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Research Article

Optimizing solutions with competing anisotropic (p, q)-Laplacian in hemivariational inequalities

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ABSTRACT. For differential inclusions and hemivariational inequalities driven by anisotropic differential operators, we establish the existence of generalized variational solutions and weak solutions. The main novelty consists in allowing that the driving operators might not satisfy any ellipticity condition, which is achieved for the first time in the anisotropic and nonsmooth context. The approach is based on a finite dimensional approximation process.

Keywords: Differential inclusion, hemivariational inequality, anisotropic p-Laplacian, competing operators, generalized variational solution, weak solution.

2020 Mathematics Subject Classification: 35J87, 35J92, 47J30.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

In this paper, we study the following differential inclusion with the Dirichlet boundary condition

$$\begin{cases} -\Delta_{\vec{p}}u + \mu\Delta_{\vec{q}}u \in \partial F(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

on a bounded domain Ω in $\mathbb{R}^N$ with $N \geq 2$ and boundary $\partial\Omega$. Here $\mu \in \mathbb{R}$ is a parameter and we have $\vec{p} = \{p_1, \cdots, p_N\}$ and $\vec{q} = \{q_1, \cdots, q_N\}$, where $1 < p_1, \cdots, p_N < \infty$, $1 < q_1, \cdots, q_N < \infty$, and $q_i < p_i$ for all $i = 1, \cdots, N$. The driving operator $-\Delta_{\vec{p}} + \mu\Delta_{\vec{q}}$ in (1.1) is formed with the anisotropic $\vec{p}$-Laplacian $\Delta_{\vec{p}}$ and the anisotropic $\vec{q}$-Laplacian $\Delta_{\vec{p}}$. We recall that the anisotropic $\vec{r}$-Laplacian with $\vec{r} = (r_1, \cdots, r_N)$ is defined as

$$\Delta_{\vec{r}} := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\cdot)}{\partial x_i}\right|^{r_i-2}\right)\frac{\partial(\cdot)}{\partial x_i}.$$

In (1.1), we take $\vec{r} = \vec{p}$ and $\vec{r} = \vec{q}$. For our purpose, the most relevant case of driving operator in (1.1) is the competing anisotropic operator $-\Delta_{\vec{p}} + \Delta_{\vec{q}}$. We assume that

$$\sum_{i=1}^{N} \frac{1}{p_i} > 1. \tag{1.2}$$

Set

$$p^+ := \max\{p_1, \cdots, p_N\},\ p^- := \min\{p_1, \cdots, p_N\},\ p^* := \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1},$$

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and further assume

$$p^+ < p^*. \tag{1.3}$$

In the right-hand side of inclusion (1.1), we have the generalized gradient ∂F of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ (see [9]). The multivalued expression $\partial F(u)$ means that pointwise $\partial F(u(x))$ is a subset of $\mathbb{R}$ for any $x \in \Omega$. Without loss of generality, we may suppose that $F(0) = 0$. We assume that the following condition is satisfied:

(H) There exist positive constants c_0 and c_1 with $c_1 < \lambda_{1,\vec{p}}p^-$ such that

$$|\xi| \leq c_0 + c_1|t|^{p^- -1}$$

for all $t \in \mathbb{R}$ and $\xi \in \partial F(t)$, where

$$\lambda_{1,\vec{p}} := \inf_{u\in W_0^{1,\vec{p}}(\Omega),u\neq 0} \frac{\sum_{i=1}^{N} \frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}}^{p^-}}{\|u\|_{L^{p^-}}^{p^-}}. \tag{1.4}$$

The definition of the generalized gradient ∂F implies that each solution $u \in W_0^{1,\vec{p}}(\Omega)$ to (1.1) is a solution of the inequality problem

$$\langle -\Delta_{\vec{p}}u, v\rangle + \mu\langle -\Delta_{\vec{q}}u, v\rangle \leq \int_\Omega F^\circ(u(x); v(x))dx \tag{1.5}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$, where F° denotes the generalized directional derivative of the locally Lipschitz function F. Problem (1.5) is a hemivariational inequality in the Banach space $W_0^{1,\vec{p}}(\Omega)$. A brief presentation of the space $W_0^{1,\vec{p}}(\Omega)$ will be done in Section 2.

