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A SURVEY OF MULTI-ECHELON INVENTORY MODELS(*)

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The purpose of this paper is to summarise the more important techniques used to analyse the multi-echelon inventory control problem. An introductory section defines the term "multiechelon" and establishes the kinds of problems involving multi-echelon considerations. Subsequent two sections provide review of work in multi-echelon inventory theory with respect to selections from the literature. Deterministic-Stochastic dichotomy, is used as a distinguishing feature to categorise models. The penultimate section discusses the recent work on the subject and the last section contains conclusions and a direction for further research.

1. INTRODUCTION [1],[2]

In accordance with systems theory if a complex system consists of a finite number of interacting and explicitly recognisable subsystems, there is a need to preciselv define vertical arrangements between the subsystems. The vertical position of subsystems in the system is defined either in reference to priority of action or in the sense that the system parts on lower-level positions are subsystems of higher-level parts. A level in such a system is called an echelon,

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(1) Petrovic, R., A. Senborn, and M. Vujosevic, Hierarchical Spare Parts Inventory Systems, Studies in Production and Engineering Economics, No.5, Elsevier Science Publishers B.V.: Amsterdam, 1986.

(2) Schwarz, L.B., (ed.), Multi-Level Production/Inventory Control Systems. Theory and Practice, Studies in the Management Sciences, Vol. 16, North-Holland: Amsterdam, 1981.

and represents an organisational level. Generally, there are many subsystems at a given level, except the highest one where, as a rule, only a single subsystem exists.

The concepts of echelons and organisational hierarchy can be defined in a precise formal manner by using abstract setting. This can be done for any system in whatever context or discipline. Here, it is done by employing the terms and words from inventory systems. Let an inventory system S have a finite number of subinventories S_n, $n \varepsilon N$ where N is a finite set of indices. If α is a strict partial ordering of N, then (S,α) is a hierarchy of subinventories. When the relation α is defined as $n \alpha n'$; $n, n' \varepsilon N$ if and only if Sn', is a subinventory of S_n, S is said to be a multi-echelon system, and (S,α) is a multi-echelon hierarchy.

First-echelon subinventories are the minimal units of S. The family $S^1=\{S_n: n \varepsilon N_1\}$, where $N_1= \{n : n$ is a minimal element of $N\}$, is the first echelon. Going further in the same way, second-, third-echelon etc., subinventories are defined. The m-th echelon subinventories are the minimal units of S when all lower echelons are omitted, i.e., the family $S^m=\{S_n \; n \varepsilon Nm\}$, where $N_m=\{n :$ is a minimal element of $N-(N_1 \cup N_2 \cup ... \cup N_{m-1})\}$ is the m-th echelon.

A multi-echelon or organisational hierarchy is a very common type of hierarchy. In reality, organisational hierarchy exists in any complex ,system. An attribute specific for multi-echelon systems lies in the partially conflicting goals and objectives between decision problems on different echelons. These partial conflicts are not only, a result of the composition of the multi-echelons system, but are also necessary for efficient functioning of the overall system. A block diagram of such a system with three echelons is given in Fig. 1. The arrows indicate the pattern for the flow of goods through the system. In the system shown, customer demands occur only at the stocking points in echelon 1. Echelon n has its stocks replenished by shipments from the echelon (n+1). A multi-echelon inventory system can also be portrayed as a directed network wherein the nodes represent the various stocking points in the system and the linkages represent flows of goods. If the network has at most one incoming link for each node and flows are acyclic (no loops in the network) it is called an *"arborescence"* or inverted tree structure. More complex interconnected systems of facilities can exist however, most of the work in multi-echelon inventory theory, has been confined to arborescence structures.[3]

The problem is to determine what inventories, if any, should be maintained at the various stocking points, and what the operating doctrine should be for controlling the stocks at all the stocking points. Clark has defined the multi-echelon inventory control problem in his informal survey of multi-echelon inventory theory (Ref 3):

> "...viewed in terms of a network of activities, with external demands occurring at some or all of them, the basic multi-activity inventory control problem for a given product is one of establishing rules or policies which, if followed, cause flows of the product through the network as functions of time and which satisfy a prescribed performance objective, such as, minimising expected costs or meeting a prescribed level of customer service. The set of such policies, for any given system, usually contain ordering policies (resupply, procurement production, repair), which prescribe amounts over time that each activity orders from its supplier(s), and supply policies (issuing, delivery, distribution, allocation) which control amounts over time that each activity ships to those activities designated as its customers. A common situation which warrants this distinction is one where there is insufficient stock at a particular supplier to fill all the orders it receives and some kind of rationing, or supply policy, is thereby required. To solve this "inventory control problem, a variety of models have been formulated which are these distinguishing features are expressed by the following dichotomies: Deterministic-Stochastic, Single Product-Multi Product Stationary- Non stationary, Continuous Review-Periodic Review, Consumable Product- Reparable Product, Backlog- No Backlog"

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(3) Clark, a.J., "An Informal Survey of Multi-Echelon Inventory Theory", Naval Research Logistics Quarterly, Vol. 19, 1972, pp. 621-650.

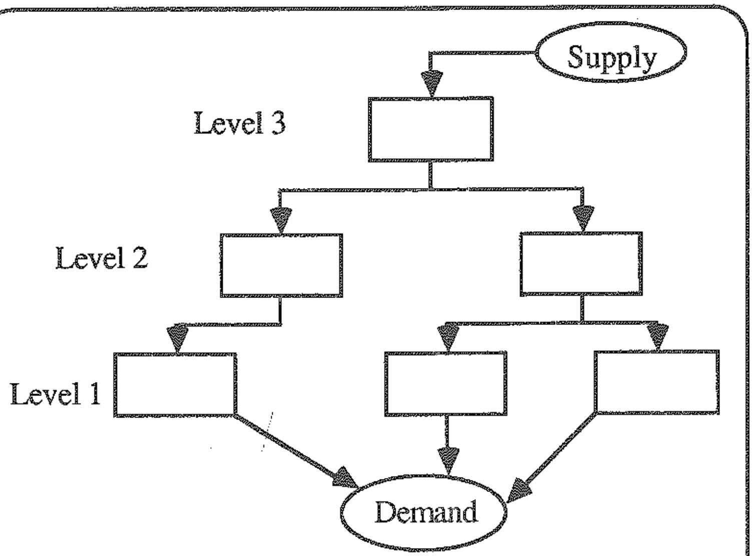

Fig.1 Multi-Echelon System

II. DETERMINISTIC MULTI-ECHELON MODELS

One of the early investigations of the deterministic multi-echelon problem was by Evans.(4) To overcome the limitations of this work, Zangwill(5) analysed a deterministic single-activity, multi-period production and inventory model which has led to the development of a multiactivity model. The single-activity model has concave production costs and piecewise concave inventory costs. Model permits backlogging which violates concavity assumptions of the previous works. Instead Zangwill considers piecewise concave cost functions to find the form of the minimum cost production schedule. An efficient dynamic programming algorithm to calculate the minimum cost schedule is presented. In a later paper(6) , Zangwill analysed the first significant deterministic multi-product, multi-activity, multi-period production and inventory model that is a linking together of the single facility models developed in Ref.[5]. The

(4) Evans, G.W., II, "A Transportation and Production Model", Naval Research Logistics Quarterly Vol.5, 1958, pp. 137-154.
(5) Zangwill, W.I., "A Deterministic Multi-Period Production Scheduling Model with Backlogging, Management Science, Vol. 13, No. 1, 1966, pp. 1.05-119.
(6) Zangwill, W.I., "A Deterministic Multifacility Production and Inventory Model", Operations Research,Vol.14,No.3,pp.486-507.

