left(right) convergent to a point x^*, if for every $\varepsilon > 0$ there exists $n_* = n_*(\varepsilon) \in \mathbb{N}$ such that $x_n \in \mathbb{B}^-(x^*, r) \, (x_n \in \mathbb{B}^+(x^*, r))$ holds for all $n > n_*$.

One of the important problems that arise as a result of the lack of symmetry property is that left(right) limit of a sequence is not unique, in generally. Let's give an example to see this defect of asymmetric metric space:

Example 2.3. Let us consider a real valued sequence $\tilde{x} = (x_n)$ as

$$x_n := \begin{cases} \frac{1}{2^n}, & n \textit{ is odd}, \\ \frac{1}{3^n}, & n \textit{ is even}, \end{cases}$$

and asymmetric metric as

$$\mathfrak{q}(a, b) := \begin{cases} 0, & a \leq b, \\ 1, & a > b. \end{cases}$$

Hence, it is evident that every point of $(-\infty, 0)$ serves as a left limit point of the sequence.

Definition 2.4. [9] A subset $\mathfrak{B}$ of $\mathbb{N}$ is considered to have natural density natural density (or asymptotic density) denoted by $d(\mathfrak{B})$ if following limit exists

$$d(\mathfrak{B}) := \lim_{n\to\infty} \frac{|\mathfrak{B}(n)|}{n},$$

where $\mathfrak{B}(n) := \{j \in \mathfrak{B} : j \leq n\}$ and the symbol $|.|$ denotes the cardinality of the inside set.

Definition 2.5. [13] Let $\mathbf{X}$ be a non-empty set. A family $\mathbf{I} \subset 2^{\mathbf{X}}$ is termed an ideal on $\mathbf{X}$ if (i) $U \cup V \in \mathbf{I}$ holds for all $U, V \in \mathbf{I}$ and (ii) $U \in \mathbf{I}$ and $V \subset U$, then $V \in \mathbf{I}$ holds.

An ideal $\mathbf{I}$ is is referred to as non-trivial if $\mathbf{I}$ is not equal to $\emptyset$ and $\mathbf{X}$ is not an element of $\mathbf{I}$. A non-trivial ideal is termed admissible if it includes the singleton set $\{x\}$ for every $x \in \mathbf{X}$.

Definition 2.6. [13] Let $\mathbf{X} \neq \emptyset$. Then, $\mathfrak{F} \subset 2^{\mathbf{X}}$ is defined as a filter on $\mathbf{X}$ if it meets these criteria: (i) $U \cap V \in \mathfrak{F}$ for all $U, V \in \mathfrak{F}$ and (ii) $U \in \mathfrak{F}$ and $U \subset V$ implies that $V \in \mathfrak{F}$ holds.

For any non-trivial ideal $\mathbf{I} \subset 2^{\mathbf{X}}$ it can be defined a filter as follows

$$\mathfrak{F}(\mathbf{I}) := \{U \subset \mathbf{X} : U^c \in \mathbf{I}\}$$

and it is called a filter associated with $\mathbf{I}$. Following families

$$\mathbf{I}_d = \{U \subset \mathbb{N} : d(U) = 0\};\ \mathfrak{F}(\mathbf{I}_d) := \{U \subset \mathbb{N} : d(U) = 1\}$$

are well known nontrivial admissible ideal and filter.

Definition 2.7. [10] Let $\mathbf{I}$ be an admissible ideal.It is referred to as Good, for any sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets where $A_n \notin \mathbf{I}$ for all $n \in \mathbb{N}$, if there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of mutually disjoint sets such that $B_n \subset A_n$, $B_n \in$ Ł and $\bigcup_{n=1}^{\infty} B_n \notin \mathbf{I}$ hold.

A condition equivalent to this definition will be given in the following lemma:

Lemma 2.8. *An ideal* $\mathbf{I}$ *is Good iff for every* $\{D_n\}_{n\in\mathbb{N}} \notin \mathfrak{F}(\mathbf{I})$ *there exists pairwise disjoint sets* $\{P_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}(\mathbf{I})$ *such that* $P_n \supset D_n$ *and* $\bigcap_{n=1}^{\infty} P_n \notin \mathfrak{F}(\mathbf{I})$ *hold.*