We are interested in two types of solutions for inclusion (1.1) and a fortiori for hemivariational inequality (1.5), namely the weak and generalized variational solutions.

Definition 1.1. *A function $u \in W_0^{1,\vec{p}}(\Omega)$ is called a weak solution to (1.1) if*

$$\langle -\Delta_{\vec{p}}u, v\rangle + \mu\langle -\Delta_{\vec{q}}u, v\rangle = \int_\Omega z(x)v(x)dx \tag{1.6}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$, with $z \in L^{\vec{p}'}(\Omega) \in \partial F(u)$ a.e. on Ω.

Definition 1.2. *A function $u \in W_0^{1,\vec{p}}(\Omega)$ is called a generalized variational solution to inclusion (1.1) if there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{1,\vec{p}}(\Omega)$ such that*

- (a) *$u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ as $n \to \infty$;*
- (b) *$-\Delta_{\vec{p}}u_n + \mu\Delta_{\vec{q}}u_n - z_n \rightharpoonup 0$ in $W^{-1,\vec{p}'}(\Omega)$ as $n \to \infty$ with $z_n \in L^{\vec{p}'}(\Omega)$ and $z_n \in \partial F(u_n)$ a.e. on Ω;*
- (c) $\lim_{n\to\infty}\langle \Delta_{\vec{p}}u_n + \mu\Delta_{\vec{q}}u_n, u_n - u\rangle = 0.$

From Definitions 1.1 and 1.2, we see that any weak solution $u \in W_0^{1,\vec{p}}(\Omega)$ to problem (1.1) is a generalized variational solution. In order to confirm this, it suffices to take $u_n = u$ in the definition of the generalized variational solution. The converse assertion is generally not valid.

Our main results are formulated as follows. Note that the part played by the parameter μ is fundamental.

Theorem 1.1. *Under the stated assumptions, there exists a generalized variational solution to problem (1.1) for every $\mu \in \mathbb{R}$. In particular, there exists a solution of the hemivariational inequality (1.5).*

Theorem 1.2. *Under the stated assumptions, if $\mu \leq 0$ then each generalized variational solution to problem (1.1) is a weak solution. Moreover, if $\mu \leq 0$, problem (1.1) admits a weak solution which is a global minimizer of the minimization problem*

$$\inf_{v \in W_0^{1,\vec{p}}(\Omega)} \left[\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{\mu}{q_i} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{q_i}}^{q_i} - \int_{\Omega} F(v(x))dx \right]. \tag{1.7}$$

The main novelty in our study is the presence of the anisotropic operator $-\Delta_{\vec{p}}u + \mu\Delta_{\vec{q}}u$ in the nonsmooth problem, which loses the ellipticity when $\mu > 0$. This extends to an anisotropic nonsmooth setting the use of competing operators considered until now in completely different situations [12, 15, 16, 17, 19]. We mention that the concept of generalized solution for equations involving competing operators and convection terms was developed in [11, 14, 15, 16, 23] (see also [1, 2, 7, 26]). In the present work, we explore the existence of generalized solutions to hemivariational solutions driven by competing anisotropic operators.

The rest of the paper, has the following structure. In Section 2, we outline the needed background of anisotropic spaces and operators and provide auxiliary results regarding the nonsmooth analysis for inclusion (1.1). In Section 3, we present our approach based on finite dimensional approximate solutions. In Sections 4 and 5, we prove Theorems 1.1 and 1.2, respectively.

2. Mathematical background and auxiliary results

The anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is defined as the completion of the set of smooth functions with compact support $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}},$$

where $\|\cdot\|_{L^r}$ is the usual norm of the space $L^r(\Omega)$. It is separable and uniformly convex, thus a reflexive Banach space. The dual of $W_0^{1,\vec{p}}(\Omega)$ is denoted $W^{-1,\vec{p}\,'}(\Omega)$. The following embedding theorem can be found in [10, Theorem 1].