linking is arranged to form an acyclic network of the facilities. Each facility can receive inputs from either raw material or lower numbered facilities and supply only higher numbered facilities or market demand for its own product. The object is to determine a production schedule that specifies how much each facility in the network should produce so that the total cost is minimised. The notation is given below:

- r^j_i, $r^j_i \geq 0$, is the market requirements for facility j's (j= 1,....,N) production in the period i, (i=i, ...n) where n is the number of periods under consideration and there are N facilities. r^j_i is known in advance. $r^j=(r^j_1, r^j_2,...... r^j_n)$ represents total market requirements for facility j,

- $x^j= (x^j_1,........,x^j_n)$is a production schedule for facility j, where x^j_i. $x^j_i \geq 0$, is the production completed in period i facility j,

- a^{jh}, $a^{jh} \geq o$, is the number of units of facility j's production required to produce one unit of facility h's product. Since the model is an acyclic network $a^{jh}=0$ for $h \leq j$,

- λ_j is a non negative integer that represents the number of periods lag from the start of production in facility j until the completion of production.
- y^j_i is the total demand on facility j in period i, $y^j_i=r^j_i+\Sigma^N{}_{h = j+1} (\alpha^{jh}x^h{}_{i+\lambda_h})$, $y^j=(y^j_1,y^j_2,......,y^j_n)$ represents the total demand for facility j,

- α_j is a non negative integer denoting the number of periods of backlog permitted for facility j,

- I^j_i is the inventory at the, end of period i in facility j, $I^j_i=\Sigma^i{}_{h=1}(x^j_h-y^j_h)$ and.$I^j_i \geq -\Sigma^i{}_{h=i-\alpha_j+1}(y^j_h)$,

- $Z=(x^1, x^2,.....,x^N)=(X^1_1, X^2_2,....X^1_n, X^2_1, x^2_2,.....X^2_n,...., x^N_1,...x^N_n)$ is a schedule vector for the entire networks, and $Z^j=(x^J, x^{j+i},....x^N)$ is a partial production vector,

- P(z) is a concave function of the schedule vector z,

- $H^{Ji}(I^J_i)$ is an inventory cost function, $H^ji(Iji)=H^ji(Z)$,

• F(z) is total cost function. $F(z)=P(z)+=\sum^{N}_{j=1}\sum^{N}_{i=1}(H^{j}i(z))$ Zangwill shows that the total cost function is piecewise concave.

Zangwill states the entire model as follows:

Minimise

$$F(z) = P(z) + \sum^{N}j=1\sum^{n}i = 1(H^{j}i(z))$$

Subject To

$I^{j}i=\sum^{i}h=1(x^{j}h-y^{j}h)$ (i=1,......n)
$Ji\geq- \sum^{i}h=i-\alpha j+1(y^{j}h)$
$y^{j}i=r^{j}i+ \sum^{N}h=j+1(\alpha^{jh}x^{h}i+\lambda h)$ (j=1,.....N)
$I^{j}_{n}=0$
$x^{j}i\geq O$

The total cost function is shown to be concave on certain bounded polyhedral sets called basic sets. Then, by the theory of concave functions, the total cost considered as a function on a particular basic set is minimised on that set at an extreme point of that set. The union of all basic sets is proven to be the set of all feasible production schedules. The total cost function now considered as a function of all feasible production schedules must be minimised on some basic set, and hence at an extreme point of some basic set. Defining the dominant set as the set of all extreme points of all basic sets, an optimal production schedule must be in the dominant set. The principal result of the paper characterises the dominant set. For the two special cases of "series" and "parallel" networks efficient dynamic programming algorithms are developed that search the dominant set for the optimal production schedule.

In a later work Zangwill[7] analysed the multi-echelon system as a dynamic economic lot-size system with no backlogging on demand permitted. In particular, it is shown that the multiechelon structure can be represented as a single source network and can thereby be analysed by applying the theory of concave cost networks. The notation for the model is:

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(7) Zangwill, W.I., "A Backlogging Model and a Multi-Echelon Model of Dynamic Economic Lot Size Production System-A Network Approach", Management Science, Vol. 15, No.9, 1969, pp. 506-527.

• r_i, $r_i \geq 0$, is the market demand for the finished product in period i, i=1....n

• X_{ij} represents the production in period i of facility j, i=1....n, j=1....m

• I_{ij} is the inventory stored in facility j at the end of period $I_{ij} \geq 0$

$I_{i-1,j} - I_{ij} + x_{ij} - x_{i,j+1} = 0$ j=1,.....,m-1 i=1,......n

$I_{i-1,m} - I_{im} + x_{im} = r_i$ i=1....,n

$I_{0j} = I_{nj} = 0$ for all j

• $P_{ij}(x_{ij})$ is the cost of producing x_{ij} units

• $H_{ij}(I_{ij})$ is the cost of holding I_{ij} units in stock, $H_{ij}(I_{ij}) = H_{ij}(x_{ij})$

Zangwill states the entire model as follows:

Minimize

$$\Sigma_{ij}\{P_{ij}(x_{ij}) + H_{ij}(x_{ij})\}$$

Subject To

$$\Sigma^n{}_{i=1}(x_{i1}) = \Sigma^n{}_{i=1}(r_i)$$

$I_{i-1,j} - I_{ij} + x_{ij} - x_{i,j+1} = 0$	j=1,......,m-1	i=1,.......,n
$I_{i-1,m} - I_{im} + x_{im} = r_i$	i=1,......,n	
$I_{oj} = I_{nj} = 0$	for all j	
$I_{ij} \geq 0$, $x_{ij} \geq 0$	for all and j	

An equivalent network representation of the mathematical model is given. Under the concavity assumptions on costs, there exists an optimal schedule which is an extreme flow in the associated single-source network. An extreme flow is an extreme point of the convex set of feasible solutions for the problem and as Zangwill previously demonstrated has the property that any node in the network can have at most one positive input. Exploiting these result, Zangwill presents an extremely efficient algorithm that is superior the one in Ref.[6].

It is pointed out by Veinott[8] that some network models, such as the one in Ref.[7], are equivalent to transhipment Leontief substitution systems. For these models the charicterisation of the extreme flow follows alternatively from the characterisation of the extreme points of the system's solution set. Zangwill in Ref.[7]. has shown that a number of existing qualitative results on deterministic inventory models with concave costs can be deduced immediately from the characterisation of extreme flows in networks having exactly one source. Recognising the network interpretation of the deterministic inventory problem and exploiting the equivalency of these network models and transhipment Leontief substitutions systems, Veinott[9], expands upon the formulation of the problem as a Leontief substitution model with concave costs and shows that the solution algorithm developed by Zangwill extends to this case, but that the amount of computation, depending linearly upon the number of wholesale facilities but to the fourth power of the number of time periods, can still be extensive. With rather severe assumptions about the cost functions, Veinott presents a simpler and more efficient solution algorithm for the general arborescence model.