Proof. Assume $\mathbf{I}$ is a Good ideal and consider $\{D_n\}_{n\in\mathbb{N}} \notin \mathfrak{F}(\mathbf{I})$. Then, $\mathbb{N} \setminus D_n \notin \mathbf{I}$. Since $\mathbf{I}$ is Good ideal, there exists $A_n \subset \mathbb{N} \setminus D_n$ such that $A_n \in \mathbf{I}$ and $\bigcup_{n=1}^{\infty} A_n \notin \mathbf{I}$. If P_n is chosen as such $P_n := \mathbb{N} \setminus A_n \in \mathfrak{F}(\mathbf{I})$, then

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} \mathbb{N} \setminus A_n = \mathbb{N} \setminus \bigcup_{n=1}^{\infty} A_n \notin \mathfrak{F}(\mathbf{I}).$$

□

Definition 2.9. [13] A sequence $\{x_k\}_{k\in\mathbb{N}} \subset (\mathbf{X}, \mathfrak{q})$ is described as left(right) **I**-convergent to $x_* \in \mathbf{X}$ if

$$\{k \in \mathbb{N} : \mathfrak{q}(x_k, x_*) \geq \varepsilon\} \in \mathbf{I}; (\{k \in \mathbb{N} : \mathfrak{q}(x_*, x_k) \geq \varepsilon\} \in \mathbf{I})$$

holds for every $\varepsilon > 0$, respectively.

In this case,it is denoted by symbolically $x_k \xrightarrow{\mathbf{I}^-} x_*$ and $x_k \xrightarrow{\mathbf{I}^+} x_*$, respectively.

Definition 2.10. [10] Function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is defined as left(right) pointwise **I**-convergent to a function $f \in \mathbf{Y}^{\mathbf{X}}$ if $f_k(x) \xrightarrow{\mathbf{I}^-} f(x)$ $(f_k(x) \xrightarrow{\mathbf{I}^+} f(x))$ holds for each $x \in \mathbf{X}$.

Definition 2.11. [10] Function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) **I**-convergent uniformly to $f \in \mathbf{Y}^{\mathbf{X}}$ if for each $\varepsilon > 0$ there exists $A \in \mathfrak{F}(\mathbf{I})$ such that $\mathfrak{p}(f_k(x), f(x)) < \varepsilon$ $(\mathfrak{p}(f(x), f_k(x)) < \varepsilon)$ holds for all $k \in A$ and $x \in \mathbf{X}$.

Definition 2.12. [10] A function $f \in \mathbf{Y}^{\mathbf{X}}$ is referred to as left continuous (f^--continuous) at a point $\xi \in \mathbf{X}$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{p}(f(y), f(\xi)) < \varepsilon$ satisfies for all $y \in \mathbb{B}^-(\xi, \delta)$.

Similarly, right continuous (f^+-continuous) at $\xi \in \mathbf{X}$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{p}(f(\xi), f(y)) < \varepsilon$ satisfies for all $y \in \mathbb{B}^+(\xi, \delta)$.

Definition 2.13. (Sequential continuity at a point) A function $f \in \mathbf{Y}^{\mathbf{X}}$ is said to be (i) $f^{-,-}$ continuous at $x^* \in \mathbf{X}$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ left converges to x^* in $(\mathbf{X}, \mathfrak{q})$, then corresponding sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ left converges to $f(x^*)$ in $(\mathbf{Y}, \mathfrak{p})$;

(ii) $f^{+,+}$ continuous at a point $x_* \in \mathbf{X}$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ right converges to x_* in $(\mathbf{X}, \mathfrak{q})$, then corresponding sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ right converges to $f(x_*)$ in $(\mathbf{Y}, \mathfrak{p})$.

Definition 2.14. Let $(\mathbf{X}, \mathfrak{q})$ be an asymmetric metric space, $\{x_n\} \subset \mathbf{X}$ be a sequence and $a^* \in \mathbf{X}$. A sequence $\{x_n\}$ is said to be left (right) $\mathbf{I}^*$-convergent to a^*, if there exists $K = \{m_1 < m_2 < ... < m_n < ...\}$ such that

$$\lim_{n\to\infty} \mathfrak{q}(x_{m_n}, a^*) = 0 \ (\lim_{n\to\infty} \mathfrak{q}(a^*, x_{m_n}) = 0)$$

holds.

It is denoted by symbolically $x_n \xrightarrow{\mathbf{I}^{*-}} a^*$ $(x_n \xrightarrow{\mathbf{I}^{*+}} a^*)$, respectively.

