Gümüşhane University Journal of Science

GUFBD / *GUJS* (2025) 15(1): 245-259 doi: 10.17714/gumusfenbil.1567137 Research Article

Analytical and numerical solutions of some differential equations: a comparative study using Natural transform and Runge-Kutta method

Bazı diferansiyel denklemlerin analitik ve numerik çözümleri: Natural dönüşüm ile Runge-Kutta yöntemi kullanılarak karşılaştırmalı bir çalışma

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• Received: 14.10.2025 • Accepted: 03.03.2025

Abstract

This paper focuses on Natural transform and the Runge-Kutta numerical method. These techniques have been used to analyze blood glucose concentrations and electrical circuits. These examples have been selected to demonstrate the applicability of Natural transform in many different areas. It was aimed to solve electrical circuits, which are generally solved by Laplace transform in engineering literature, with Natural transform and to obtain a comparative solution with Runge-Kutta method, which is a numerical method. These engineering problems defined with differential equations were analyzed using Natural transform. Firstly, the differential equations were written using the Natural transform, then the new equations were solved and applying the inverse transform, the results of the equations were obtained. The fourth-order Runge-Kutta numerical method was the second method employed in this study. The results found with this methods were presented in tables and compared graphically. The results applying Natural transform and Runge–Kutta method are equivalent to exact solution.

Keywords: Glucose concentration, Natural transform, RLC circuit, Runge-Kutta

Öz

Bu çalışma, Naturel dönüşüm ve Runge-Kutta sayısal yöntemi üzerine odaklanmaktadır. Bu teknikler kan şekeri konsantrasyonlarının analizinde ve devre analizlerinde kullanılmıştır. Örnekler, Naturel dönüşümün birçok farklı alanda uygulanabilirliğini göstermek için seçilmiştir. Mühendislik literatüründe genellikle Laplace dönüşümü ile çözülen elektrik devrelerinin Naturel dönüşüm ile çözülmesi ve sayısal bir yöntem olan Runge-Kutta yöntemi ile karşılaştırmalı bir çözüm elde edilmesi amaçlanmıştır. Diferansiyel denklemlerle tanımlanmış bu mühendislik problemleri Natural dönüşümü ile analiz edilmiştir. İlk olarak diferansiyel denklemler Naturel dönüşüm kullanılarak yazılmış, daha sonra yeni denklemler çözülmüş ve ters dönüşüm uygulanarak denklemlerin sonuçları elde edilmiştir. Bu çalışmada kullanılan ikinci yöntem, dördüncü dereceden Runge-Kutta sayısal yöntemidir. Bu metotlarla elde edilen sonuçlar tablolar halinde sunulmuş ve grafiksel olarak karşılaştırılmıştır. Naturel dönüşüm ve Runge-Kutta yöntemi uygulanarak elde edilen sonuçların tam çözüme eşdeğer olduğu görülmüştür.

Anahtar kelimeler: *Glikoz konsantrasyonu, Naturel dönüşüm, RLC devresi, Runge-Kutta*

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1. Introduction

Integral transforms are essential mathematical techniques that are frequently used to solve differential equations and analyze complicated systems in many scientific and technical fields. They are essential in applications including signal processing, control theory, fluid dynamics, and quantum physics because of their capacity to translate differential equations into algebraic equations (Debnath & Bhatta, 2016). Numerous integral transforms, each designed to handle a particular difficulty, have been introduced as a result of the physical science and engineering models' rapid evolution throughout time. These transforms are derived from the different kernel functions in the fundamental integral transform. Fourier and Laplace transforms, which are the oldest and most popular transforms, have been frequently used in many problems before. In recent years, studies in this field have been continuing rapidly. In addition to these, many other transforms have been obtained and used: Sumudu, Natural, Anuj, Sawi, Shehu etc. (Gardner & Barnes, 1942; Bateman, et al., 1954; Lado, 1971; Watugala, 1998; Khan & Khan, 2008; Doetsch, 2013; Mahgoub & Mohand, 2019; Maitama & Zhao, 2019; Kumar et al., 2021). Additionally, generalized forms of these transforms have been extensively studied in the literature. For instance, Natural transform is the particular case of Upadhyaya transform (Upadhyaya, 2019; Upadhyaya et al., 2021).

Among the extensive suite of integral transforms, Natural transform has garnered significant attention due to their properties and wide-ranging applications. This transform not only simplify mathematical problems but also preserve essential characteristics such as scale and units, making them useful in practical applications. This transform has broad applications in fields such as signal processing, control theory, and financial mathematics, exemplifying their versatility and effectiveness in addressing complex mathematical problems. The duality features of the Natural transform with the Laplace and Sumudu transforms, as well as the scale change, weight shift, multiple shift, and convolution theorem, are thoroughly examined by Belgacem and Silambarasan (Belgacem & Silambarasan, 2012). The Natural transform has also dualities with Khalouta integral transforms (Khalouta, 2023).

The Natural transform has been extensively studied in the solution of lineer / nonlinear ordinary differential equations (ODEs) and partial differential equations (Al-Omari, 2013; Rawashdeh & Maitama, 2014, 2015, 2016). It is especially useful in problems involving initial and boundary conditions, and in the realm of fractional calculus for solving Fractional differential equations (FDEs). Alkan and Anaç studied numerical approximation techniques particularly designed for the studied equations are presented. The objective of this study is to use the fractional natural transform decomposition method (FNTDM) to solve the nonlinear time-fractional Korteweg–De Vries (KdV) equation, the nonlinear time-fractional Klein-Gordon equation, and the nonlinear time-fractional Fornberg Whitham equation. By using the recently created hybrid technique, this study seeks to provide novel numerical solutions for the aforementioned equations (Alkan & Anaç, 2024). Köklü studied that the Natural transform converged to the Laplace and Sumudu transform (Köklü, 2020). In 2019, Maitama and Zhao studied that the Shehu transform, which converges to the Natural transform, converges to Laplace and Sumudu transforms (Maitama & Zhao, 2019). Moreover, Kiliçman and Omran generalize the concept of one-dimensional Natural transform to two-dimensional Natural transform namely, double Natural transform (Kiliçman & Omran, 2017). The Natural transform, sharing a dual relationship with the Laplace transform, facilitates the solution of integral and differential equations by maintaining many of the Laplace transform's properties while offering advantages.