Theorem 2.3. *Assume that conditions (1.2) and (1.3) hold. Then for all $r \in [1, p^*]$, there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^r(\Omega)$. For $r < p^*$, the embedding is compact.*

From Theorem 2.3, we have the compact embedding

$$W_0^{1,\vec{p}}(\Omega) \subset L^{p^-}(\Omega). \tag{2.8}$$

In particular, by (2.8) we infer that there exists a constant $S_1 > 0$ such that

$$\|v\|_{L^1} \leq S_1 \|v\|_{W_0^{1,\vec{p}}(\Omega)}, \quad \forall v \in W_0^{1,\vec{p}}(\Omega). \tag{2.9}$$

The quantity $\lambda_{1,\vec{p}}$ in (1.4) is finite due to the compact embedding (2.8). Since the space $W_0^{1,\vec{p}}(\Omega)$ is separable, there exists a Galerkin basis for $W_0^{1,\vec{p}}(\Omega)$, that is, a sequence of vector subspaces $\{X_n\}_{n\geq 1}$ of $W_0^{1,\vec{p}}(\Omega)$ such that

(i) $dim(X_n) < \infty$ for all n;
(ii) $X_n \subset X_{n+1}$ for all n;
(iii) $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,\vec{p}}(\Omega)$.

For various aspects involving anisotropic Sobolev spaces, we refer to [3, 4, 5, 10, 13, 18, 20, 23, 24, 21, 22, 25].

We continue with a brief survey of basic elements of nonsmooth analysis that are needed in the sequel.

Given a locally Lipschitz function $F : X \to \mathbb{R}$ on a normed space X, the generalized directional derivative of F at $u \in X$ in the direction $v \in X$ is defined as

$$F^\circ(u;v) := \limsup_{w\to u, t\to 0^+} \frac{1}{t}\left(F(w+tv) - F(w)\right).$$

The generalized gradient of F at $u \in X$ is the subset of X^* given by

$$\partial F(u) := \{u^* \in X^* : \langle u^*, v\rangle \le F^\circ(u;v) \quad \text{for all } v \in X\}.$$

A case of major interest for us in connection with the resolution of problem (1.1) is when $X = \mathbb{R}$. In this case, a relevant realization of the preceding notions is as follows. Let $f \in L^\infty_{loc}(\mathbb{R})$ and its primitive $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \int_0^t f(s)ds, \quad \forall t \in \mathbb{R} \tag{2.10}$$

which is locally Lipschitz. The explicit expression of the generalized gradient $\partial F(t)$ is $\partial F(t) = [\underline{f}(t), \overline{f}(t)]$, where

$$\underline{f}(t) = \lim_{\delta\to 0} \operatorname{ess\,inf}_{|\eta - t|<\delta} f(\eta) \text{ and } \overline{f}(t) = \lim_{\delta\to 0} \operatorname{ess\,sup}_{|\eta - t|<\delta} f(\eta)$$

for every $t \in \mathbb{R}$. With the choice in (2.10), inclusion (1.1) becomes

$$\begin{cases} -\Delta_{\vec{p}} u + \mu\Delta_{\vec{q}} u \in [\underline{f}(u), \overline{f}(u)] & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

which is important for equations with discontinuous nonlinearities (see [8]).

Now, we return to our general case of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfying hypothesis (H). It follows from hypothesis (H) that the function F verifies the growth condition

$$|F(t)| \le c_0|t| + \frac{c_1}{p^-}|t|^{p^-}, \quad \forall t \in \mathbb{R}. \tag{2.11}$$

Indeed, note that $F(0) = 0$ and F is differentible almost everywhere due to Rademacher's theorem, thus

$$F(t) = \int_0^t F'(s)ds, \quad \forall t \in \mathbb{R}.$$

Since $F'(s) \in \partial F(s)$ for all $t \in \mathbb{R}$ (refer to [9, p. 32])), it turns out from hypothesis (H) that (2.11) holds true.

It is straightforward to check that the functional $\Phi : L^{p^-}(\Omega) \to \mathbb{R}$ given by

$$\Phi(v) = \int_\Omega F(v(x))dx, \quad \forall v \in L^{p^-}(\Omega) \tag{2.12}$$

is Lipschitz continuous on the bounded subsets of $L^{p^-}(\Omega)$, thus locally Lipschitz on $L^{p^-}(\Omega)$. Therefore the generalized gradient $\partial\Phi$ is well defined on $L^{p^-}(\Omega)$.