Definition 2.15. A sequence of function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left (right) $\mathbf{I}^*(\alpha)$ convergent to $f \in \mathbf{Y}^{\mathbf{X}}$ if for any sequence $\{x_k\}$ that left(right) $\mathbf{I}^*$ converges to point x in $\mathbf{I}$, the sequence $(f_k\{x_k\})$ is also left $\mathbf{I}^*$-convergence to $f(x)$.

Definition 2.16. A sequence of function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) $\mathbf{I}^*$-exhaustive at a point $\acute{a} \in \mathbf{X}$ if there exists $\mathrm{A} = \mathrm{A}(\acute{a}) \in \mathbf{I}$ such that for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \acute{a}) > 0$ and $n_0 = n_0(\varepsilon, \acute{a}) \in \mathbb{N}$ such that $\mathfrak{q}(\acute{a}, x) < \delta$ $(\mathfrak{q}(x, \acute{a}) < \delta)$ implies $\mathfrak{p}(f_n(\acute{a}), f_n(x)) < \varepsilon$ $(\mathfrak{p}(f_n(x), f_n(\acute{a})) < \varepsilon)$ for all $n \in \mathbb{N} \setminus \mathrm{A}$ and $n \geq n_0$.

Definition 2.17. A function sequence $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is called left(right) pointwise $\mathbf{I}^*$-convergent to a function $f \in \mathbf{Y}^{\mathbf{X}}$ if $f_k(\acute{x}) \xrightarrow{\mathbf{I}^{*-}} f(\acute{x})$ $(f_k(\acute{x}) \xrightarrow{\mathbf{I}^{*+}} f(\acute{x}))$ satisfies for all $\acute{x} \in \mathbf{X}$.

Theorem 2.18. *Let $x \in \mathbf{X}$ and $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$. If $\{f_k\}_{k\in\mathbb{N}}$ is right pointwise $\mathbf{I}^*$-convergent to f at every point $z \in \mathbf{X} \setminus \{x\}$ and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then f is f^--continuous*

Proof. Owing to the fact that $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$-exhaustive at $x \in \mathbf{X}$, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathbf{I}$ such that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and $\exists n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y, x) < \delta$ implies

$$\mathfrak{p}(f_n(y), f_n(x)) < \varepsilon$$

for all $n \geq n_0$ and $n \in \mathbb{N} \setminus \mathrm{A}$.

Let $y \in B^-(x, \delta) \setminus \{x\}$. Since, $\{f_k\}$ is right pointwise $\mathbf{I}^*$-convergent to f, then we have $f_k(x) \stackrel{\mathbf{I}^{*+}}{\to} f(x)$ for all $y \in \mathbf{X}$.

So, there exists $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{n\to\infty} \mathrm{p}(f(y), f_{k_n}(y)) = 0$. Then, for all $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that $\mathfrak{p}(f_n(y), f_n(x)) < \frac{\varepsilon}{3}$ holds for every $k_n \geq n_1$.

Since, $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$, then there exists $M_2 \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{m\to\infty} \mathrm{p}(f_{k_m}(x), f(x)) < \frac{\varepsilon}{3}$ holds.

Now, $K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A}) \in \mathfrak{F}(\mathbf{I})$ and this implies that $K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A}) \neq \emptyset$.

Hence, we can choose $j \in K_1 \cap K_2 \cap (\mathbb{N} \setminus \mathrm{A})$. Then, for all $y \in B^-(x, \delta) \setminus \{x\}$ we have

$$\mathrm{p}(f(y), f(x)) \leq \mathrm{p}(f(y), f_j(y)) + \mathrm{p}(f_j(y), f_j(x)) + \mathrm{p}(f_j(x), f(x)) < \varepsilon.$$

Therefore, f is left continuous. □

Theorem 2.19. *If $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$ and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*(\alpha)$ convergent to $f \in \mathbf{Y}^{\mathbf{X}}$ at $x \in \mathbf{X}$.*

Proof. For the reason that $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$, then $f_k(x) \stackrel{\mathbf{I}^{*-}}{\to} f(x)$. So, it can be find a set $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that

$$\lim_{m\to\infty} \mathfrak{p}(f_{k_m}(x), f(x)) = 0$$

holds. Hence, for all $\varepsilon > 0$ there exists natural number n_0 such that $\mathfrak{p}(f_{k_m}(x), f(x)) < \frac{\varepsilon}{2}$ holds for every $k_m \geq n_0$. Given that $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$, then there exists $K' = K'(x) \in \mathbf{I}$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and $n_1 = n_1(\varepsilon, x) \in \mathbb{N} \ni$ for every $\mathfrak{q}(y, x) < \delta$ implies $\mathfrak{p}(f_n(y), f_n(x)) < \frac{\varepsilon}{2}$ for all $n \in \mathbb{N} \setminus K'$ and $\forall n \geq n_1$.