The other deterministic multi-echelon system control models are developed by the following researchers. Love[10] shows that if, in addition to concavity assumptions, per-unit ordering costs are non increasing over time for each activity and per-unit holding costs for each activity are always greater than or equal to those for the next activity in the series structure, then there exists a nested extreme optimal solution. He defines a nested production schedule as being one where if any activity produces in a given time period, then so does the next facility in the series structure. This result is exploited to develop a more efficient solution algorithm. A decomposition

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(8) Veinott, A.F., Jr., "Extreme Points of Leontief Substitution Systems", Linear Algebra and Its .Applications, Vol. 1, 1968, pp. 181-194.Veinott., A.F., Jr. "Minimum Concave-Cost Solution of Leontief Substitution Models of Multi-Facility Inventory Svstems", Operations Research, Vol. 17,1969, pp.262-291.
(9) Veinott., A.F., Jr. "Minimum Concave-Cost Solution of Leontief Substitution Models of Multi-Facility Inventory Svstems", Operations Research, Vol. 17,1969, pp.262-291.
(10) Love, S-.F., "Dynamic Deterministic Production and Inventory Models with Piecewise Concave Costs", Stanford University, Department of Operations Research, Tecnical Report No.3, Stanford, Calif, 1968.

algorithm, which is computationally feasible for arboresconce structures that were previously too large to solve, is developed by Kalymon[11]. Von Lanzenauer[12] constructed a mixed bivalent linear programming model where all costs are assumed linear with set-up costs for production for each activity and product. For each product a sequence of the activities is specified to indicate the technological ordering of production stages for the product. Each activity can process only one product in each time period. The problem is to determine the sequence of production at all activities as well as the lot-size and number of lots for each product such that total costs are minimised. Crowston and Wagner[13] examined complex systems involve an assembly structure where each intermediate facility has exactly one successor but possibly several predecessors. Their first model is based on dynamic programming where, through efficient sequencing, computational savings can be obtained conpared to complete enumeration. Their second model uses branch and bound approach, where the subproblems are solved by dynamic programming. Lambrecht[14] shows that an optimal production-inventory schedule has the property that for each facility, if there is production in period i, then the incoming inventory must be zero; and conversely, if the incoming inventory in period i is positive, then the production must be zero given that in a basic feasible solution the vectors representing the coefficients of the basic variables are linearly independent, and that a basic feasible solution cannot be written as a convex combination of two non-basic feasible solutions. Afentakis et al.[15] presents a new formulation of the lot-sizing problem in multi stage assembly systems which leads to an effective optimisation algorithm for the problem. The problem is reformulated in terms of *"echelon stock"* which simplifies its decomposition by a Lagrangean relaxation method. A branch and

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(11) Kalymon, B.A., "A Decomposition Algorithm for Arborescence Inventory Systems", University of California, Western Management Science Institute, Working Paper No. 167, Los Angeles, Calif., 1970.

(12)Von Lanzenauer, C.H., "A Production Scheduling Model by Bivalent Linear Programming, Management Science, Vol. 17, 1970, pp. 105-111.

(13) Crowston W.B., and M.H. Wagner, " Dynamic Lot Size Models for Multi-Stage Assembly Systems", Management Science, Vol.2O, No. 1, 1973, pp. 14-21.

(14) Lambrecht, M.R., "Capacity Constrained Multi-Facility Dynamic Lot-Size Problem", Unpublished Doctoral Dissertation, Katholieke Universiteit Leuven, 1976.

(15) Afentakis, P., B. Gavish and U. Karmarkar, "Computationally Efficient Optimal Solutions to the LotSizing Problem in Multistage Assembly Systems", Management Science, Vol.30, No.2,1984, pp.222-239.

bound algorithm which uses the bounds obtained by the relaxation was developed and tested.

As a result of the widespread interest in the deterministic dynamic production and inventory models many heuristics have been developed. Some of the most popular are WagnerWhitin[16] , Part-Period Balancing,[17] Silver-Meal[18], and Least Unit Cost[19] These heuristics use single level information in order to determine scheduling pattern. Due to the complexity of general multi-stage problem, many heuristic procedures are proposed for this structure. The most common form of heuristic is to consider the stages sequentially, starting with some single-stage procedure which may itself be a heuristic. Examples of such multi-stage heuristics are given in McLaren[20], McLaren and Whybark -Order Moment Heuristic[21] , Bigg et al. [22], and Blackburn and Millen[23] , Carlson et al. [24] For an N stage system, the amount of work necessary for these heuristics is comparable to that needed for solving N single-stage problems. All of the reported work has been restricted to assembly systems. Graves[25] considered the lot-sizing problem in a general multi-stage, discrete-time inventory system. A heuristic,

(16) Wagner, M.H., and T.M. Wbtin, "Dynamic Version of the Economic Lot-Sizing Model", Management Science, Vol.5,1958, pp.89-96.
(17)De Matters, J.J., and G. Mendoza, "An Economic Lot-Sizing Technique", IBM Systems Journal Vol.7, 1969.
(18) Silver, E.A., and H.C. MeaL "A Heuristic for Selecting Lot-Size Quantities for the Case of Deterministic Time-Varying Demand Rate and Discrete Opportunity for Replenishment", Production and Inventory Management, Second Quarter 1973, pp.64-77.
(19) Love, S.F., Inventory Control, McGraw- Hill New-York, 1979.
(20) McLaren, B.J., "A Study of Multiple Level Lot Sizing Techniques for Material Requirements, Unpublished Doctoral Dissertation, Purdue University,1976.
(21) McLaren B.J., and D.C. Whybark, "Multi-Level Lot Sizing Procedures in a Material Requirements Planning Environment, Discussion Paper No.64, Indiana University, 1976.
(22) Biggs, J.R., S.H. Goodman, and S.T. Hardy, "Lot Sizing Rules in a Hierarchical Multi-Stage Inventory System Production and Inventory Management, First Quarter 1977, PP. 104-115.
(23) Blackburn, J., and R-A Millen, "Lot Sizing in Multi-Level Inventory Systems", Proceedings of 1978 AIDS Conference, 1978, p.314.
(24) Carlson, R.C., D.H. Kropp, M.C. Burstern, P.G. Hanson, and L.J, Rodler, "An Algorithm for Lot Sizing in the MRP Product Hierarchy", Technical Report No.80-2, Stanford Universty, 1980.
(25) Graves, S.C., "Multi-Stage Lot Sizing: An Iterative Procedure", in Schwarz, L.B. (ed), Multi-Level Production/Inventory Control Systems: Theory and Practice, Studies in the Management Science, Vol.16, North-Holland: Amsterdam, 1981.

iterative procedure is proposed and tested for finding a periodic review schedule. Lambrecht et al.[26] consider the lot-size problem for serial production/inventory systems operating with deterministic, dynamic, periodic demand. They review the characteristics of the optimal policy for the uncapacitated and capacitated versions of this problem present two algorithms for optimisation in the capacitated problem, and examine the performance of several heuristics for both problems. They conclude that the costs of the heuristically based policies differ only slightly from the costs of the optimal policies and are far more efficient computationally.

III. STOCHASTIC MULTI-ECHELON MODELS

One of the early work on multi-echelon systems is due to Simpson[27]. Simpson investigated an allocation problem. The model assumes that there are N warehouses that are controlled and supplied by a central agency. Each warehouse is faced with an independent external random demand. Single item that is reordered from time to time is handled by the warehouses and supply system. This means that present allocation has to last the warehouses only until the material from the next allocation arrives. It is assumed that a fixed penalty cost is incurred every time the emergency replenishment is invoked. Emergency procedure is invoked whenever a warehouse inventory gets down to a previously established emergency trigger level. The problem then is to find the minimum cost allocation policy. Under these assumptions a simple allocation rule is obtained. The rule states that "a necessary condition that an allocation have minimum total expected cost is that the weighted probabilities, $p_i P(S_i=Q_i-a_i)$ be equal for all warehouses. Here pi is the penalty associated with the emergency replenishment action, and $P(S_i=Q_i-a_i)$ is the probability that sales will be exactly equal to the quantity allocated minus the emergency trigger level". Another allocation policy is obtained for the "no emergency replenishments" case. In this case a warehouse remains out until the next regular replenishment if it runs out of stock. Any demand that occurs when a warehouse is not out of stock is considered to be lost. Then the problem is to find the allocation

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(26) Lambrecht, M.R., j.V. Eecken, and H. Vanderveken, "Review of Optimal and Heuristic Methods for a Class of Facilities in Series Dynamic Lot-Size Problems" in: L.B. Schwarz(ed), Multi-Level Production/Inventory Control Systems: Theory and Practice, Studies in the Management Science, Vol.16, North-Holland: Amsterdam, 1981.

