Journal of Universal Mathematics
To Memory Assoc. Prof. Dr. Zeynep Akdemirci Şanlı
Vol.7, pp.102-108 (2024)
ISSN-2618-5660
DOI: 10.33773/jum.1564979

HEMI-SLANT SUBMANIFOLDS OF LORENTZIAN KENMOTSU SPACE FORMS

RAMAZAN SARI

0000-0002-4618-8243

Abstract. In this paper, we study curvature properties of hemi-slant submanifolds of Lorentzian Kenmotsu space forms. We define Lorentzian Kenmotsu space forms and study their curvature properties. We give an example for hemi-slant submanifold of Lorentzian Kenmotsu space forms. Finally, the curvature properties of distributions are analyzed and the conditions for Einstein are investigated.

1. Introduction

Bishop and O'neill investigated negative curvature manifolds [3]. They studied these manifolds using warped product. From the second half of the twentieth century, the warped product began to be used in contact manifolds. Kenmotsu investigated a different class of an almost contact manifold. He defined new conditions by

$$
\begin{aligned}
(\nabla_X\varphi)Y &= -\eta(Y)\varphi X - g(X,\varphi Y)\xi \\
\nabla_X\xi &= X-\eta(X)\xi
\end{aligned} \tag{1.1}
$$

He showed that the contact manifold satisfying these two conditions is normal. But this manifold was not Sasakian [7]. A differentiable manifold called Lorentzian manifold with a Lorentzian metric of index 1. A Lorentzian manifold has lightlike, timelike and spacelike vector fields. Therefore, the Lorentzian metric can also be used on odd dimensional manifolds. So we can study Lorentzian contact manifolds. Firstly, Takahashi defined and studied Lorentzian Sasakian manifolds using the Lorentzian metric on Sasakian manifold [13]. After, Duggal has investigated the space time manifolds [6]. From all these studied, Rosça investigated Lorentzian Kenmotsu manifolds [9]. Many authors have been studied on Lorentzian Kenmotsu manifolds [2, 4, 5, 8, 14, 15].

Date: **Received:** 2024-10-10; **Accepted:** 2024-12-01.
2000 *Mathematics Subject Classification.* 53C15,53C40.
Key words and phrases. Hemi-slant submanifolds, Lorentzian Kenmotsu manifolds, Einstein manifolds.

In this paper, we are studied curvature properties of hemi-slant submanifolds of Lorentzian Kenmotsu space form. Firstly, we are defined Lorentzian Kenmostsu space forms and study their curvature properties. After, the definition of a hemi-slant submanifold of an Lorentzian Kenmotsu space form is given and an example is presented. Finally, the curvature properties of distributions are analyzed and the conditions for Einstein are investigated.

2. Lorentzian Kenmotsu Manifolds

Let B be almost contact manifold with an almost contact structure (φ, η, ξ), where ξ is a vector field on B, η is a $1-$form and φ is a tensor field of type $(1, 1)$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

If a semi-Riemannian metric g on almost contact manifold B by

$$g(\varphi X, \varphi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \qquad g(\xi, \xi) = \epsilon = -1$$

therefore $(B, \varphi, \eta, \xi, g)$ is called a Lorentzian almost contact manifold. Then we get $\eta(X) = \epsilon g(X, \xi)$. Moreover, ξ is never a spacelike vector field and a lightlike vector field on B. We consider a local basis $\{e_1, ..., e_{2n}, \xi\}$ in TB i.e.

$$g(e_i, e_j) = \delta_{ij} \text{ and } g(\xi, \xi) = -1$$

that is $e_1, ..., e_{2n}$ are spacelike vector fields, and ξ is timelike.

We not that, for all $X, Y \in \Gamma(TB)$, if $\Phi(X, Y) = g(X, \varphi Y)$, Φ is said to be fundamental $2-$form.

On the other hand, manifold is normal if

$$N = [\varphi, \varphi] + 2d\eta \otimes \xi = 0$$

where $[\varphi, \varphi]$ is Nijenhuis tensor field of φ.

Definition 2.1. Let B be a Lorentzian almost contact manifold. B is called a Lorentzian Kenmotsu manifold if normal and $d\eta = 0$ and $d\Phi = 2\epsilon\eta \wedge \Phi$.

Theorem 2.2. [10] *Let B be a Lorentzian contact manifold. Therefore for all $X, Y \in \Gamma(TB)$, B is a Lorentzian Kenmotsu manifold if and only if*

$$\left(\overline{\nabla}_X \varphi\right) Y = \epsilon\{g(Y, \varphi X)\xi - \eta(Y)\varphi X\}. \tag{2.1}$$

.