Let $x_n \stackrel{\mathbf{I}^{*-}}{\to} x, n \to \infty$. We must show that $f_n(x_n) \stackrel{\mathbf{I}^{*-}}{\to} f(x), n \to \infty$. Since $x_n \stackrel{\mathbf{I}^{*-}}{\to} x, n \to \infty$, then there exists

$$K'' = \{\mathrm{m}_1 < \mathrm{m}_2 < ... < \mathrm{m}_\mathrm{k} < ...\} \in \mathfrak{F}(\mathbf{I})$$

such that $\lim_{k\to\infty} \mathfrak{q}(x_{m_k}, x) = 0$.

So, for all $\delta > 0$ there exists $n_1(\delta) \in \mathbb{N}$ such that $\mathfrak{q}(x_{m_k}, x) < \delta$ holds for all $m_k \geq n_1$.

Let us take $K^* := K' \cap K'' \in \mathfrak{F}(\mathbf{I})$ and $n^* := max\{n_0, n_1\} \in \mathbb{N}$. Thus, we have $\mathfrak{p}(f_n(x_n), f_n(x)) < \frac{\varepsilon}{2}$ for all $n \geq n^*$ where $n \in K^*$. Also, for any $j \in K^*$ following inequality

$$\mathfrak{p}(f_j(x_j), f(x)) < \mathfrak{p}(f_j(x_j), f_j(x)) + \mathfrak{p}(f_j(x), f(x)) < \varepsilon$$

holds.

This gives left pointwise $\mathbf{I}^*$-convergence of $\{f_k\}$. So, proof is ended. □

Theorem 2.20. *Assume that left $\mathbf{I}^*$-convergence signifies right $\mathbf{I}^*$-convergence in $\mathbf{Y}$. If $\mathbf{I}$ is Good and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*(\alpha)$ convergent to $f \in \mathbf{Y}^{\mathbf{X}}$ at $x \in \mathbf{X}$, then $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$ and $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$.*

Proof. Obviously, $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is left pointwise $\mathbf{I}^*$-convergent to f at point $x \in \mathbf{X}$. Assume that $\{f_k\}$ does not left $\mathbf{I}^*$- exhaustive at a point x. Then, for every $A = A(x) \in \mathfrak{F}(\mathbf{I})$ there exists an $\varepsilon^{'} > 0$ such that for all $\delta = \delta(\varepsilon^{'}, x) > 0$ and $n_0 = n_0(\varepsilon, x) \in \mathbb{N}$ there exists $k \in A$ $(k \geq n_0)$ such that $\mathfrak{q}(z, x) < \delta$ implies

$$\mathfrak{p}(f_k(z), f_k(x)) \geq \varepsilon^{'}.$$

Especially, let us choose $A = \mathbb{N}$ and $\delta = \frac{1}{k}$. Then, there exists $n_k \in \mathbb{N}$ such that for some $x_k \in \mathbb{B}^-(x, \frac{1}{k})$, implies

$$\mathfrak{p}(f_{n_k}(x_k), f_{n_k}(x)) \geq \varepsilon^{'}.$$

We consider only one such x_k corresponding to each such n_k. Let A_k denote all such $n_k \in \mathbb{N}$ satisfying the above inequality and B_k denote the collection of corresponding unique $x_k^{\prime}s$. We claim that $\mathbb{N}\setminus\{A_k\} \notin \mathfrak{F}(\mathbf{I})$. Suppose $\mathbb{N}\setminus\{A_k\} \in \mathfrak{F}(\mathbf{I})$. Then, $\{A_k\} \in \mathbf{I}$. Thus, there exists $n_0^k \in A_k$ such that

$$\mathfrak{p}(f_{n_0^k}(x_0^k), f_{n_0^k}(x)) \geq \varepsilon^{'}$$

for some $x_0^k \in \mathbb{B}^-(x, \frac{1}{k})$, which is inconsistent with the definition of A_k.