(27) Simpson, K-F., Jr., "A Theory of Allocation of Stocks to Warehouses", Operations Research, Vol.7, 1959, pp.797-805.

policy that minimises weighted number of unsatisfied demands. The allocation rule derived for this case states that "a necessary condition that an allocation minimises the, weighted number of lost sales is that the weighted probabilities $w_iP(S_i \geq Q_i)$ be equal for all warehouses. Here w_i is the weight given to a lost customer at a particular warehouse, and $P(S_i \geq Q_i)$ is the probability that demand is equal to or greater than the quantity allocated to that warehouse". For the both cases Simpson gives total expected cost expressions and minimises cost functions netting out the first derivatives. Thus, he proves aforementioned rules by contradiction.

A significant contribution to the multi-echelon inventory theory is made by Clark and Scarf[28] . The model assumes that there are N installations, where installation N supplies stock to installation N-1, N-1 supplies stock to N-2,......, installation 2 supplies stock to installation 1. The highest installation in the series, N, receives its stock from the source of production. It is important to note the following distinction between an installation and an echelon. The stock at installation i refers only to the stock physically at that location, stock at echelon i refers to the sum of all the stocks at installations i, i-1,...,2,1 plus all the stock in transit between installations i, i-1,... 2, 1. It is also assumed that *(1)* demand, exogenous to the system occurs at installation 1 only, *(2)* the purchasing cost between installations is linear without a fixed cost of ordering (the only exception to this assumption is at the highest installation, where a fixed cost of ordering is allowed), *(3)* demand in excess of supply at any installation is backlogged, *(4)* delivery at each installation is instantaneous, *(5)* in addition to the purchase cost holding cost, h-proportional to the stock on hand at the beginning of the period- and shortage cost p -proportional to the deficit of available stock at the end of the period- are charged during each period, *(6)* delivery of an order occurs λ periods after the order is placed, *(7)* the one period cost function, L is convex for all echelons. As a continuation of the classic dynamic programming approach used in single-activity periodic review problems, Clark and Scarf formulated and solved the aforementioned problem. The model given below is for single-installation problem.

$$\text{if } x>O \text{ then } L(x)=hx+p\int_x^{\infty}(t-x)\,\varnothing(t)dt$$

$$\text{else } L(x)=p\int_x^{\infty}(t-x)\,\varnothing(t)dt$$

(28) Clark, A.J., and H. Scarf, "Optimal Policies for a Multi-Echelon Inventory Problem", Management Science, Vol.6, No.4, 1960, pp.475-490.

In this one period cost function x_i is the stock on hand at the begining of the period i. $C_n,(x_1,w_1, \ldots , w,_{\lambda-1})$ represents the expectation of the discounted costs, begining with x_1 units of stock on hand and following an optimal provisioning scheme, where w_i is the units to be delivered i periods in the future. This sequence of functions satisfy the following functional equation:

$$C_n(x_i,w_i, \ldots\ldots, w_{\lambda-1})=\min(z\geq 0)\{c(z)+L(x_1)+ \\ \alpha\int C_{n-1}\,(x_1+w_1-t, w_2,\ldots,w,_{\lambda-1},z)\varnothing(t)dt\}$$

where the minimising value of z is the optimal purchase quantity for the given stock configuration.

$$u=x_1,w_1,\ldots\ldots,w,_{\lambda-1}$$

$y=u+z$ (in fact, z is w_λ)

$$C_n(x_1,,w_1, \ldots.w,_{\lambda-1}) = L(x_1)+a\ \alpha\int L(x_1+w_{1-t})\varnothing(t)dt+ \\ +a^{\lambda-1}\int\ldots..\int L(x_1+w_1+w_{\lambda-1}\ -t_1-\ldots\ldots-t_{\lambda-1})\varnothing(t_1)\ldots.\ dt1 \\ \ldots\int n(x_1+\ldots..+w_{\lambda-1})$$

$$f_n(u)=\min y\geq u\{c(y-u)+\alpha^\lambda\int\ldots..\int L(y-t1-\ldots.-t_\lambda))\varnothing(t_1) \\ \varnothing(t_\lambda)dt_1\ldots..dt_\lambda+\alpha\int f_{n-1}(y-t)\varnothing(t)dt\}$$

For this formulation of single installation problem y* -u is the optimal purchase quantity, where y* is the minimising value in the above equation. Following the approach for the single installation model one gets the recursive relation for multi-echelon model The problem with this approach is that in the general case C~n is a function of N variables. Therefore, the recursive calculations of dynamic programming would be prohibitively long, if the function C~n is left in this form. Clark and Scarf proved that the function C~n can be decomposed into N functions, each of a single variable, one for each echelon in the system. Each of these problems can then be solved by the usual single activity technique. The set of one state variable problems are interconnected by "implied shortage costs" generated at echelon (excepted the highest one) and passed on (included in the cost function) to the next higher echelon. Thus the optimal policy is first established for the lowest echelon, from which implied shortage costs are obtained. These costs are then included in the cost function for the next higher echelon for which the process is repeated. The ordering policies determined in this fashion take the form of periodic (S-1,S) policies at lower echelons and an (s,S) type policy at the highest echelon. And it is shown that, in general, the parallel echelon structure cannot be broken down into a set of single activity problem.

Clark, and Scarf's paper is significant because it introduces the concepts of system stock (echelon stock) and implied shortage costs to demonstrate the optimality of a simple ordering rule. Extensions of these results to general arborescent structures, using a different mode of analysis, was accomplished by Bessler and Veinott(29) . In previous papers, Veinott developed a technique for analysing inventory problems, first for the single-product, single-activity case(30) and then for the multi-product, single-activity problem(31) . Both papers are concerned with determining an optimal ordering policy for a single commodity, in a dynamic multi-period inventory model in which the demand pattern and the cost structure may change from period to period. The criterion of optimisation is the minimum expected discounted cost over an infinite time horizon. Ordering policy is to order up to a critical stock level in each time period. Further underlying assumptions involve: partial or complete backlogging of excess demand; deterioration of stock in storage for single-product case. The cost parameters are non stationary. If demand is assumed backlogged a constant delivery lag can be accommodated. The functional equation approach of dynamic programming is not used in proofs. Instead, a direct analysis, which is called "dynamic process analysis" by Clark in Ref.[3], of the underlying stochastic process is used. Bessier and Veinott (Ref.29), extended these results to the multi-activity inventory problem. In this seminar paper a general multi-period multi-echelon supply system consisting of n facilities each stocking a single product is studied. At the begining of a period each facility may order stock from an exogenous source with no delivery lag and proportional ordering costs. Demand during each period at each activity are satisfied by available stocks at the facility, with excess demands

(29) Bessler, S.A., and A.F. Veinott Jr., "Optimal Policy for a Dynamic Multi-Echelon Inventoriv Model", Naval Research Logistics Quarterly, Vol. 13,1966, pp.355-389.