Using that the domain Ω is bounded, Hölder's inequality ensures the continuous embedding $W_0^{1,\vec{p}}(\Omega) \subset W_0^{1,\vec{q}}(\Omega)$ (note that $q_i < p_i$ for all $i = 1, \dots, N$). Then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ in (2.8) allows us to define the functional $J : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ by

$$J(v) = \sum_{i=1}^N \frac{1}{p_i}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^N \frac{\mu}{q_i}\left\|\frac{\partial v}{\partial x_i}\right\|_{L^{q_i}}^{q_i} - \int_\Omega F(v(x))dx \tag{2.13}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$.

Proposition 2.1. *Assume that condition* (H) *holds. The functional* J *given by* (2.13) *is locally Lipschitz on* $W_0^{1,\vec{p}}(\Omega)$ *with the generalized gradient*

$$\partial J(v) = \sum_{i=1}^{N} \left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i} - \mu \sum_{i=1}^{N} \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i} - \partial \Phi(v) \tag{2.14}$$

for all $v \in W_0^{1,\vec{p}}(\Omega)$. *Moreover, the functional* J *is coercive on* $W_0^{1,\vec{p}}(\Omega)$, *which means that*

$$\lim_{\|v\|_{W_0^{1,\vec{p}}(\Omega)} \to \infty} J(v) = +\infty. \tag{2.15}$$

Proof. The first part of the statement is a direct consequence of (2.13) and of what was said about the functional Φ introduced in (2.12).

We pass to the proof of (2.15). Hypothesis (H) in conjunction with (2.9), (1.4), (2.8), (2.13) and Hölder's inequality, leads to

$$\begin{aligned} J(v) &\geq \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{|\mu|}{q_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{q_i}}^{q_i} - \int_\Omega \left(c_0 |v| + \frac{c_1}{p^-} |v|^{p^-} \right) dx \\ &\geq \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p_i} - \sum_{i=1}^{N} \frac{|\mu|}{q_i} |\Omega|^{\frac{p_i - q_i}{p_i}} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{q_i} \\ &\quad - c_0 S_1 \sum_{i=1}^{N} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}} - \frac{c_1 \lambda_{1,\vec{p}}^{-1}}{p^-} \sum_{i=1}^{N} \frac{1}{p_i} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p_i}}^{p^-}, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. As it was assumed that $1 < q_i < p_i$ for all $i = 1, \cdots, N$, and $c_1 < \lambda_{1,\vec{p}} p^-$, we arrive at (2.15), so the functional J is coercive. □

3. Sequence of approximate solutions

In order to simplify the notation, for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r), and we can set $\vec{p}' := (p_1', \cdots, p_N')$ for $\vec{p} = (p_1, \cdots, p_N)$.

As noticed in Section 2, there exists a Galerkin basis $\{X_n\}_{n\geq 1}$ for the space $W_0^{1,\vec{p}}(\Omega)$ that we now fix. We construct approximate solutions to inclusion (1.1) on each finite dimensional subspace X_n.

Proposition 3.2. *Assume that hypothesis* (H) *holds. Then, for each* n, *there exist* $u_n \in X_n$ *and* $z_n \in L^{p^{-\prime}}(\Omega)$ *with* $z_n \in \partial F(u_n)$ *almost everywhere on* Ω *such that*

$$J(u_n) = \inf_{v \in X_n} J(v) \tag{3.16}$$

and

$$\langle -\Delta_{\vec{p}} u_n, v\rangle + \mu \langle -\Delta_{\vec{q}} u_n, v\rangle - \int_\Omega z_n v dx = 0 \tag{3.17}$$

for all $v \in X_n$.