Different transform methods have been used in many engineering and applied mathematics studies. Vashi and Timol used Sumudu and Laplace transform to solve applications in physics and circuit theory (Vashi & Timol, 2016). Jadhav et al solved the RL, RC and LC circuit using Sumudu transform (Jadhav et al., 2022). Peker and Çuha used Kashuri Fundo transform for exact solutions of some cardiovascular models (Peker & Çuha, 2023). Silambarasan and Belgacem presented the Natural transform for Maxwell equations (Silambarasan & Belgacem, 2011). Chindhe and Kiwne used Natural transform for Cryptography (Chindhe & Kiwne, 2017). Higazy et al determined the number of infected cells and concentration of infected particles in plasma during HIV-1 infections using Shehu transform (Higazy et al., 2020).

The second method used in this study is the 4th-order Runge Kutta method, which is a numerical method. It is a powerful and versatile tool for solving ODEs. Due to its accuracy and ease of application, it is widely used in scientific and engineering applications. It is widely recognized that the foundational work on Runge-Kutta methods can be traced back to the contributions of Carl Runge (1895) (Runge, 1895), who extended the Euler method by incorporating multiple derivative evaluations within a single integration step. This extension laid the groundwork for more sophisticated numerical integration techniques. Subsequent advancements were made by Heun (1900) (Heun, 1900) and Wilhelm Kutta (1901) (Kutta, 1901). Heun's work introduced enhancements to the initial Runge method, while Kutta further improved these techniques, particularly with his development of the fourth-order Runge-Kutta methods and initial approaches for higher-order methods (Butcher, 2008).

Runge–Kutta methods have attracted interest, and a number of researchers have contributed to the development of particular approaches as well as more recent developments in the theory. Second-order differential equations have many applications as well. The transform of a second-order initial value problem into an equivalent system of first-order initial value problems has been comprehensively addressed to ensure thoroughness (Iyengar & Jain, 2009). In the realm of second-order differential equations, the contributions of E. J. Nyström (Nyström, 1925) were particularly notable. Nyström not only advanced the theory of numerical methods for first-order differential equations but also devised specialized techniques for effectively addressing second-order problems. The pursuit of higher-order methods continued, culminating in the introduction of sixth-order numerical techniques by Hušta in the mid-20th century (Hušta 1956, 1957). Hušta's work represented a significant leap in the precision and efficiency of numerical integration methods, extending the range of applications for Runge-Kutta type methods. The 4th-order Runge-Kutta method has a local truncation error of O (h^5) and a global error of O (h^4). This makes it highly accurate for most problems, but the step size h must be chosen carefully to balance accuracy and computational cost.

Our aim is to evaluate the accuracy and applicability of both numerical and analytical techniques in solving ODEs by comparing the Runge-Kutta method and the Natural transform. It is also aimed to show that the Runge-Kutta method converges to the exact solution. In this study, for the first-order differential equation, the application of blood glucose concentration was chosen, and for the second-order differential equation, electrical circuits were selected. The results of the Runge-Kutta method and Natural transform are presented in the tables and shown graphically.

2. Methods

The definition and properties of the Natural transform are firstly explained in this section. Then, the 4th-order Runge-Kutta method for first and second-order initial value problems is explained.

2.1. Natural transform

Khan and Khan defined the Natural transform (Khan & Khan, 2008). Belgacem and Silambarasan defined inverse Natural transform and studied some properties (Silambarasan & Belgacem, 2011; Belgacem & Silambarasan, 2012). The Natural transform, on the other hand, shares a dual relationship with the Laplace transform, which enhances its utility in solving integral and differential equations.

On the set of definitions, Natural transform was defined (Khan & Khan, 2008; Belgacem & Silambarasan, 2012):

$$D=\left\{ f\left(t\right):\ \exists N,\, k_1, k_2>0,\, \left|f\left(t\right)\right|<Ne^{t/k_i},\ if\ \ t\in\left(-1\right)^{i}\times\left[0,\,\infty\right)\ \right\},$$
$$R(u,s)=N(f)=\int_0^\infty f\left(ut\right)e^{-st}dt, \qquad s>0,\ \ u>0 \tag{1}$$

The Natural transform retains many properties of the Laplace transform while offering unique advantages, such as its straightforward application to problems involving initial conditions and boundary value problems. This transform is particularly useful in fractional calculus, where it aids in the resolution of fractional differential equations by transforming them into algebraic equations, thus simplifying their analysis and solution (Debnath, & Bhatta, 2016).

The most commonly used transformations of Natural transform are given in Table 1 and some of their properties are given in Table 2.

Table 1. Special Natural transforms (Belgacem & Silambarasan, 2012)

$f(t)$	$N[f(t)]$	$f(t)$	$N[f(t)]$
1	$\frac{1}{s}$	$\frac{\sin at}{a}$	$\frac{u}{s^2+a^2u^2}$
t	$\frac{u}{s^2}$	$\cos at$	$\frac{s}{s^2+a^2u^2}$
$\frac{t^{n-1}}{(n-1)!}, n=1,2,\ldots$	$\frac{u^{n-1}}{s^n}$	$\frac{e^{bt}\sin at}{a}$	$\frac{u}{(s-bu)^2+a^2u^2}$
e^{at}	$\frac{1}{(s-au)}$	$e^{bt}\cos at$	$\frac{s-bu}{(s-bu)^2+a^2u^2}$

Table 2. Properties of Natural transform (Khan & Khan, 2008; Belgacem & Silambarasan, 2012)

Comment	Formula of Natural transform
Duality with Laplace transform	$R(s,u)=\frac{1}{u}F(\frac{s}{u})$
Linearity property	$N[af(t)+bg(t)]=aN[f(t)]+bN[g(t)]$
	$N[f'(t)]=\frac{s}{u}R(s,u)-\frac{f(0)}{u}$
Function derivatives	$N[f''(t)]=\frac{s^2}{u^2}R(s,u)-\frac{s}{u^2}f(0)-\frac{f'(0)}{u}$
	$N[f^n(t)]=\frac{s^n}{u^n}R(s,u)-\sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^k(0)$
First scale preserving theorem	$N[f(at)]=\frac{1}{a}R(\frac{s}{a},u)$
First shifting theorem	$N[e^{at}f(t)]=\frac{s}{s-au}R\left(\frac{us}{s-au}\right)$