(30) Veinott, A.F., Jr., "Optimal Policy in a Dynamic, Single-Product Non-Stationary Inventory Model with Classes", Operations Research, Vol 13, 1965, pp.761-778.

(31) Veinott, A.F., Jr., "Optimal Policy for a Multi-Product, Dynamic, Nonstationary Inventory Problem". Management Science, Vol.I2, No.3,1965, pp.206-222.

being immediately transmitted to its supplier for possible satisfaction. Excess demands are successively passed up, with backlogging occurring only at the top supplier. When the stock at each facility is viewed as a product, results of the multi-product single-activity problem have corresponding interpretations for the single-product, multi-facility case. Exploiting the correspondence one proves that if there is a y~i which minimes $G_i(y)$ over $Y_i(i=1, ...n)$, if $x_l \leq$y~i, and if S_i(y~i,Di) $\leq$y~i +l (for all D_i and i=l, ... N-1) then one optimal policy is given by Y~i(H_i), (i=1,...,N-1); where $x_i=(x_{ij})$ is the vector of inventories of the item on hand at each of the n facilities at the begining of period i; $D_i=(D_{ij})$ is the demand for the item at facility j in period i; $y_i=(y_{ij})$ is the vector of inventories on hand after orders have been in placed in period i at each of the n facilities; $S_i(y_i,D_i)=(S_{ij}(y_i,D_i))$ is a vector function called a supply policy which specifies the amount of stock on hand at each of the n facilities after the demand occurs in period i; $G_i(y)$ is total cost function; $Y=(y_1,y_2, ---)$ is a sequence of vector value functions which specifies the ordering policy such that at the begining of period i, after having observed the past history H_i, order quantity is Y~i i(H_i)-x_i. This result reduces the problem of determining the optimal policy to that of solving N n-dimensional minimisation problems, where N is the number of periods and n is the number of activities. In the following sections, the arrangement of the activities in an arborescence one-period cost function, investigation into effects of parameter variations, establishment of bounds for the optimal stock levels and an algorithm for computing approximations to the optimal levels based upon the values for the lower bounds are considered. Ignall and Veinott(32), removed the restrictions on the initial stocks and proposed a supply policy for specific networks.

In a later work(33) , Clark, and Scarf extended their previous work (Ref 28) to include fixed order cost at lower installations with all other characteristics of the problem the same as before. Since the problem could not be broken down into a sequence of single-state variable problems, the optimal value for the cost function is bounded from both above and below.

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(32) Ignall E., and A.F. Veinott Jr., "Optimality of Myopic Inventory Policies for Several Substitute Products", Management Science, 15, 1969, pp.284-304.
(33) Clark, A.J., and H. Scarf "Approximate Solutions to a Simple Multi-Echelon Inventory Problem", Chap.5 in K.Arrow, S. Karlin, and H.Scarf(eds.) : Studies in Applied Probability and Management Science, Stanford University Press, Stanford, Calif., 1962, pp.88-110.

As a continuation of the Clark-Scarf approach, Fukuda[34] combined ordering and disposal policies. A model is described in which the decision to be made at the beginning of each period is always one of ordering fresh stock, disposal of surplus stock, or doing nothing. Four costs are considered: shortage and holding costs, which are both linear functions of the number of items concerned, and ordering and disposal costs, each of which may be either of type A or type B. Type A costs are proportional to the number of units purchased and type B costs include an additional fixed reorder cost. Disposal cost may be negative (when revenue is obtained from disposal). It is also assumed that excess demand is backlogged, that items for disposal are withdrawn at the beginning of the period, and that stock is delivered one period after it is ordered. The optimum policy for minimising total expected costs is first determined for a single installation system by a dynamic programming formulation. Then multi-echelon system is considered. Type A costs are assumed for ordering and disposal at each echelon. If a decision is made to dispose of units at any echelon, they are immediately withdrawn from that echelon, but one period is required for units to move from one echelon to another. When stock reaches the highest echelon, it leaves the system. The second echelon in a three echelon system is analysed in detail. Similar to Clark-Scarf model the implied shortage costs for ordering are passed upward in the structure. Using the same reasoning the implied shortage costs for disposal are passed downward.

A much simpler proof of optimality for Clark and Scarf model is given by Veinott[35] . Veinott used an approach which exploited a convexity theorem of Karush[36] . Hochstaedter[37] established upper and lower bounds for the optimal cost function of the system, in which activities are in parallel with a common supplier. In this model Hochstaedter permitted fixed reorder costs. Zacks[38] formulated a Bayesian model of the two-echelon parallel activity structure, assuming that Poisson distribution demands occur at the

(34) Fukuda, Y., "Optimal Disposal Pohcies", Naval Research Logistics Quarterly, Vol.8, 1961, pp.221-227.
(35) Veinott, A.F., Jr, "The Status of Mathematical Inventory Theory", Management Science, Vol. 12, 1966,pp.745-777.
(36) Karush W., "A Theorem in Convex Programming", Naval Research Logistics Quarterly, Vol.6,1957, pp.245-260.
(37)Hochstaedter, D., 'An Approximation of the Cost Function for Multi-Echelon Inventory Model",Management Science, Vol. 16, 1970, pp.716-727.
(38) Zacks, S., "A Two-Echelon, Multi-Station Inventory Model for Navy Applications", Naval Research Logistics Quarterly, Vol. 17,1970, pp.79-85.

lower activities and that the prior distribution of the Poisson parameters is a Gamma distribution. Linear holding and shortage costs are assumed, ordering costs are not included. With these assumptions, a multi-state variable dynamic programming solution procedure is developed. In a later paper, Zacks[39] removes the restriction "no return of stock to the higher activity" and hence the later model allows unrequired stock at lower activities to be returned to the higher facility. The optimal policy is derived from a dynamic progamming formulation. The principle result is that the optimal ordering policy of the lower activities is obtained by solving an integer convex programming problem with linear constraints. Another extension of ClarkScarf model was given by Williams[40]. In this paper a multi-state variable dynamic programming model is developed for both the backlog and lost sales case. A series structure, where each activity has a fixed ordering cost in addition to the usual inventory cost is considered.

Another important technique, other than the dynamic programming used to investigate inventory systems is expected cost minimisation technique. One of the oldest papers, which used this approach to examine multi-echelon inventory systems was published by Berman and Clark[41]. The work is devoted to procurement policies in a single-product inventory system consisting of several bases supplied by a depot. The paper provided more than an extension of the previous research on single-stocking point problems in that it announced many real alternatives in complex inventories such as: consumable or repairable items, transhipment or no lateral resupply among subinventories at the same level, normal and emergency resupply, life-of-type (sufficient amounts are purchased all at once to satisfy all expected future demands) and periodic procurement, etc. Expressions for the expected avarage costs were obtained. From these cost functions, expressions for minimising values of the policy variables were derived.

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(39) Zacks, S., "Bayes Adaptive Control of Two-Echelon Multi-Station Inventory Systems", The George Washington University, Institute for Management Sicience and Engineering, Programs in Logistics, TN-61541, Washington, D.C., 1970.

(40) Williams, J.F., Multi-Echelon Production Scheduling When Demand is Stochastic", University of Wisconsin, School of Business Administration, Milwaukee, Wisc., Wisc., 1971.

(41) Berman E.B., and A.J. Clark "An Optimal Inventory Policy for a Military Organization", The Rand Corporation, P-647, Santa Monica., 1955.