Proof. Proposition 2.1 ensures that the restriction $J|_{X_n}$ of the functional $J : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R}$ to the finite dimensional subspace X_n is locally Lipschitz and coercive. Therefore there exists $u_n \in X_n$ satisfying (3.16). We derive from (3.16) the necessary optimality condition

$$0 \in \partial \left(J|_{X_n}\right)(u_n). \tag{3.18}$$

In view of (2.14), we have that (3.18) results in (3.17). The Aubin-Clarke theorem (see [9, p. 83]) applied to the integral functional Φ on $L^{p^-}(\Omega)$ in (2.12) yields that $z_n \in \partial F(u_n)$ almost everywhere on Ω. This completes the proof. $\square$

Corollary 3.1. *Assume that condition (H) holds. Then the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ constructed in Proposition 3.2 satisfies*

$$\lim_{n\to\infty} J(u_n) = \inf_{w\in W_0^{1,\vec{p}}(\Omega)} J(w). \tag{3.19}$$

Proof. Recall that $X_n \subset X_{n+1}$ for all n. Then (3.16) shows that the sequence $\{J(u_n)\}$ is nonincreasing, while the proof of Proposition 3.2 provides that is bounded from below. Hence the limit $l := \lim_{n\to\infty} J(u_n)$ exists.

Arguing by contradiction, admit that

$$l > \inf_{w\in W_0^{1,\vec{p}}(\Omega)} J(w).$$

This amounts to saying that there exists $\hat{w} \in W_0^{1,\vec{p}}(\Omega)$ such that $J(\hat{w}) < l$. Consequently, there exists a neighborhood U of $\hat{w}$ in $W_0^{1,\vec{p}}(\Omega)$ such that

$$J(w) < l \text{ for all } w \in U. \tag{3.20}$$

Since $W_0^{1,\vec{p}}(\Omega) = \overline{\cup_{n=1}^{\infty} X_n}$, there exists m such that $\tilde{w} \in U \cap X_m$. Then (3.16) and (3.20) yield

$$\min_{v\in X_m} J(v) \le J(\tilde{w}) < l \le \min_{v\in X_m} J(v).$$

The obtained contradiction proves (3.19), thus completing the proof. $\square$

We focus on the sequence $\{u_n\}$.

Proposition 3.3. *Assume that condition (H) holds. Then the sequence $\{u_n\}$ constructed in Proposition 3.2 is bounded in $W_0^{1,\vec{p}}(\Omega)$, so there is a constant $M_1 > 0$ such that*

$$\|u_n\|_{W_0^{1,\vec{p}}(\Omega)} \le M_1 \textit{ for all } n \ge 1. \tag{3.21}$$

Proof. Set $v = u_n$ in (3.17) (note that $u_n \in X_n$). Then, as in the proof of Proposition 2.1, we use $z_n(x) \in \partial F(u_n(x))$ for almost all $x \in \Omega$ to infer that

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i} \\
=&\mu \sum_{i=1}^{N} \frac{1}{q_i} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i} + \int_\Omega z_n u_n dx \\
\le& \sum_{i=1}^{N} \frac{|\mu|}{q_i} |\Omega|^{\frac{p_i - q_i}{p_i}} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{q_i} + c_0 S_1 \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}} + \frac{c_1 \lambda_{1,\vec{p}}^{-1}}{p^-} \sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p^-}.
\end{aligned}$$

Since $1 < q_i < p_i$ and $p^- \le p_i$ for all $i = 1, \dots, N$, and $c_1 < \lambda_{1,\vec{p}} p^-$, we get the stated result. $\square$

Corollary 3.2. *Assume that condition (H) hods. Then for the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ in Proposition 3.2 there is a constant $M_2 > 0$ such that*

$$\| -\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \|_{W^{-1,\vec{p}\,'}(\Omega)} \le M_2 \tag{3.22}$$

for all n, with z_n as described in Proposition 3.2.

Proof. For each $v \in W_0^{1,\vec{p}}(\Omega)$, by Hölder's inequality, hypothesis (H), (2.9) and (1.4), we find the estimate

$$\begin{aligned}
&|\langle -\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n, v\rangle| \\
&= \left| \sum_{i=1}^N \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \mu \sum_{i=1}^N \int_\Omega \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_i} dx - \int_\Omega z_n v dx \right| \\
&\le \sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}} + |\mu| \sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{q_i}} \\
&+ \int_\Omega (c_0 + c_1 |u_n|^{p^- - 1}) |v| dx \\
&\le \left(\sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} + |\mu| \sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} + c_0 S_1 + \lambda_{1,\vec{p}}^{-\frac{1}{p^-}} \|u_n\|_{L^{p^-}}^{p^- - 1} \right) \|v\|_{W_0^{1,\vec{p}}(\Omega)}.
\end{aligned}$$