Thus, $\mathbb{N} \setminus \{A_k\} \notin \mathfrak{F}(\mathbf{I})$. Since $\mathbf{I}$ is Good ideal, then from Lemma 2.8 for every $\mathbb{N}\setminus\{A_k\} \notin \mathfrak{F}(\mathbf{I})$ there exist $P_k \supset \mathbb{N}\setminus\{A_k\}$ pairwise distinct sets such that $\mathbb{N}\setminus P_k \in \mathfrak{F}(\mathbf{I})$ for every $k \in \mathbb{N}$ and $\bigcap_{k=1}^{\infty} \mathbb{N}\setminus P_k \notin \mathfrak{F}(\mathbf{I})$.

Now, let $P_k = \left\{p_1^k < p_2^k < ...\right\}$. Examine a sequence $\{z_n\}$ as follows:

$$\{z_n\} := \begin{cases} x, & n \notin \bigcap_{k=1}^{\infty} \mathbb{N}\setminus P_k, \\ x_j^k, & n \in P_k, \end{cases}$$

and $n = p_j^k$, $x_j^k \in B_k$ corresponds to the natural number $p_j^k \in A_k$. Let $\varepsilon > 0$ be given. As a result, there is a least $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$. Now,

$$\{n \in \mathbb{N} : \mathfrak{q}(z_n, x) \geq \varepsilon\} \subset \bigcup_{k=1}^{k_0-1} \mathbb{N}\setminus P_k \in \mathbf{I}$$

Thus, $z_n \xrightarrow{\mathbf{I}^{*-}} x, n \to \infty$. On the flip side,

$$\left\{n \in \mathbb{N} : \mathfrak{p}(f_n(z_n), f_n(x)) \geq \varepsilon^{'}\right\} = \mathbb{N}\setminus P_k \in \mathfrak{F}(\mathbf{I})$$

holds which is a contradiction. Hence, $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at $x \in \mathbf{X}$. □

Definition 2.21. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is described as left(right) uniformly $\mathbf{I}^*$- convergent to a function f if for every $\varepsilon > 0$ and for all $x \in \mathbf{X}$ there exists $K \notin \mathbf{I}$ with $n_0 = n_0(\varepsilon) \in K$ such that $\mathfrak{p}(f_n(x), f(x)) < \varepsilon$ ($\mathfrak{p}(f(x), f_n(x)) < \varepsilon$) holds for all $n \geq n_0$ and $n \in K$.

Similarly, right uniformly $\mathbf{I}^*$- convergence can also be defined.

Theorem 2.22. *Assume that left $\mathbf{I}^*$-convergence implies right $\mathbf{I}^*$-convergence in $\mathbf{Y}$ and $x \in \mathbf{X}$. If for every $\varepsilon > 0$ there exists $\delta > 0$ and $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that for all $y \in \mathbb{B}^-(x, \delta)$ we have*

$$\mathfrak{p}(f_{k_n}(y), f_{k_n}(x)) < \varepsilon$$

then, $\mathfrak{p}(f_{k_n}(x), f_{k_n}(y)) < \varepsilon$ *holds for all* $y \in \mathbb{B}^-(x,\delta)$.

Proof. The proof is clear. So, it is omitted here. □

Definition 2.23. Let $(\mathbf{X}, \mathfrak{q})$ be an asymmetric metric space and $K \subset \mathbf{X}$ be a set. Then, K is said to be left(right) compact if every open cover of K in left(right) topology has a finite sub-cover.

Theorem 2.24. *Assume that left* $\mathbf{I}^*$*-convergence implies right* $\mathbf{I}^*$*-convergence in* $\mathbf{Y}$*. If sequence of functions* $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ *is left pointwise* $\mathbf{I}^*$*-convergent to* f *and* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*-$*exhaustive on* $\mathbf{X}$*, then* f *is left continuous on* $\mathbf{X}$ *and* $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ *is left uniformly* $\mathbf{I}^*-$*convergent to the function* f *on every left compact subset of* $\mathbf{X}$.

Proof. Initially, we will establish that f is left continuous on $\mathbf{X}$. Let $x \in \mathbf{X}$ be an arbitrary element. Since $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$- exhaustive at x, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathfrak{F}(\mathbf{I})$ such that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ and there exists $n^{'} = n^{'}(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y,x) < \delta$ implies $\mathfrak{p}(f_k(y), f_k(x)) < \varepsilon$ for all $n \in \mathrm{A}$ and $n \geq n^{'}$.