Another important paper which used the expected cost approach to analyse the inventory problem for a low-demand item was published by Hadley and Whitin(42). In this paper a supply system consisting of N depots and a central control point studied. It is assumed that:

- instantaneous information concerning inventory levels is available;
- items are ordered one at a time;
- the supply lead time, and times required for either of two available modes of redistribution are constants;
- demand comes from a stationary Poisson process at each of the depots;
- demand at one depot is independent of that at other depots.

The amount of stock on hand plus on order minus backorders for the system remains constant throughout time. The optimal system stockage objective and depot stockage objectives are derived by balancing carrying costs against the costs of stockout and redistribution. If the system stockout cost is neglected, then the depot stockage objectives can be determined independently of the mode of transportation used from the source. However, the optimal system stockage objective depends on the mode of transportation. It is assumed that the decision as to where to allocate each unit ordered is made at the time a unit is ready to be shipped from the source. A dynamic programing model is developed to obtain the optimal allocation for minumum costs of stockout and redistribution, but it is noted that for many low cost items, it is not worthwhile to use the dynamic programming model. For some cases, it is sufficient simply to allocate the unit to the depot which has the greatest probability of using it before the next allocation. A redistribution is to be considered each time there is a demand in the system, provided the distribution time is not greater than the time until the next allocation, rules are developed for deciding whether and how to redistribute. Using the same approach, Hadley and Whitin also considered the case of higher demand items(43). It is assumed that the system as a whole uses an (s,S) type policy and that redistribution is

(42) Hadley, G., and T.M. Whitin, "A Model for Procurement Allocation and Redistribution for Low Demand Items", Naval Research Logistics Quarterly, Vol.8, 1961, pp.395-414.
(43) Hadley, G., and T.M. Whitin "An Inventory-Transportation Model with N Locations", Chap.5 in H. Scarf, D. Gilford, and M. Shelly (eds.): Multistage Inventory Models and Techniques, Stanford University Press, Stanford, Calif., 1963.

considered whenever a depot's stock falls to critical levels that are set by external criteria. Furthermore, only that depot triggering the redistribution decision is considered as a receiver each time. All other assumptions for this model are the same as for the previous. Again, cost minimising expressions are obtained for determining stockage objectives at the depots, values for the system procurement policy, and sources and amounts for redistribution decisions. A dynamic progamming algorithm is presented for allocating system procurements to the depots upon receipt from the external source.

Gross[(44)] investigated the same problem, confining the attention to a single time period and no a priori assumptions are made concerning the form of stockage and redistribution policies. A total cost function is first formulated for the case of two locations. From this, minimising values for ordering and transhipment amounts are derived. An iterative procedure is then given to generalise the results to an arbitrary number of locations.

The problem formulated by Gross was also considered by Krishnan and Rao[(45)]. Demand at each warehouse is independent of other warehouses and is continuously distributed with a known density function. All costs are linear and include: holding cost, which is directly proportional to excess demand over available stock during the period; cost of transportation,which is incurred when delivering at the end of a period, from a warehouse with excess stock to a warehouse with a shortage (which is the most important difference between the Gross model and the Krishnan-Rao model). The expressions for total expected costs are developed for a two-warehouses and for an N-warehouses distribution system. In both cases the minimum total expected cost is found by partially differentiating with respect to the required stock levels. Treating the second case as two centres, one consisting of the first warehouse and the other comprising the remaining N-1 warehouses; successive iterations provide solutions for the N optimal inventory replenishment levels.

The results of many disciplines such as queuing theory, Markov analysis, reliability analysis etc. can be employed to perform

(44) Gross, D., "Centralized Inventory Control in Multilocation Supply Systems", Chap.3 in H. Scarf, D. Gilford, and M. Shelly (eds.): Multistage Inventory Models and Techniques, Stanford University Press, Stanford, Calif., 1963.
(45) Krishnan, K.S., and Rao, V.R.K., "Inventory Control in N Warehouses" Journal of Industrial Engineering, Vol. 16, No.3,1965, pp.212-215.

stationary analysis of multi-echelon inventory systems considered in steady-state conditions. Stationary process analysis was applied by Rosenman and Huckster[46] in considering a two-level supply/repair system. It is here that it was assumed for the first time that failed items can be repaired at the peripheral level on at a higher level according to given rates. The problem of distributing a given system stock among the subinventories which minimises the total expected customer waiting time was solved. It is assumed that items can be repaired locally or centrally according to given rates and that losses to the system are negligible. It is further assumed that external demands are Poisson distributed that lower-level facilities use continuous review, one for one (S-1,S) ordering policies, and that replenishment times and repair cycles are given constants. Under these assumptions, a cost-free model is developed.

Love[47] treated the simplest two-subinventory cascade, both subinventories using continuous review (S-1,S) policies. External Poisson demand occurred only at the peripheral subinventory. Resupply times at both locations were exponentially distributed. By applying the results of queuing theory Love obtained the expected number of backorder days and stock on hand per unit time. The expected total system cost was shown to be convex with S and an algorithm for determining the optimal policies was given.

One among the most frequently cited papers from the field of multi-echelon inventory systems is the paper by Sherbrooke[48] which presented METRIC model (an acronym for "The Multiechelon Technique for Recoverable Item Control") based on a stationary process analysis. Originally, two-echelon system was considered in which the bases with repair capabilities follow continuous review (S-1,S) policies and the depot (with repair capabilities as well) performs no reordering since all failed units are always repairable. The two-echelon system is pictured in Fig.2.

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(46) Rosenman, B., and D. Hockstra, "A Management System for High-Value Army Aviation Components", U.S.Army, Advanced Logistic Research Office, Report TR 64-1, Philadelphia, Pennsylvania, 1964.
(47) Love, R.F., "A Two-Station Stochastic Inventory Model with Exact Method of Computing Optimal Policies", Naval Research Logistics Quarterly, Vol. 14,1967, pp.185-217.
(48) Sherbrooke, C.C., "METRIC: A Multi-Echelon Technique for Recoverable Item Control", Operations Research,Vol.16,1968,pp.122-141.

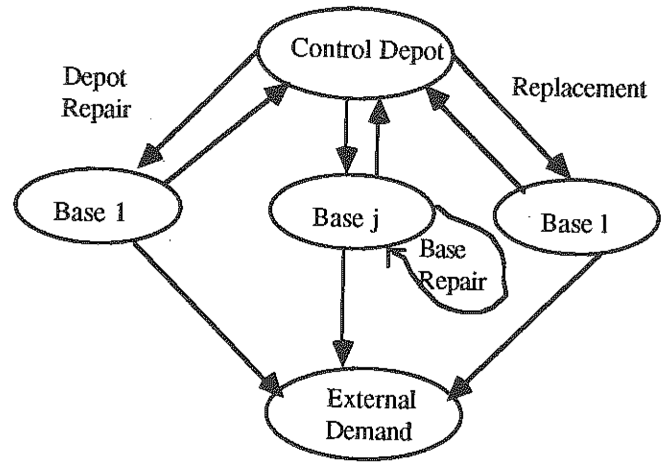

Fig.2 The Two-Echelon System Considered by METRIC

Each of I bases stock J spare parts. At the occurrence of a demand (that is, failure of one or more items in the field), the following takes place: the demand is either satisfied from available base stock or backordered and the failed item is inspected to determine the extent of the repair required. If the repair can be made at the base the unprepared item enters base repair. If the item cannot be repaired at the base level it is shipped to the depot. Simultaneously with shipping the item (or items) to the depot, the base places an order for a replacement (or replacements), so that the inventory position for item i at base j is a fixed constant, S_{ij}. The model assumes that demand requests are filled by the depot in the same order that they are received. The goal is to find values for S_{ij} at the bases and the depot that minimise the total expected level of backorders for spare parts at a random point in time subject to a constraint on the total investment available. The following assumptions are made:

- Demands for item i at base j are generated by a stationary compound Poisson process with rate λ_{ij} and compounding distribution with mean F_{ij}.
- With probability r_{ij} a failed item i at base j can be repaired at the base. With probality $(1-r_{ij})$ that item must be repaired at the depot.