Corollary 2.3. *Let B be a Lorentzian Kenmotsu manifold. Therefore we get*

$$\overline{\nabla}_X \xi = \epsilon\varphi^2 X \tag{2.2}$$

for all $X, Y \in \Gamma(TB)$.

3. Lorentzian Kenmotsu Space Forms

Let Lorentzian Kenmotsu manifold B has constant $\varphi-$holomophic section curvature k. Therefore it is called Lorentzian Kenmotsu-space form. If constant $\varphi-$holomophic section curvature is k, manifold B is denoted by $B(k)$. Therefore, curvature tensor satisfied,

$$R(X,Y,Z,W) = \frac{k+3}{4}\{g(Z,Y)g(W,X)-g(W,Y)g(Z,X)\}$$
$$+\frac{k-1}{4}\{g(Z,\varphi Y)g(W,\varphi X)-g(W,\varphi Y)g(Z,\varphi X)$$
$$-2g(W,\varphi Z)g(Y,\varphi X)+g(Z,X)\eta(W)\eta(Y)$$
$$-g(Z,Y)\eta(W)\eta(X)+g(W,Y)\eta(Z)\eta(X)\}. \tag{3.1}$$

Theorem 3.1. *Let B be a Lorentzian Kenmotsu manifold. If B have constat $\varphi-$ holomophic sectional curvature, therefore the Ricci tensor is not parallel.*

Proof. We using (3.1). For all $X,Y \in \Gamma(TB)$, we get

$$S(X,Y) = \frac{(k-1)+(k+3)n}{2}g(\varphi Y,\varphi X) - 2n\eta(Y)\eta(X)$$

which proves the assertion. □

Corollary 3.2. *Let B be a Lorentzian Kenmotsu manifold. Therefore we get*

$$\tau = \frac{((k-3)n-2)(2n+1)}{4}$$

where τ is the scalar curvature.

4. Hemi-Slant Submanifolds of Lorentzian Kenmotsu Space Forms

Let B be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$ and ∇ be the Levi-Civita connection of B. For all $X,Y \in \Gamma(TB)$ and $N \in \Gamma(TB)^{\perp}$, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{4.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N. \tag{4.2}$$

This equations is called Gauss and Weingarten formulas, respectively. Moreover, from (4.1) and (4.2), we get

$$g(A_N X,Y) = g(h(X,Y),N). \tag{4.3}$$

For any $X \in \Gamma(TB)$, we give

$$\varphi X = TX + NX$$

where NX and TX is the normal and tangential components, respectively.

For any $V \in \Gamma(T^{\perp}B)$, we have

$$\varphi V = tV + nV$$

where nV and tV is the normal and tangential components, respectively [12].

Lemma 4.1. *Let B be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$. Therefore, for all $K,L \in \Gamma(TB)$*

$$(\nabla_K T)L = A_{NL}K + th(K,L) + \epsilon\{g(TK,L)\xi - \eta(L)TK\} \tag{4.4}$$

$$(\nabla_K N)L = nh(K,L) - h(K,TL) - \epsilon\eta(L)NK. \tag{4.5}$$

From now on, we accept that the ξ is tangent to the submanifold B. Therfore, we can consider the orthogonal direct decomposition

$$TB = D \oplus \xi,$$

where D is the orthogonal distribution to ξ.

Definition 4.2. Let B be a submanifold of Lorentzian Kenmotsu manifold $\overline{B}$. Therefore B is called anti-invariant if and only if $\varphi(T_x B) \subset T_x^{\perp} B$ for all $x \in B$.

Definition 4.3. Let B be a submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$. If angle betwen φB and TB is a constant, submanifold B is called slant submanifold.

In [1], $Sp\{\xi\}$ defines the timelike vector field distribution. Let W is a spacelike vector field. If vector field W is orthogonal to ξ, we get

$$g(\varphi W, \varphi W) = g(W, W) \geq 0.$$

For spacelike vector fields the Cauchy-Schawrz inequality

$$g(W, W) \leq \|W\| \, \|W\|$$

is verified.

Then we have

$$\cos\theta = \frac{g(\varphi W, TW)}{\|\varphi W\| \, \|TW\|}.$$

Definition 4.4. Let B be submanifold of of a Lorentzian Kenmotsu manifold $\overline{B}$. Therefore B is called a hemi-slant submanifold which D_1 and D_2 two orthogonal spacelike distributions such that

(i) $TB = D_1 \oplus D_2 \oplus sp\{\xi\}$
(ii) D_1 is anti-invariant.
(iii) D_2 is slant with angle $\theta \neq 0$.