2.2. The Runge-Kutta 4th order method

The Runge - Kutta 4th order (RK4) method is a numerical technique for resolving ordinary differential equations:

$$z'=g(t,z),\quad z(t_0)=z_0 \tag{2}$$

The Runge-Kutta 4th order method is based on the following

$$z_{i+1}=z_i+\tfrac{1}{6}(l_1+2l_2+2l_3+l_4)h \tag{3}$$

where knowing the value of $z=z_i$ at t_i, we find the value of $z=z_{i+1}$ at t_{i+1} and $h=t_{i+1}-t_i$, $l_1=g(t_i,z_i)$, $l_2=g\left(t_i+\frac{1}{2}h,z_i+\frac{1}{2}l_1h\right)$, $l_3=g\left(t_i+\frac{1}{2}h,z_i+\frac{1}{2}l_2h\right)$, $l_4=g(t_i+h,z_i+l_3h)$.
Let the linear second-order initial value problem be given as

$$m_0(t)z'' + m_1(t)z' + m_2(t)z = q(t), \quad z(t_0) = b_0, \quad z'(t_0) = b_1 \tag{4}$$

Define $w_1 = z$, then we have the system $w_1' = z' = w_2$, $w_1(t_0) = b_0$. If we substitute them in Equation (4), we get $w_2' = z'' = \frac{1}{m_0(t)}[q(t) - m_1(t)z'(t) - m_2(t)z(t)]$ and the following first-order equation is obtained

$$w'_2 = \frac{1}{m_0(t)}[q(t) - m_1(t)w_2 - m_2(t)w_1], \qquad w_2(t_0) = b_1 \tag{5}$$

The system is given by

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} w_2 \\ g_2(t, w_1, w_2) \end{bmatrix}, \quad \begin{bmatrix} w_1(t_0) \\ w_2(t_0 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

where $g_2(t, w_1, w_2) = \frac{1}{m_0(t)}[q(t) - m_1(t)w_2 - m_2(t)w_1]$.

Generally, we may have a system as well

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}' = \begin{bmatrix} g_1(t, z_1, z_2) \\ g_2(t, z_1, z_2) \end{bmatrix}, \quad \begin{bmatrix} z_1(t_0) \\ z(t_0) \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \tag{6}$$

In vector notation, denote $\boldsymbol{z} = [z_1 \quad z_2]^t$, $\boldsymbol{g} = [g_1 \quad g_2]^t$, $\boldsymbol{b} = [b_0 \quad b_1]^t$

Then we can create the system

$$\boldsymbol{z}' = \boldsymbol{g}(t, \boldsymbol{z}), \quad \boldsymbol{z}(t_0) = \boldsymbol{b} \tag{7}$$

By expressing the answer in vector form, the techniques developed for the solution of the first-order initial value problem in Equation (2) may be applied to the system of Equations (7), i.e., the second-order initial value problem Equation (6) (Iyengar & Jain, 2009).

3. Application of Natural transform

In this study, differential equations of first and second-order are examined. Blood glucose concentration application for the first order differential equation and circuit analysis applications for the second order differential equation are presented. These illustrations have been chosen to show how Natural transform can be applied in a wide range of contexts.

3.1. The blood glucose concentration

A person's blood glucose concentration is determined at any given time. The blood glucose concentration is modelled mathematically as below (Khidir et al., 2023; Peker & Çuha, 2023):

$$\frac{dC(t)}{dt} + kC(t) = \frac{\beta}{P}, \quad t > 0 \qquad \text{and} \quad C(0) = C_i \tag{8}$$

where *C(t)* is the blood glucose concentration at time *t*, *k* is the constant velocity of the elimination, β is the proportion of the infusion, P is the volume in which glucose is distributed, C_i is the initial concentration of glucose in the blood. We will find the concentration of glucose in the blood by using the Natural transform method. The Natural transform of *C(t)* is denoted by $C(s,u)$. Applying the Natural transform on both sides of Equation (8),

$$\frac{s}{u}C(s,u) - \frac{C_i}{u} + kC(s,u) = \frac{\beta}{P}\frac{1}{s} \tag{9}$$

is obtained. Using partial fraction decomposition, *C(s,u)* is found

$$C(s,u) = \frac{C_i}{s+ku} + \frac{\beta}{P}\left(\frac{1}{sk} - \frac{1}{k(s+ku)}\right) \tag{10}$$

Operating inverse Natural transform on both sides of Equation (10), we obtain the concentration of glucose in the blood:

$$C(t) = C_i e^{-kt} + \frac{\beta}{Pk}\left(1 - e^{-kt}\right) \quad (11)$$

The results of both obtained with the Kashuri Fundo transform by Peker and Çuha and obtained with the Anuj transform by Kumar et al. are consistent with those obtained with the Natural transform (Peker & Çuha, 2023; Kumar et al, 2021). Solving this problem greatly helps doctors to determine the patient's exact blood glucose level at any time when receiving continuous intravenous injections.

3.2. Electrical circuits

In this section, circuit analysis for the second order differential equations are presented: series RLC circuit, parallel RLC circuit and two-meshed RL circuit.

3.2.1. Series RLC circuit

In a series RLC circuit, the resistor (R), inductor (L), capacitor (C) and source are connected in series with each other. In this structure, all currents ($i(t)$) are the same. Using Kirchoff's voltage law (KVL), the following equation for the circuit of Figure 1 is obtained:

$$V_R(t) + V_L(t) + V_c(t) = V_k(t) \quad (12)$$

where $V_R(t), V_L(t), V_c(t)$ and $V_k(t)$ are the voltage of resistor, inductor, capacitor and source, respectively and defined as $V_R(t) = Ri(t)$, $V_L(t) = L\frac{di(t)}{dt}$ and $V_C(t) = \frac{1}{C}\int V(t)dt$. When the voltage formulas are substituted in Equation (12),

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = V_k(t) \quad (13)$$

is obtained. When the derivative of both sides of the equation is taken, the second order differential equation is obtained as:

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dV_k(t)}{dt} \quad (14)$$

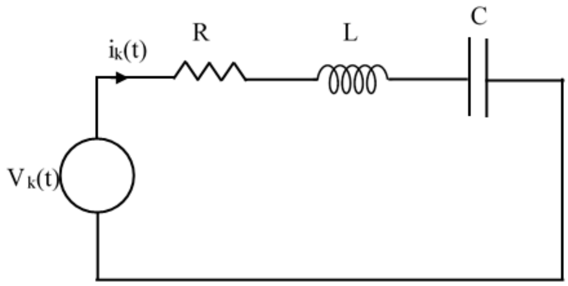

Figure 1. A series RLC circuit (Arifoğlu, 2013)

The circuit of Figure 2 models the ignition system of a car (Irwin & Nelms, 2015). V_S, C, L and R represent voltage source (battery), capacitor, inductor and resistor, which is the internal resistance of the inductor, respectively. And there is also a switch in the circuit. Initially, the resistor, inductor, capacitor are connected to the battery by switching and the capacitor charge up to voltage source. At time t=0, the switch is closed and the capacitor discharges.

We calculate $i_L(t)$ using Natural transform assuming that $V_s = 12V$, $C = \frac{1}{3.8}F$, $L = 200mH$ and $R = 4\Omega$ for the circuit of Figure 2*. The initial conditions are $i_L(0) = 0A, V_c(0) = 12\,V, i_L'(0) = 60A$. Using the initial values the following initial value problem is obtained:

*: In (Irwin & Nelms, 2015), the C value is calculated in the conditions below: it is required to be overdamped, the current (i(t)) reaches at least 1mA within 100ms after switching, and it remains above 1A between 1-1.5s. In this question, the values in (Irwin & Nelms, 2015) are used and $i_L(t)$ is calculated with Natural transform.

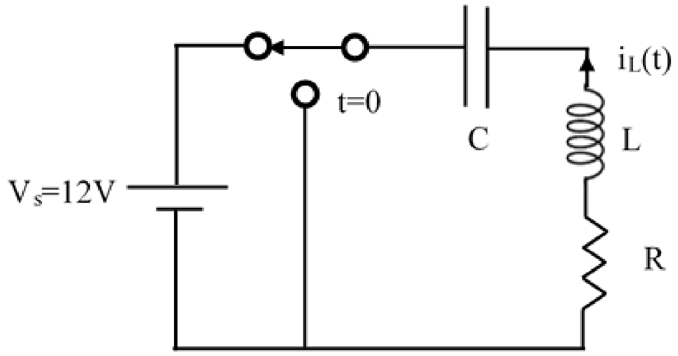

Figure 2. A circuit model of ignition system of a car (Irwin & Nelms, 2015)

$$\frac{d^2i(t)}{dt^2} + 20\frac{di(t)}{dt} + 19i(t) = 0,\ i(0) = 0A,\ i'(0) = 60A \tag{15}$$

The Natural transform of $i(t)$ is denoted by $I(s,u)$. If the Natural transform is applied to Equation (15), we will get

$$\frac{s^2}{u^2}I(s,u) - \frac{s}{u^2}i(0) - \frac{i'(0)}{u} + 20\frac{s}{u}I(s,u) - 20\frac{i(0)}{u} + 19I(s,u) = 0 \tag{16}$$

Substituting the initial values in the Equation (16), $I(s,u)$ is obtained:

$$\frac{s^2}{u^2}I(s,u) - \frac{60}{u} + 20\frac{s}{u}I(s,u) + 19I(s,u) = 0 \tag{17}$$

$$I(s,u) = \frac{60u}{s^2+2su+19u^2} \tag{18}$$

After decomposition, the inverse Naturel transform is taken and the current is obtained:

$$i(t) = \frac{-10}{3}e^{-19t} + \frac{10}{3}e^{-t} \tag{19}$$

3.2.2. Parallel RLC circuit

In a parallel RLC circuit, the resistor, inductor and capacitor are connected in parallel to each other and the source. In this configuration, while the voltage across each component is the same, the currents are different and can be found using Kirchoff's Current Law (KCL). Using KCL, the currents for the circuit of Figure 3 are written as:

$$I_R(t) + I_L(t) + I_c(t) = I_k(t) \tag{20}$$

where $I_R = \frac{V(t)}{R}$, $I_L(t) = \frac{1}{L}\int V(t)dt$ and $I_c(t) = C\frac{dV(t)}{dt}$. Substituting these formulas in Equation (20)

$$\frac{V(t)}{R} + \frac{1}{L}\int V(t)dt + C\frac{dV(t)}{dt} = I_k(t) \tag{21}$$

is obtained. Taking the derivative of both sides, the second-order differential equation is obtained:

$$\frac{d^2V(t)}{dt^2} + \frac{1}{RC}\frac{dV(t)}{dt} + \frac{1}{LC}V(t) = \frac{1}{C}\frac{di_k(t)}{dt} \tag{22}$$

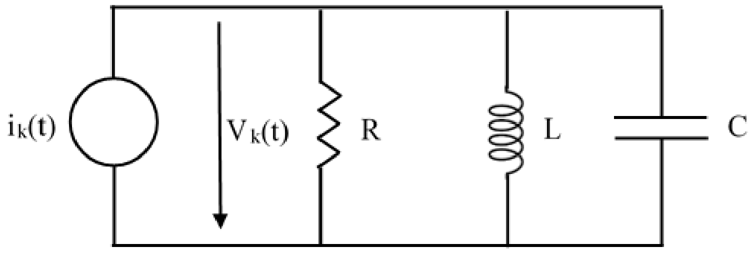

Figure 3. A Parallel RLC circuit (Arifoğlu, 2013)

We determine $V(t)$ in the circuit of Figure 3 using Natural transform assuming that $i_k(t) = 5e^{-2t}$, $R = 4\Omega$, $L = 8H$, $C = \frac{1}{8}F$, $V'(0) = 0$ and $V(0)$=0 (Arifoğlu, 2013). Using the initial values the following initial value problem is obtained:

$$\frac{d^2V(t)}{dt^2} + 2\frac{dV(t)}{dt} + V(t) = -80e^{-2t},\ V(0) = 0,\ V'(0) = 0 \tag{23}$$