- The expected base repair time, A_{ij}, the expected order and ship time from the depot to base j, O_{ij}, and the expected depot repair time, D_i for item i are known constants.
- All items can be repaired. That is, the system is completely conservative with no condemnations allowed. Sherbrooke argues that since the condemnation rate is only around 5 % and procurement decisions are determined separately from levels for spares, assuming a zero condemnation rate should be satisfactory.
- There is no lateral resupply (transhipment) among bases.
- Successive repair times are independent identically distributed random variables. This is basically the same as saying that there are infinitely many servers at the repair channels so there is no queuing at the repair stations.

The principal computation requires obtaining an expression for the expected number of backorders on the books for item i at base j at a random point in time. The computation of the expected lead-time (A_{ij}) when items are shipped to the depot (the depot resupply time) is affected by a direct application of the classic formula $L=\lambda W$ from queuing theory, which says that the expected queue length is the product of the arrival rate and the expected waiting time of an entering customer, independent of the form of the interarrival or service distribution. The expected waiting time at the depot for an arriving order is the expected number of backorders (expected queue length of backorders) divided by the expected rate of the demand on the depot. The total demand on the depot is compound Poison with rate $\lambda= \Sigma^J{}_1 f_i(1-r_i)$ and compounding distribution with mean $f= \Sigma_1{}^J fi(1-ri)$since it is the superposition of the demand process at the bases. Since the number of busy servers has distribution $p(x \mid \lambda D)$[49], the expected number of unfilled requests at the depot at a random point in time when depot stock is S_O is

$$B(So \mid \lambda D)=\Sigma^{\infty}{}_{x=So+1}[(X-S_O)p(x \mid \lambda D)]$$

The total expected depot demand per unit time is λ f or $B(O \mid \lambda D)/D$. It follows from $L=\lambda W$ that the expected waiting time per demand at the depot is $D.B(S_O \mid \lambda D)/B(O \mid \lambda D)$ or $\delta\ (S_O)D$, where $\delta(So)=B(S_O \mid \lambda D)/B(O \mid \lambda D)$. Hence, the total expected resupply time for an item shipped to the depot is $O+\delta(S_O)D$. Combining this with the computation of the expected base resupply time and including

(49) Feeney, G.J:, and C.C. Sherbrooke, "The (S-1,S) Inventory Policy Under Compound Poisson Demand", Management Science, Vol.12, No.5,1966, pp.391-411.

subscripts for clarity, it follows that the lead time for item i at base j, say $T_{ij}(S_{io})$, is given by

$$T_{ij}(S_{io})=r_{ij}A_{ij}+(1-r_{ij})[O_{ij}+\delta(S_{io})Di]$$

By applying Feeney and Sherbrooke extension of Palm's Theorem (Ref.49), the expected number of backorders of item i at base j at a random point in time when depot stock for item i is S_{io} and base stock is S_{ij} is given by $\beta_{ij}(S_{io}, Sij)$, where

$$\beta_{ij}(S_{io}, Sij)=\Sigma^{\infty}Sij(x-Sij)p[x \mid \lambda_{ij}T_{ij}(S_{io})]$$

It should be pointed out that this expression for expected backorders will be exact only if lead times are independent random variables. However, since orders from the bases are assumed to be filled in the same sequence in which they were placed, successive lead times will generally be correlated. Fox and Landi[50], however, state that simulation experiments indicate that this expression gives relatively good agreement with backorder levels occurring in actual application. c_i is the cost of item i and C the total investment available. The optimisation problem is: find S_{ij}, $1 \leq i \leq I$, $0 \leq j \leq J$ to

Minimise

$$\Sigma^{I}i=1\Sigma^{J}j=1[\beta_{ij}(S_{io}, S_{ij})]$$

Subject To

$$\Sigma^{I}i=l(C_i)\Sigma^{J}j=(S_{ij}\leq C)$$
$$S_{ij}\geq 0,\ 0\leq i\leq I,\ 0\leq j\leq J$$

A five step procedure is given for finding optimal solutions. First, using the expression for expected number of base backorders, the average delay per demand against the depot is found for each items as a function of depot stock. Second, for each level of depot stock and each base, expected base backorders are calculated as a function of the base stock. Third, for each level of depot stock, an allocation to the bases is made which minimises the total expected backorders; this is done by a marginal analysis method. Fourth, the minimum expected system backorders are found as a function of total system

(50) Fox, B.L., and D.M. Landi, "Searching for the Multiplier in One Constraint Optimization Problems", Operations Research, Vol. 18, No. 12,1970, pp.253-262.

stock (bases plus stock). Fifth the multi-item aspect is considered by the use of a marginal value method to allocate a given investment across items: each additional increment of investment is assigned to that item for which the largest reduction in expected system backorders win result.

An interesting continuous review model was developed by Simon(51) which treats the same base/depot supply system considered in METRIC. Simon's model is more general than METRIC in that both a positive condemnation rate and external procurements at the depot level are allowed, but less general in that item demands at the bases are assumed to be generated by simple Poisson processes and all resupply times are assumed deterministic. As with METRIC, it is assumed that each base follows a continuous review (S-1,S) policy. For an item which fails at base j, there is a probability of r_j that the item will be base repairable and a probability p that an item which is not base repairable will be depot repairable. Hence $(1-r_j)(1-p)$ is the probability that on item which fails at base j must be condemned. Because the system is no longer conservative, outside replenishments are required. This is accomplished by assuming that the depot follows a continuous review (s_o,S_o) policy. For each base, Simon obtains exact expressions for the steady-state number of backorders and proves that the number of units in repair is stationary and Poisson distributed. Similar results are obtained for the depot.

Sherbrooke's paper has inspired many authors to propose modified or extended models which are referred to as METRIC-based models in the literature. The most commonly cited among them is MOD-METRIC model by Muckstadt(52) which introduced a concept of high relevance for practical purposes a multi-indenture inventory. The fact that many of the endproducts being maintained are complex and consist of assemblies has motivated the incorporation of the multi-indenture aspect of the problem into inventory model. This has been reflected in the procedure for calculating the average base repair time. The notions of line replaceable unit (LRU) and shop replaceable unit (SRU) were introduced. Muckstadt derived expressions for the expected delay in the LRU base repair time due to

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(51)Simon, R.M., "Stationary Properties of a Two-Echelon Operations Research, Vol.19,1971,pp.761-773.

(52)Muckstadt J.A., " A Model for a Multi-Item, Multi-Echelon Multi-Indenture Inventory System", Management Science, Vol.20, No.4, 1973, pp.472 -481.

a shortage of a given type of SRU and, further on, the average resupply time for a failed LRU at each base. Muckstadt formulated and solved the very realistic problem of how to allocate a given budget for the procurement of spares between LRU and SRU which maximises the operational availability of end-products. The solution obtained is reported to be more suitable than the one obtained by original METRIC. Comparison was performed by simulation.