Therefore, the angle θ is called the slant angle of a submanifold B.

On the other hand, let d_i be dimension of the distribution D_i for $i = 1, 2$. Therefore we have the following cases:

If $d_2 = 0$, therefore B is an anti-invariant submanifold.
If $d_1 = 0$ and $\theta = 0$, therefore B is an invariant submanifold.
If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, therefore B is a proper slant timelike submanifold.
If $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$, B is a proper hemi-slant timelike submanifold.
For a local orthonormal frame $\{e_1, ..., e_{2p}, e_{2p+1}, ..., e_{2p+2q}, \xi\}$,

$$D_1 = sp\{e_1, ..., e_{2p}\}, D_2 = sp\{e_{2p+1}, ..., e_{2p+2q}\}$$

where $dimD_1 = 2p$ and $dimD_2 = 2q$.

Example 4.5. In what follows, $\mathbb{R}^{2m+1}$ with Lorentzian Kenmotsu structure given by

$$\varphi(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + Y_i y_i \frac{\partial}{\partial z}$$

$$g = e^{-2z}(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i) - \epsilon \eta \otimes \eta$$

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz$$

where $(x_1, ..., x_n, y_1, ..., y_n, z)$ are Cartesian coordinates on $\mathbb{R}^{2m+1}$.

Now, a submanifold B of $\mathbb{R}^7$ defined by

$$B = F(s,l,k,u,t) = (s,0,k,l,u,0,t).$$

Therefore local frame of TB

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1}, \qquad e_2 = \frac{\partial}{\partial y_1}, \qquad e_3 = \frac{\partial}{\partial x_3}, \\ e_4 &= \frac{\partial}{\partial y_2}, \qquad e_5 = \frac{\partial}{\partial z} = \xi \end{aligned}$$

and

$$e_1^* = \frac{\partial}{\partial x_2}, \qquad e_2^* = \frac{\partial}{\partial y_3}$$

from a basis of $T^{\perp}B$.

We choose

$$D_1 = sp\{e_1, e_2\}$$

and

$$D_2 = sp\{e_3, e_4\},$$

then D_1, D_2 are anti-invariant and slant distribution. Thus

$$TB = D_1 \oplus D_2 \oplus sp\{\xi\}$$

B is a hemi-slant submanifold of $\mathbb{R}^7$.

5. Curvature Properties of Distributions

[11], Let B be a hemi-slant submanifold of a Lorentzian Kenmotsu manifold $\overline{B}$. From (3.1) and (4.1), a hemi-slant submanifold B has constat φ-sectional curvatre k if and only if the Riemanian curvatre tensor R satisfied

$$\begin{aligned} R(X,Y,Z,W) &= \frac{k+3}{4}\{g(Z,Y)g(W,X) - g(W,Y)g(Z,X)\} \\ &+\frac{k-1}{4}\{g(\varphi Y,Z)g(\varphi X,W) - g(\varphi Y,W)g(\varphi X,Z) \\ &-2g(\varphi Z,W)g(\varphi X,Y) + g(Z,X)\eta(W)\eta(Y) \\ &-g(Z,Y)\eta(W)\eta(X) + g(W,Y)\eta(Z)\eta(X)\} \\ &+g(h(Z,Y),h(W,X)) - g(h(Z,X),h(W,Y)). \end{aligned} \tag{5.1}$$

Proposition 1. Let B be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$.Therefore we get

$$\begin{aligned} R(X,Y,Z,W) &= \frac{k+3}{4}\{g(Z,Y)g(W,X) - g(W,Y)g(Z,X)\} \\ &+g(h(Z,Y),h(W,X)) - g(h(Z,X),h(W,Y)) \end{aligned} \tag{5.2}$$

for all $X,Y,Z,W \in \Gamma(D_1)$.