$V(s,u)$ denotes the Natural transform of $V(t)$. Taking the Natural transform of Equation (23), then we will reach

$$\frac{s^2}{u^2}V(s,u) - \frac{s}{u^2}V(0) - \frac{V'(0)}{u} + 2\frac{s}{u}V(s,u) - 2\frac{V(0)}{u} + V(s,u) = \frac{-80}{s+2u} \tag{24}$$

Substituting the initial values in the equation, $V(s,u)$ is obtained:

$$V(s,u) = \frac{-80\ u^2}{(s+2u)(s^2+2su+u^2)} \tag{25}$$

Taking inverse Natural transform, we get

$$V(t) = -80e^{-2t} + 80e^{-t} - 80te^{-t} \tag{26}$$

3.2.3. Two-mesh circuit

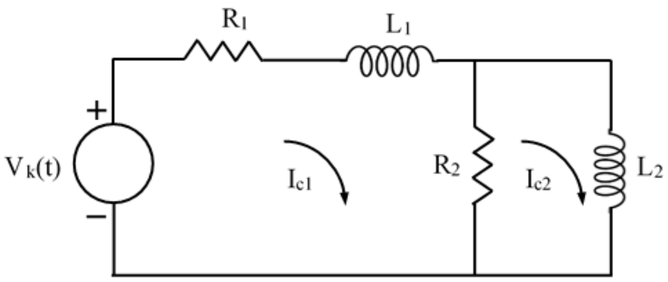

Figure 4. A two-mesh circuit (Arifoğlu, 2013)

We find the current of inductor (I_{L2}) in the circuit of Figure 4 assuming that $V_K(t) = 100\sin(3t)$, $R_1 = 4\Omega$, $R_2 = 2\Omega$, $L_1 = 1H$ and $L_2 = 3H$. The initial conditions of $I_{c1}(t=0) = 0, I_{c2}(t=0) = 0$ (Arifoğlu, 2013). Firstly, applying KVL to each mesh, the mesh equations are written as:

$$(R_1 + R_2)i_{c1}(t) + L_1\frac{di_{c1}(t)}{dt} - R_2 i_{c2}(t) = V_K(t) \tag{27}$$

$$R_2\big(i_{c2}(t) - i_{c1}(t)\big) + L_2\frac{di_{c2}(t)}{dt} = 0 \tag{28}$$

After writing the values in the Equations (27)-(28),

$$6i_{c1}(t) + \frac{di_{c1}(t)}{dt} - 2i_{c2}(t) = 100sin3t \tag{29}$$

$$2\big(i_{c2}(t) - i_{c1}(t)\big) + 3\frac{di_{c2}(t)}{dt} = 0 \tag{30}$$

are obtained. In Equation (30), $i_{c1}(t)$ is written in terms of $i_{c2}(t)$ and substituted in Equation (29) to obtain second order initial value problem:

$$\frac{d^2 i_{c2}(t)}{dt^2} + \frac{20}{3}\frac{di_{c2}(t)}{dt} + \frac{8}{3} i_{c2}(t) = \frac{200}{3} sin3t, \;\; i_{c2}(0) = 0, \;\; i_{c2}{}'(0) = 0 \tag{31}$$

The Natural transform of $i_{c2}(t)$ is denoted by $I(s,u)$. If the Natural transform is applied to Equation (31), we get

$$\frac{s^2}{u^2} I(s,u) - \frac{s}{u^2} i(0) - \frac{i'(0)}{u} + \frac{20}{3}\left(\frac{s}{u} I(s,u) - \frac{i(0)}{u}\right) + \frac{8}{3} I(s,u) = \frac{200}{3}\frac{3u}{s^2+9u^2} \tag{32}$$

Substituting the initial values in the equation, $I(s,u)$ is written as below:

$$I(s,u) = \frac{600u^3}{(s^2+9u^2)(3s^2+20su+8u^2)} \tag{33}$$

After the inverse Naturel transform is taken, the current is obtained:

$$i_{c2}(t) = 3.7475e^{-0.4274t} - 0.7180e^{-6.2393t} - 0.9594\sin(3t) - 3.0295\cos(3t) \tag{34}$$

The solutions of Equation (8), Equation (15), Equation (23) and Equation (31) obtained with Natural transform correspond exactly to the analytical solutions.

4. Application of Runge-Kutta numerical method

The Ruge-Kutta method is used to calculate the blood glucose concentration and circuit's voltage and current. The results of the methods are given in Figure 5-8 and in Table 3-6, respectively. The number of samples obtained with the Runge-Kutta method is inversely proportional to the step size. Because of many samples, Table 3-6 shows the results at particular times. The error values in the tables are calculated by taking multiple numbers after the comma.

Firstly, in Equation (11) the parameters $C_i = C(0) = 320, \beta = 280, P = 45, k = 0.058$ (Khidir et al., 2023) were used to calculate the blood glucose concentration and $C(t) = 320e^{-0.058t} + \frac{280}{45\ 0.058}\left(1 - e^{-0.058t}\right)$ is obtained. Using the same parameters, the RK4 method was applied to Equation (8) for 0≤t≤90s and h=1. Blood glucose concentration was calculated for each cycle. Before determining the step size, blood glucose concentration was calculated for step size values 0.1, 0.25, 0.5 and 1. For the step size selected (h=1), it has been observed that the exact solution and the numerical solution converge. Choosing the step size differently caused the number of cycles to change. The solution of Natural transform (exact solution), the numerical solution (RK4 solution) and error for the concentration are given in Table 3. The results obtained graphically are also supported with the results given in Table 3. Figure 5 shows the blood glucose concentration (blue: result obtained with Natural transform, red: result obtained with the Runge-Kutta method).