This entails

$$\begin{aligned}
&\| -\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \|_{W_0^{-1,\vec{p}\,'}(\Omega)} \\
&\le \sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i - 1} + |\mu| \sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{q_i}}^{q_i - 1} + c_0 S_1 + \lambda_{1,\vec{p}}^{-\frac{1}{p^-}} \|u_n\|_{L^{p^-}}^{p^- - 1}.
\end{aligned} \tag{3.23}$$

By (3.23), (3.21) and Theorem 2.3, we obtain the validity of (3.22), which completes the proof. □

4. Proof of Theorem 1.1

Proposition 3.3 provides the sequence $\{u_n\} \subset W_0^{1,\vec{p}}(\Omega)$ which is bounded in $W_0^{1,\vec{p}}(\Omega)$ as demonstrated in (3.21). Therefore, thanks to the reflexivity of the space $W_0^{1,\vec{p}}(\Omega)$, up to a subsequence it holds $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}}(\Omega)$ for some $u \in W_0^{1,\vec{p}}(\Omega)$. Corollary 3.2 ensures that the sequence $\{-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n\}$ is bounded in $W^{-1,\vec{p}\,'}(\Omega)$, with $z_n \in L^{p^{-\prime}}(\Omega)$ satisfying $z_n \in \partial F(u_n)$ almost everywhere on Ω. Then along a relabeled subsequence we have $-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \rightharpoonup \eta$ in $W^{-1,\vec{p}\,'}(\Omega)$ for some $\eta \in W^{-1,\vec{p}\,'}(\Omega)$.

We claim that $\eta = 0$. In order to prove the claim, let $v \in \cup_{n=1}^\infty X_n$, so $v \in X_m$ for some m. Note that for each $n \ge m$, we have $v \in X_n$, which enables us to insert v in (3.17). Letting $n \to \infty$ in (3.17) renders $\langle \eta, v\rangle = 0$. Using that $\cup_{n=1}^\infty X_n$ is dense $W_0^{1,\vec{p}}(\Omega)$, we are able to conclude that $\eta = 0$. Therefore we have

$$-\Delta_{\vec{p}} u_n + \mu \Delta_{\vec{q}} u_n - z_n \rightharpoonup 0 \text{ in } W^{-1,\vec{p}\,'}(\Omega). \tag{4.24}$$

Combining (3.17) and (4.24) results in

$$\lim_{n\to\infty} \left[\langle -\Delta_{\vec{p}} u_n, u_n - u\rangle + \mu \langle \Delta_{\vec{q}} u_n, u_n - u\rangle - \int_\Omega z_n (u_n - u) dx \right] = 0. \tag{4.25}$$

We stress that in the above arguments $\mu \in \mathbb{R}$ is arbitrary. We are thus in a position to assert that $u \in W_0^{1,\vec{p}}(\Omega)$ is a generalized variational solution to problem (1.1) whose sequence required in Definition 1.2 is $\{u_n\}$. As noticed before, we deduce that $u \in W_0^{1,\vec{p}}(\Omega)$ is a solution to the hemivariational inequality (1.5). The proof of Theorem 1.1 is completed.

5. Proof of Theorem 1.2

Now we assume that $\mu \le 0$. Theorem 1.1 applies producing a generalized weak solution for problem (1.1).

Let $u \in W_0^{1,\vec{p}}(\Omega)$ be a generalized weak solution to problem (1.1). According to Definition 1.2, there is a sequence $\{u_n\}$ in $W_0^{1,\vec{p}}(\Omega)$ satisfying the requirements therein. In particular, it holds (4.25). The sequence $\{z_n\}$ is bounded in $L^{p^{-\prime}}(\Omega)$ due to the Lipschitz continuity of the functional Φ on the bounded subsets of $L^{p^-}(\Omega)$ (refer to the proof of Proposition 3.2). Moreover, it is true that $u_n \to u$ in $L^{p^-}(\Omega)$ owing to the compact embedding in Theorem 2.3 for $r = p^-$. Altogether this gives

$$\lim_{n\to+\infty} \int_\Omega z_n(u_n - u)dx = 0.$$

Then (4.25) leads to

$$\lim_{n\to\infty} \langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle = 0. \tag{5.26}$$