Proof. The proof follows from (5.1). □

Corollary 5.1. *Let B be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$ and anti-invariant distribution D_1 is totally geodesic. Therefore D_1 is flat if and only if $k = -3$.*

Theorem 5.2. *Let B be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$.If anti-invariant distribution D_1 is totally geodesic, therefore it is Einstein.*

Proof. Let D_1 is totally geodesic. For all $X, Y \in \Gamma(D_1)$ using (5.2), we have Ricci curvature by

$$S_1(X,Y) = \sum_{i=1}^{2p} \frac{k+3}{4}\{g(X,Y)g(E_i,E_i) - g(X,E_i)g(E_i,Y)\}.$$

Then, by elementary calculations, we get

$$S_1(X,Y) = \frac{(k+3)(2p-1)}{4} g(X,Y)$$

which proves the assertion. □

Corollary 5.3. *Let B be hemi-slant submanifold of Lorentzian Kenmotsu space form $\overline{B}(k)$. If D_1 is totally geodesic, scalar curvature of D_1 given by*

$$\tau_{D_1} = p(p-1)\frac{k+3}{4}$$

Theorem 5.4. *Let B be hemi-slant submanifold of Lorenzian Kenmotsu space form $\overline{B}(k)$. Therefore the scalar curvatre of D_2 is given by*

$$\tau_{D_2} = q\frac{(k+3)(2q-1) + 3(k-1)}{2}.$$

Proof. For all $U, V \in \Gamma(D_2)$, from (5.2), Ricci curvature of D_2 is given by

$$S_2(U,V) = \frac{3(k-1)(k+3) + (2q-1)}{4} g(U,V)$$

which proves the assertion. □

6. Conclusion

Lorentzian manifolds have potential for applications in many fields of mathematics and physics. In particular it is applicable to the theory of relativity, theory of spacetimes. Researchers have increased studies on this field from different areas in recent years. After the defination of Lorentzian Kenmotsu manifold, hemi-slant submanifolds were studied. In this paper, the idea of examining curvature of hemi-slant submanifold are emphasized. The works on this subject will be useful tools for the applications of hemi-slant submanifold with different manifolds.

7. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding The author declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest

The author declared that no conflict of interest or common interest

The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics

The author declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the author declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

References

[1] P. Alegre, Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds, Taiwanese Journal of Mathematics, Vol.17, pp.897-910 (2013).

[2] N. S. Basavarajappa, C. S. Bagewadi, D. G. Prakasha, Some results on Lorentzian $\beta-$Kenmotsu manifolds. Ann. Math. Comp. Sci. Ser, Vol.35, pp.7-14 (2008).

[3] R.L. Bishop, B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., Vol.145, pp.1-50 (1969).

[4] S. Dirik, B.Bulut, On the geometry of contact pseudo-slant submanifolds of para $\beta-$Kenmotsu manifolds, Bulletin of the International Mathematical Virtual Institute, Vol.14, No.1, pp.157-168 (2024).

[5] S. Dirik, R. Sarı, Contact Pseudo-Slant Submanifolds of Lorentzian Para Kenmotsu Manifold, Journal of Engineering Research and Applied Science, Vol.12, No.2, pp.2301-2306 (2023).

[6] K. L. Duggal, Speace time manifold and contact Manifolds, Int. J. of math. and mathematical science, Vol.13, pp.545-554 (1990).

[7] K. Kenmotsu, A class of almost contact Riemannian manifolds, TohokuMath. J. II Ser., Vol.24, pp.93-103 (1972).

[8] M. A. Khan, K. Singh, V. A. Khan, Slant submanifolds of almost LP-contact manifold, Differential Geometry - Dynamical Systems, Vol.12, pp.102-108 (2010).

[9] R. Rosca, On Lorentzian Kenmotsu manifolds, Atti Accad. Peloritana Pericolanti Cl. Aci. Fis. Mat. Natur, Vol.69, pp.15-30 (1991).

[10] R. Sarı, A. Vanlı, Slant submanifolds of a Lorentz Kenmotsu manifold. Mediterr. J. Math.,16:129,(2019).

[11] R. Sarı, İ. Ünal, Semi-invariant submanifolds a Lorentzian Kenmotsu manifold with semi-symmetric metric connection, Bingol University Journal of Technical Science, Vol.2, No.1, pp.36-42 (2021).

[12] R. Sarı, S. Dirik, Generic Submanifolds of Para $\beta-$Kenmotsu Manifold, Journal of Engineering Research and Applied Science, Vol.12, No.1, pp.2291-2294 (2023).

[13] T. Takahashi, Sasakian manifold with pseudo-Riemannian metric, Tohoku Math. J., Vol.21, No.2, pp.271-290 (1969).

[14] İ. Ünal, Generic submanifolds of Lorentzian Para Kenmotsu Manifold, KMU Journal of Engineering and Natural Sciences, Vol.3, No.2, pp.79-85 (2021).

[15] İ. Ünal, A classification of para-Kenmotsu space forms, Palestine Journal of Mathematics, 10(S.I,II), pp.197-203 (2021).

Department of Mathematics, Amasya University, Amasya, Türkiye
Email address: `ramazan.sari@amasya.edu.tr`