Table 3. Error of the blood glucose concentration in RK4 method

t	NT solution	RK4 solution	Absolute Error	t	NT solution	RK4 solution	Absolute Error
0	320	320	0	50	118.9842	118.9843	3.36E-06
5	266.4505	266.4506	4.57E-06	55	116.0378	116.0378	2.77E-06
10	226.3814	226.3815	6.84E-06	60	113.8331	113.8331	2.26E-06
15	196.3992	196.3992	7.67E-06	65	112.1833	112.1833	1.83E-06
20	173.9646	173.9646	7.66E-06	70	110.9489	110.9489	1.47E-06
25	157.1776	157.1776	7.16E-06	75	110.0252	110.0252	1.18E-06
30	144.6164	144.6165	6.43E-06	80	109.3341	109.3341	9.43E-07
35	135.2174	135.2174	5.61E-06	85	108.8169	108.8169	7.5E-07
40	128.1845	128.1845	4.8E-06	90	108.4299	108.4299	5.94E-07
45	122.922	122.922	4.04E-06				

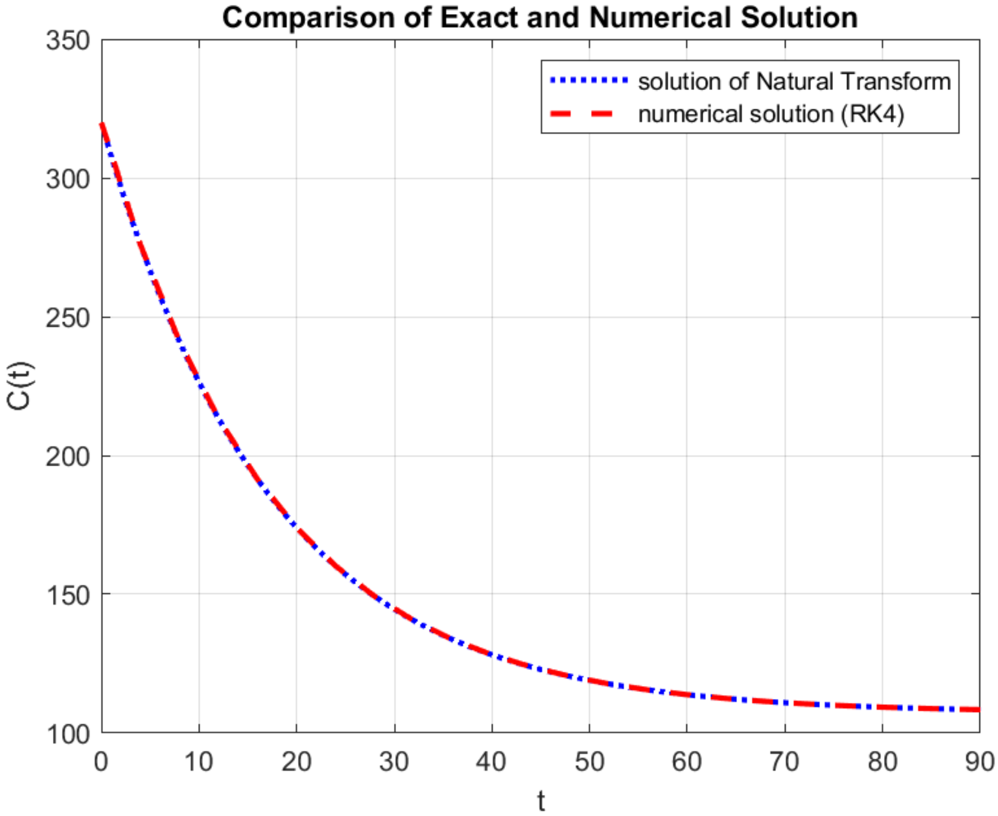

Figure 5. Exact and numerical solutions (RK4) of the blood glucose concentration
$(C_i = 320, \beta = 280, P = 45, k = 0.058, h = 1)$

Secondly, taking the initial conditions (i(0)=0, i' (0)=60) in Equation (15), comparison of the analytical solution and the numerical solution of the current in Figure 2 were given in Table 4 when RK4 was applied for h = 0.01 and 0≤t≤5s. The current in the series RLC circuit is given in Figure 6. In (Irwin & Nelms, 2015), it is aimed that i(t) reaches at least 1mA within 100ms after switching, and it remains above 1A between 1-1.5s. All are seen in the Figure 6 and Table 4. Also it can be seen that as t goes to infinity, the current goes to zero. For the RK4 method, the results were obtained by changing the step size as 0.01, 0.05, 0.075, 0.1, respectively and step size was selected as 0.01. The reason for this is that at other values of the step size, the solutions created by the RK4 method for t < 0.5s are not the same with the results obtained with Natural transform and exact solution.

Table 4. Error of the current in the circuit of Figure 2 in RK4 method

t	NT solution	RK4 solution	Absolute Error	t	NT solution	RK4 solution	Absolute Error
0	0	0	0	2.75	0.213093	0.213093	4.92E-11
0.25	2.567164	2.567162	1.7E-06	3	0.165957	0.165957	4.18E-11
0.5	2.021519	2.021519	3E-08	3.25	0.129247	0.129247	3.53E-11
0.75	1.574553	1.574553	2.9E-10	3.5	0.100658	0.100658	2.96E-11
1	1.226265	1.226265	9.85E-11	3.75	0.078392	0.078392	2.47E-11
1.25	0.955016	0.955016	1E-10	4	0.061052	0.061052	2.05E-11
1.5	0.743767	0.743767	9.37E-11	4.25	0.047547	0.047547	1.7E-11
1.75	0.579246	0.579246	8.52E-11	4.5	0.03703	0.03703	1.4E-11
2	0.451118	0.451118	7.58E-11	4.75	0.028839	0.028839	1.15E-11
2.25	0.351331	0.351331	6.64E-11	5	0.02246	0.02246	9.44E-12
2.5	0.273617	0.273617	5.75E-11				