Using that $\mu \le 0$ and the monotonicity of the operator $-\Delta_{\vec{q}}$ on $W_0^{1,\vec{q}}(\Omega)$, we are able to write

$$\begin{aligned}&\langle -\Delta_{\vec{p}} u_n, u_n - u\rangle \\ =&\langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u_n + \Delta_{\vec{q}} u, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u, u_n - u\rangle \\ \le&\langle -\Delta_{\vec{p}} u_n + \mu\Delta_{\vec{q}} u_n, u_n - u\rangle + \mu\langle -\Delta_{\vec{q}} u, u_n - u\rangle.\end{aligned}$$

By (5.26) and $u_n \rightharpoonup u$ in $W_0^{1,\vec{q}}(\Omega)$, we find that

$$\limsup_{n\to\infty} \langle -\Delta_{\vec{p}} u_n, u_n - u\rangle \le 0. \tag{5.27}$$

The monotonicity of the operator $-\Delta_{\vec{p}}$ on $W_0^{1,\vec{p}}(\Omega)$ implies

$$\begin{aligned} 0 &\le \sum_{i=1}^N \int_\Omega \left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx \\ &= \langle -\Delta_{\vec{p}} u_n + \Delta_{\vec{p}} u, u_n - u\rangle. \end{aligned}$$

By (5.27) and $u_n \rightharpoonup u$ in $W_0^{1,\vec{q}}(\Omega)$, we are entitled to assert that

$$\lim_{n\to\infty} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx = 0 \quad \forall\, i = 1, \dots, N$$

which yields

$$\limsup_{n\to\infty} \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}} \le \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}} \qquad \forall\, i = 1, \dots, N.$$

Since the space $L^{p_i}(\Omega)$ is uniformly convex (see [6]), we infer the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, thus $-\Delta_{\vec{p}} u_n \to -\Delta_{\vec{p}} u$ in $W^{-1,\vec{p}\,'}(\Omega)$ and $-\Delta_{\vec{q}} u_n \to -\Delta_{\vec{q}} u$ in $W^{-1,\vec{q}\,'}(\Omega)$.

On the other hand, taking into account that $u_n \to u$ in $L^{p^-}(\Omega)$ and $z_n \in \partial\Phi(u_n) \subset L^{p^{-\prime}}(\Omega)$, the sequence $\{z_n\}$ is bounded in $L^{p^{-\prime}}(\Omega)$, so along a subsequence $z_n \rightharpoonup z$ in $L^{p^{-\prime}}(\Omega)$ for some $z \in L^{p^{-\prime}}(\Omega)$. From [9], it is known that the generalized gradient $\partial\Phi$ is weak*-closed, so we obtain $z \in \partial\Phi(u)$. Furthermore, (4.24) ensures

$$-\Delta_{\vec{p}} u + \mu\Delta_{\vec{q}} u - z = 0 \text{ in } W^{-1,\vec{p}\,'}(\Omega).$$

Under assumption (H), the Aubin-Clarke theorem (see [9]) can be applied to the functional $\Phi : L^{p^-}(\Omega) \to \mathbb{R}$ in (2.12) establishing that $z(x) \in \partial F(u(x))$ for almost all $x \in \Omega$. Consequently,

$u \in W_0^{1,\vec{p}}(\Omega)$ satisfies (1.6), thus it is a weak solution to the inclusion problem (1.1), thereby of hemivariational inequality (1.5), too.

The last step in the proof concerns to show that $u \in W_0^{1,\vec{p}}(\Omega)$ solves the global minimization in (1.7). In view of (2.13), the global minimization in (1.7) reads as $u \in W_0^{1,\vec{p}}(\Omega)$ is a global minimizer of the functional J on $W_0^{1,\vec{p}}(\Omega)$. On the basis of the strong convergence $u_n \to u$ in $W_0^{1,\vec{p}}(\Omega)$, we are allowed to pass to the limit in (3.19) finding that $\inf_{w \in W_0^{1,\vec{p}}(\Omega)} J(w)$ is achieved at $u \in W_0^{1,\vec{p}}(\Omega)$. The proof is complete.

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