Postulate that f is not left continuous function. Then, when $\{x_k\}$ is left $\mathbf{I}^*$-convergent to x, the sequence $\{f(x_k)\}$ is not left $\mathbf{I}^*$-convergent to $f(x)$. So, there exists $K = \{k_1 < k_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ such that $\lim_{n\to\infty} \ \mathfrak{q}(x_{k_n}, x) = 0$ holds but $\lim_{n\to\infty} \ \mathfrak{p}(f(x_{k_n}), f(x)) = 0$. Then, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{q}(x_{k_n}, x) < \varepsilon$ holds for all $n \geq n_0$ but there exists $n_1 \in \mathbb{N}$ such that $\mathfrak{p}\left(f(x_{k_n}), f(x)\right) \geq \varepsilon$ holds for all $n \geq n_1$. This is a conflict with the definition of being left $\mathbf{I}^*$-exhaustive and therefore f is left continuous.

Let K be a left compact subset of $\mathbf{X}$, $\varepsilon > 0$ and $x \in K$. Then, f is left continuous at $\mathbf{x}$. Therefore, there exists $\delta > 0$ such that we have $\mathfrak{p}(f(y), f(x)) < \frac{\varepsilon}{3}$ for $y \in \mathbb{B}^-(x,\delta)$. Since, left $\mathbf{I}^*-$convergence implies right $\mathbf{I}^*$- convergence in $\mathbf{Y}$. Then, we have $\mathfrak{p}(f(y), f(x)) < \frac{\varepsilon}{3}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is left $\mathbf{I}^*$-exhaustive at $\mathbf{x}$, then there exists $\mathrm{A} = \mathrm{A}(x) \in \mathfrak{F}(\mathbf{I})$ such that for all $\varepsilon > 0$ there are $\delta = \delta(\varepsilon, x) > 0$ and $n^{'} = n^{'}(\varepsilon, x) \in \mathbb{N}$ such that for every $\mathfrak{q}(y,x) < \delta$ implies $\mathfrak{p}(f_k(y), f_k(x)) < \varepsilon$ for all $n \in \mathrm{A}, n \geq n^{'}$.

Now, $K \subset \bigcup_{x\in K} \mathbb{B}^-(x, \delta_x)$ and K is left compact. Then, there exists finite number points

$$x_1, x_2, ..., x_m \in K$$

such that $K \subset \bigcup_{i=1}^m \mathbb{B}^-(x_i, \delta_{x_i})$ holds. Since $\{f_k\}_{k\in\mathbb{N}}$ is left pointwise $\mathbf{I}^*$-convergent to f for each i there are $A_i \in \mathfrak{F}(\mathbf{I})$ such that

$$\mathfrak{p}(f_k(x_i), f(x_i)) < \frac{\varepsilon}{3}$$

holds for each $k \in A_i$. Now, let us consider $B := \bigcap_{i=1}^m A_i \cap A_{x_i}$. Then, $B \in \mathfrak{F}(\mathbf{I})$. If $z \in K$, then there exists $i \in \{1, 2, ..., m\}$ such that $\mathfrak{q}(z, x_i) < \delta x_i < \delta$ implies that

$$\mathfrak{p}(f(x_i), f(z)) < \frac{\varepsilon}{3}$$

and

$$\mathfrak{p}(f_k(z), f_k(x_i)) < \frac{\varepsilon}{3}$$

hold for all $k \in B$ and $z \in \mathbb{B}^-(x_i, \delta_{x_i})$. Hence, we obtain

$$\mathfrak{p}(f_k(z), f(z)) < \mathfrak{p}(f_k(z), f_k(x_i)) + \mathfrak{p}(f_k(x_i), f(x_i)) + \mathfrak{p}(f(x_i), f(z)) < \varepsilon.$$

So, we arrived the proof. □

Definition 2.25. A sequence of left(right) continuous function $\{f_k\}_{k\in\mathbb{N}} \subset \mathbf{Y}^{\mathbf{X}}$ is said to be left(right) $\mathbf{I}^*$- Alexandroff convergent to the function f if, $\{f_k\}_{k\in\mathbb{N}}$ is left(right) pointwise $\mathbf{I}^*$-convergent to f, for all $\varepsilon > 0$ and $A \in \mathfrak{F}(\mathbf{I})$ there exists $M_A = \{m_1 < m_2 < ...\} \subset A$ and an open cover $U = \{U_k : k \in A\}$ in the left (right) topology of $\mathbf{I}$ such that for every $x \in U_k$ we have $\mathfrak{p}\left(f_{m_k}(x), f(x)\right) < \varepsilon$ $(\mathfrak{p}\left(f(x), f_{m_k}(x)\right) < \varepsilon)$.