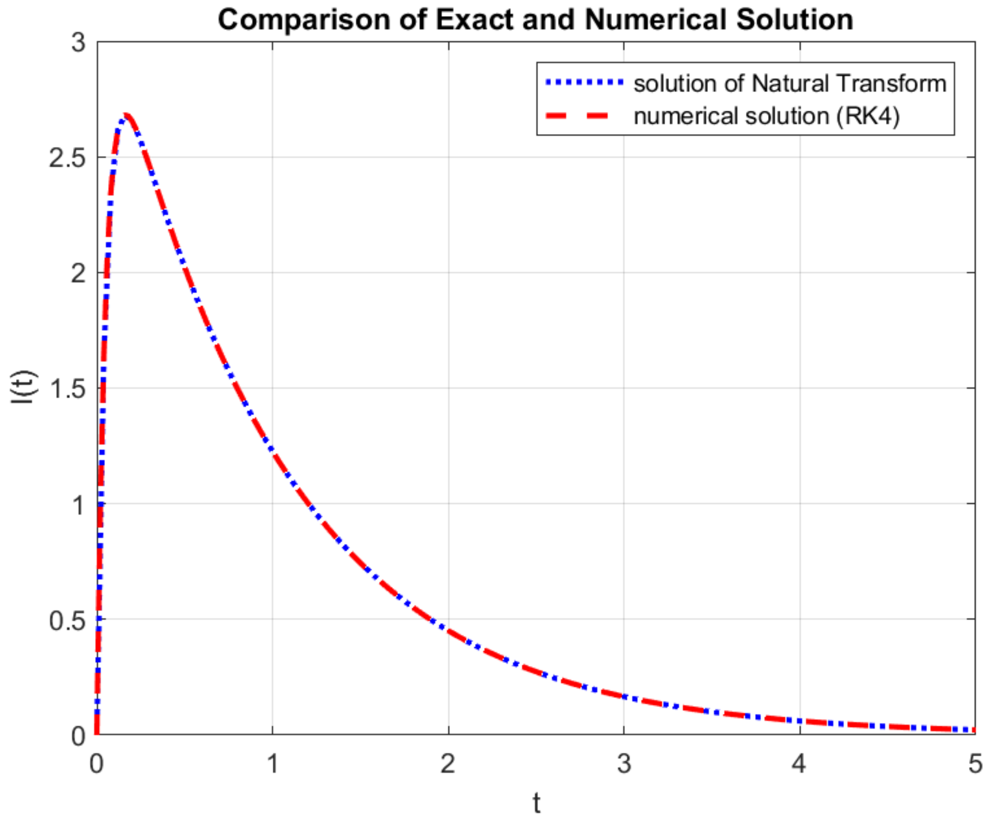

Figure 6. Exact and numerical solutions (RK4) of the current in the circuit of Figure 2

Thirdly, with initial conditions $V(0) = 0,\ V'(0) = 0$, Equation (23) was examined. The analytical solution and the numerical solution of the voltage in the parallel RLC circuit were given in Table 5 when RK4 was applied for h = 0.01 and 0≤t≤10s. The voltage in the parallel RLC circuit also goes to zero while t goes to infinity (Figure 7). When the step size is greater than 1, the solution obtained with the RK4 method does not converge to the results obtained with Natural transform and the exact solution.

Table 5. Error of the voltage in the circuit of Figure 3 in RK4 method

t	NT solution	RK4 solution	Absolute Error	t	NT solution	RK4 solution	Absolute Error
0	0	0	0	5.5	-1.47257	-1.47257	1.11E-10
0.5	-5.16913	-5.16913	1.2E-08	6	-0.99199	-0.99199	5.15E-11
1	-10.8268	-10.8268	8.3E-09	6.5	-0.66169	-0.66169	1.68E-11
1.5	-12.9082	-12.9082	3.8E-09	7	-0.43777	-0.43777	1.6E-12
2	-12.2921	-12.2921	1E-09	7.5	-0.28763	-0.28763	9.9E-12
2.5	-10.3892	-10.3892	2.52E-10	8	-0.18787	-0.18787	1.3E-11
3	-8.16423	-8.16423	6.47E-10	8.5	-0.12208	-0.12208	1.2E-11
3.5	-6.11243	-6.11243	6.4E-10	9	-0.07898	-0.07898	1.1E-11
4	-4.42259	-4.42259	4.95E-10	9.5	-0.0509	-0.0509	8.7E-12
4.5	-3.12039	-3.12039	3.34E-10	10	-0.03269	-0.03269	6.8E-12
5	-2.15978	-2.15978	2.03E-10				

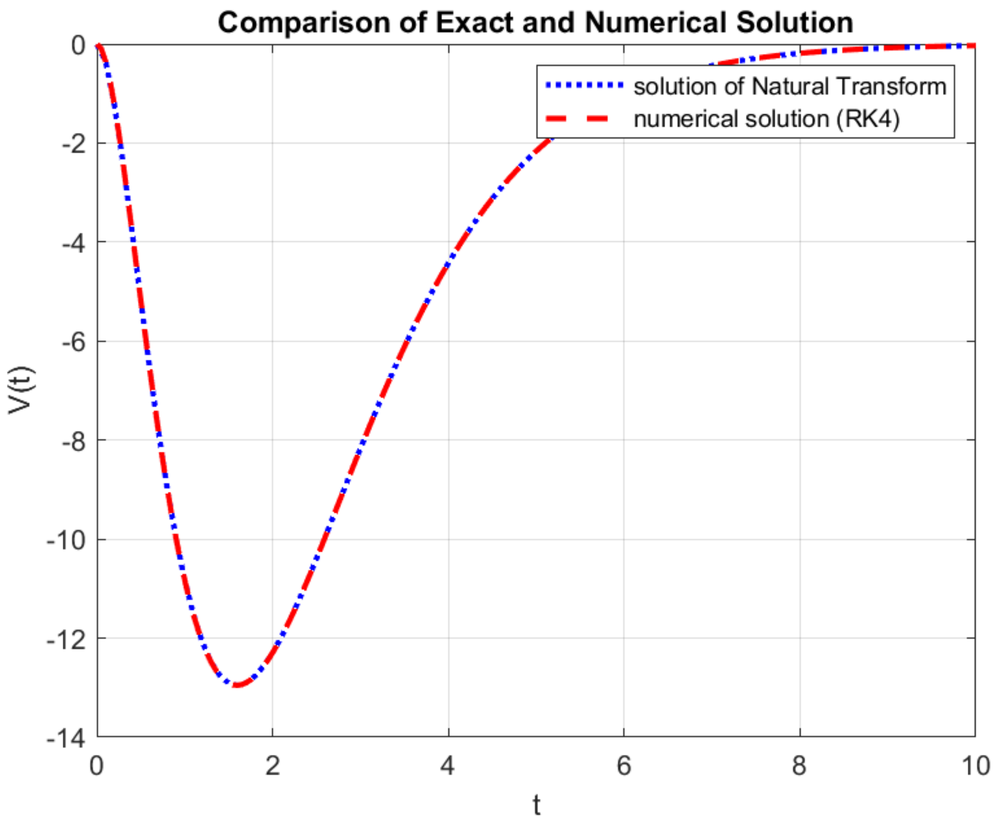

Figure 7. Exact and numerical solutions (RK4) of the voltage in the circuit of Figure 3

Finally, considering Equation (31) with initial conditions $i_{c2}(0) = 0,\ i'_{c2}(0) = 0$, the analytical solution and the numerical solution of the current around mesh 2 in Figure 4 were given in Table 6 when RK4 was applied for h = 0.01 and 0≤t≤20s. Figure 8 shows the current around mesh 2 of the two-mesh circuit. It is seen that the steady-state part of the Equation (34) is sinusoidal. When the step size is larger than 0.2, the results obtained with the RK4 method diverge from the results obtained with the Natural Transform. The step size (h) chosen for Runge-Kutta method for electrical circuits are 0.01.

Figure 5-8 demonstrate that the numerical method approximates the analytical solution well for small step sizes. As the step size increases, deviations between the two solutions may become more noticeable. Also, Table 3-6 demonstrate that the error is rather small.

Table 6. Error of the current in the circuit of Figure 4 in RK4 method

t	NT solution	RK4 solution	Absolute Error	t	NT solution	RK4 solution	Absolute Error
0	0	0	0	11	-0.88501	-0.88501	4.03E-08
1	5.306584	5.306584	4.29E-08	12	1.361337	1.361336	3.5E-08
2	-1.04674	-1.04674	4.7E-08	13	-1.71795	-1.71795	2.82E-08
3	3.404598	3.404598	5.14E-08	14	2.100482	2.100482	2.1E-08
4	-1.36365	-1.36365	5.3E-08	15	-2.40164	-2.40164	1.39E-08
5	2.119886	2.119886	5.5E-08	16	2.680386	2.680386	6.2E-09
6	-0.99156	-0.99156	5.5E-08	17	-2.88875	-2.88875	1.6E-09
7	1.044832	1.044832	5.43E-08	18	3.050211	3.050211	9.36E-09
8	-0.29361	-0.29361	5.2E-08	19	-3.1435	-3.1435	1.7E-08
9	0.04756	0.04756	4.92E-08	20	3.178519	3.178519	2.42E-08
10	0.532749	0.532749	4.5E-08				

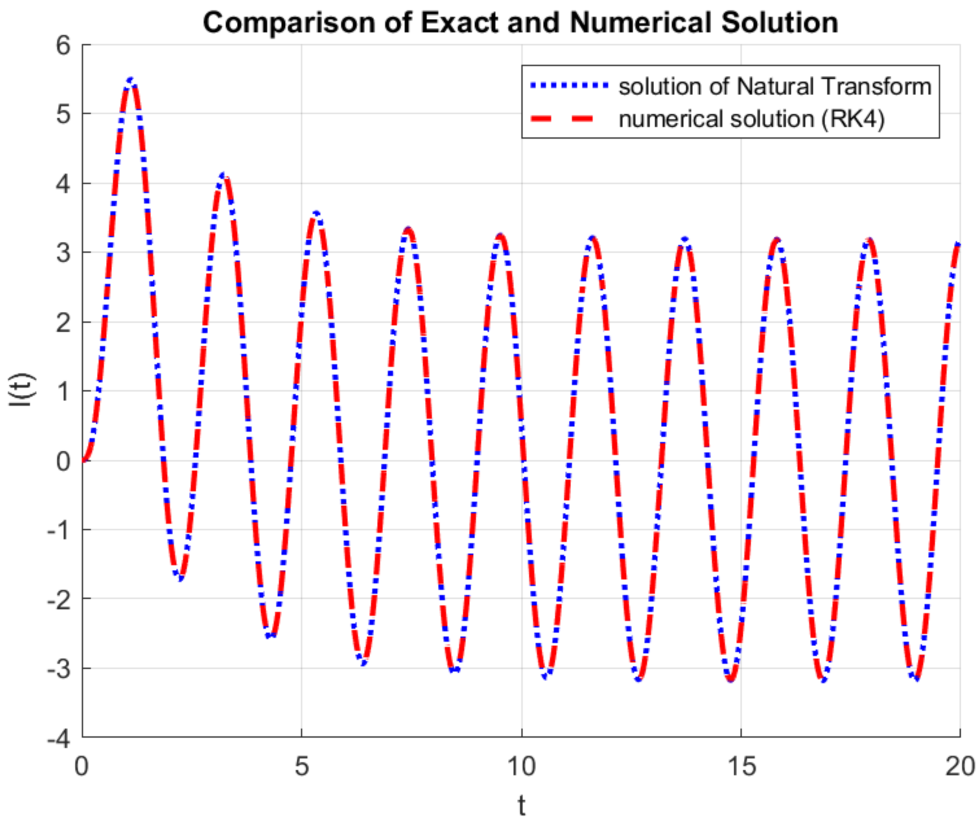

Figure 8. Exact and numerical solutions (RK4) of the current in the circuit of Figure 4

5. Discussion and conclusions

This study demonstrates a comparative analysis between the analytical solution of differential equations considered in section 3 using the Natural Transform and the numerical solution obtained through the fourth-order Runge-Kutta method. Using the Runge Kutta method, both blood glucose concentration and currents and voltage of electrical circuits were calculated for different values of the step size. The convergence of the results calculated with Runge Kutta method and obtained with Natural transform was obtained for these step size (h) which are 1 and 0.01, respectively.

The Natural Transform provided an exact solution, serving as a benchmark for evaluating the performance of the numerical approach. The Runge-Kutta method successfully approximated the analytical solution with high accuracy, especially when a small step size was used. Graphical comparisons showed that both solutions align closely across the solution domain, with only minor discrepancies due to numerical approximation errors.

Overall, the study highlights the strength of combining analytical and numerical approaches, with the Natural Transform offering insight into the exact behavior of the system and the Runge-Kutta method providing a robust tool for approximating solutions when analytical methods are infeasible. Ongoing research and development in these transforms continue to expand their utility, offering new solutions to complex scientific and engineering problems. In future studies, the results obtained with integral transform can be compared with numerical methods for different applications.

Acknowledgement

We sincerely thank the referees and journal editors for their careful review and valuable contributions.

Author contribution

All authors had the same contribution. All authors read and approved the final manuscript.

Declaration of ethical code

The authors of this article declare that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest

The authors declare that they have no conflict of interest.

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