Definition 2.26. [10] An asymmetric metric $\mathfrak{q}$ defined on $\mathbf{I}$ is said to have satisfy approximate metric axiom (AMA) if there exists a map $c : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ such that $\mathfrak{q}(y, z) \leq c(z, y).\mathfrak{q}(z, y)$ holds for every $z, y \in \mathbf{X}$ where c meets the condition described as: For all z, there exists $\delta_z > 0$ such that for all $y \in \mathbb{B}^+(z, \delta_z)$ implies that $c(z, y) \leq C(z)$ holds, where $C(z) > 0$ is a real number.

Theorem 2.27. $(\mathbf{X}, \mathfrak{q})$ *and* $(\mathbf{Y}, \mathfrak{p})$ *be asymmetric spaces. Suppose that* $(\mathbf{Y}, \mathfrak{p})$ *provides the property (AMA) and corresponding map* C *is bounded. If* $\{f_k\}_{k\in\mathbb{N}}$ *is left* $\mathbf{I}^*-$ *Alexandroff convergent to the function* f *then* f *is left continuous.*

Proof. Assume that $\{f_k\}_{k\in\mathbb{N}}$ be left $\mathbf{I}^*$- Alexandroff convergent to the function f. Then, $\{f_k\}$ is left continuous map, $\{f_k\}$ is left pointwise $\mathbf{I}^*$-convergent to f and for all $\varepsilon > 0, A \in \mathfrak{F}(\mathbf{I})$ there exists

$$M_A = \{m_1 < m_2 < ...\} \subset A$$

and open cover

$$V = \{V_k : k \in A\}$$

in the left topology of $\mathbf{X}$ such that every $x \in V_k$ we have $\mathfrak{p}(f_{m_k}(x), f(x)) < \varepsilon$.

Let $x \in \mathbf{X}$ and $\{x_n\}$ is left $\mathbf{I}^*$-convergent to x. Since $\{f_k\}_{k\in\mathbb{N}}$ is left pointwise $\mathbf{I}^*$- convergent to f, there exist $K = \{m_1 < m_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ and $n_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\mathfrak{p}(f_{m_k}(x), f(x) < \frac{\varepsilon}{3r}$$

for all m_n, n_0. Since the function corresponding to C is bounded, there exist $r > 0$ such that $C(z) < r$ holds for all $z \in \mathbf{X}$. Let $K \in \mathfrak{F}(\mathbf{I})$.

Then, there exists $M_k = \{m_1 < m_2 < ...\} \in \mathfrak{F}(\mathbf{I})$ and open cover $V = \{V_k : k \in A\}$ such that $\mathfrak{p}(f_{m_k}(x), f(x)) < \frac{\varepsilon}{3}$ for every $x \in V_k$.

Since $V = \{V_k : k \in A\}$ is open cover, then we can choose a $k \in \mathbb{N}$ such that $x \in V_k$. Because of f_{m_k} is left continuous at $\mathbf{X}$ and $\{x_n\}$ is left $\mathcal{I}^*$-convergent to $\mathbf{X}$, there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$ when $\{x_n\} \in V_k$, $\mathfrak{p}(f_{m_k}(x_n), f_{m_k}(x)) < \frac{\varepsilon}{3}$. Since $(\mathbf{Y}, \mathfrak{p})$ satisfies the property (AMA), we can see

$$\begin{aligned}\mathfrak{p}(f(x_n), f(x)) &< \mathfrak{p}(f(x_n), f_{m_k}(x_n)) + \mathfrak{p}(f_{m_k}(x_n), f_{m_k}(x)) + \mathfrak{p}(f_{m_k}(x), f(x)) \\ &< C(f_{m_k}(x_n))\mathfrak{p}(f_{m_k}(x_n), f(x_n)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

□

3. CONCLUSION

In this work, information about $\mathbf{I}^*$-convergence in asymmetric metric spaces is given. Similar results can be generalized to $\mathbf{I}^{\mathbf{K}}$ convergence, where $\mathbf{I}$ and $\mathbf{K}$ are admissible ideals.

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