Another extension of METRIC has been proposed by Miller[53]. In both METRIC and MOD-METRIC, demands upon the depot are assumed to be filled in the same sequence that they were placed originally. In Real Time METRIC (the name was used in the original RAND memorandum), this restriction is not placed upon the depot. Instead, as each item completes depot repair, the depot has the prerogative of determining to which base the item will be shipped. Miller assumes that demands for the item on the depot are generated by independent Poisson processes with respective rates λj. The time required at the depot to repair each item is independent of the repair times for other items, and repair times are exponentially distributed with mean $1/\mu$ days. In addition, Miller assumes that it requires T_j days to ship the item from the depot to base j, where the T_j's are known constants. As each item completes repair, the rule he suggests is to ship the item to that base whose marginal decrease in expected backorders will be greatest at time T_j days into the future. Miller shows this rule to be optimal for a slightly modified version of the recoverable item problem and claims that simulation of some test resulted in considerable decreases in the levels of expected backorders observed when using METRIC.

In a paper by Porteus and Lansdowne[54] the multi-location multi-item spare inventory problem was treated as a logistic subproblem. The items failed require repair. According to the authors, the time of a particular type of repair is governed by probability distribution and spares are kept on hand for replacing failed items in case of lengthy repairs. The expected weighted shortages over all items and all locations represent the measure of performance of the whole system. The optimisation of design is viewed in choosing between

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(53)Miller, B.L., "Dipatching from Depot Repair in a Recoverable Item Inventory System: On the Optimality of a Heuristic Rule", Management Science, Vol.21, No.3, 1974, pp.316-325

(54)Porteus, E., and Z.Lansdowne, "Optimal Design of a Multi-Item, Multi-Location, Multi-Repair Type Repair and Supply System", Naval Research Logistic Quarterly, Vol 21, 1974, pp.213-238.

more spares on shorter expected repair times within a budget constants. All costs are separable and a Lagrangean approach to optimisation has therefore proven efficient.

Deuermeyer and Schwarz(55) analyse a system operating under a continuous review demand replenishment (s,Q), ordering policy. The paper develops and tests an approximate model for estimating system service level performance as a function of system parameters: warehouse and subwarehouse lot-sizes, order points, lead times, and demand parameters. External demand is Poisson, with identical mean rate at all subwarehouses and the lead time to the subwarehouse is also identical and constant for all warehouses. Further, the subwarehouses have identical order points, s, and order quantities, Q. Model involves the approximation of the warehouse demand process using results from renewal theory.

Clark(56) summed up the interesting results of many-years efforts towards developing the optimal availability inventory model for Navy applications. These results are embodied in a multi-indenture, multi-echelon spare parts inventory model which was explicitly designed for practical use and implementation. The main assumptions made in the model are: demand distributions are stationary and satisfy Palm's theorem, all subinventories use continuous review (S-1,S) policies, excess demand is backlogged, there is no lateral resupply. The performance measure is the expected operational availability of end-product. It is defined as the ratio of up-time to the sum of up-time and down-time of end-product. The solution procedure is described, and the application of the model is illustrated by a number of actual two-echelon type examples. The same availability is achievable with more than three times lower total spares investment used for current spare policies in the Navy.

A review paper on various mathematical models that have appeared in the literature for determine stocking levels for

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(55) Deuermeyer, B.L., and L.B. Schwarz, "A Model for the Analysis of System Service Level in Warehouse Retalier Distribution System: The Identical Retailer Case", paper presented to Multi-Level Production/Inventory Systems Conference, Purdue University, 1979.
(56) Clark A.J., "Experiences with Multi-Indenture, Multi-Echelon Inventory Model", in Schwarz, L.B.(ed), Multi-Level Production/Inventory Control System: Theory and Practice, Studies in the Management Sicience Vol 16. North-Holland: Amsterdam, 1981, pp.299-330.

repairable item inventory system is published by Nahmias[57] Existing models are classified into three general classes: continuous review, periodic review, and models based on cyclic queuing systems.

A model to determine the inventory stockage levels in a multi-echelon inventory system for a repairable item is developed by Graves[58]. The multi-echelon system consists of a set of operating sites supported by a centrally-located repair depot. Each operating site requires a set of working items and maintains an inventory of spare items. The repair depot also holds an inventory of spare items. Item failures are infrequent and are replaced on a one-for-one basis. Failures are generated by a compound Poisson process and that the shipment time from the depot to each site is deterministic. No assumptions are made with regard to the repair cycle at the depot. Under these assumptions an exact model for finding the steadystate distribution of the net inventory level at each site is presented. Also, based on the exact model, an approximation for the steady-state distribution for the case with ample servers at the repair depot is presented.

The application of a heuristic model developed to aid Eastman Kodak management in determining safety stock allocated in a two-level, finished products distribufion system is described by Rosenbaum[59]. This distribution system consists of a central distribution centre (DC) and up to seven regional distribution centres (RDCs), depending on the given product. Within Kodak's existing management system safety stock, quantities are based on fill rates, individualy set at each stocking location, the DC and the RDCs. The model was developed to determine that combination of individual fill rates, which minimises the system's safety stock while guaranteeing a prespecified level of system performance.

Before ending the review on stochastic multi-echelon inventory systems published papers on system, design are reviewed. The design of a multi-level production and inventory system the determination

(57) Nahmias, S., "Managing Reparable Item Inventory Systems: A Review", in Schwarz, L.B. (ed), Multi-Level Production/Inventory Control System: Theory and Practice, Studies in the Management Science, Vol.16, North Holland: Amsterdam, 1981, pp. 253-277.
(58) Graves, S.C., "A Multi Echelon Inventory Model for a Repairable Item with One-for-One Replenishment", Management Science, Vol.10, 1985, pp.1247-1256.
(59) Rosenbaum, B.A., "Service Level Relationships in a Multi-Echelon Inventory System", Management Science, Vol.27, pp.926-945.

of the number of activities, their size, and network configurafion, is in many ways the most important problem that the theoretician or manager may confront.

The first paper that discusses the optimal design of multi-echelon system was by Pinkus et al[60]. This paper presents a model for designing multi-activity, multi-facility systems. Given the maximum number of facilities and their possible locations, the problem is to determine which facilities to include in the system and which activities should be carried on at each facility in order to minimise the cost of the system. A branch and bound algorithm for solving the problem is given. One of the weaknesses of this model is that it assumes that there is no limit to the storage space available at a given installation. A later paper by the same authors[61] is presented to overcome this deficiency. To use the later model it is necessary to know the optimal inventory policies for a set of multi-echelon systems. Dynamic programming approach presented by Clark and Scarf (Ref.28) is used to detemine the optimal inventory policies. In the model the echelon structures are indexed by i, i=1,...m; the products are indexed by j, j=1,....n; the installations are indexed by k, k=1,.....p. The other variables are as follows:

a_{ij}: the inventory cost of product j using echelon structure i,
b_k : the facility cost of installation k,
r_k : the storage space available at installation k,
d_{ijk} : the storage space required at installation k for product j uses echelon structure i.
x_{ij} : the decision variables, 1 if product j uses echelon structure i, 0 otherwise,
y_k : the decision variables, 1 if installation k is used, 0 otherwise,

Gross states the entire model as follows:

Minimise

$$\Sigma^{m}{}_{i}=1(a_{ij}x_{ij})+.\Sigma\ pk=1\ (b_k y_k)$$

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(60) Pinkus, C.E., D. Gross, and R.M. Soland, "Optimal Design of Multiactivity Multifacility Systems by Branch and Bound", Operations Research, Vol.21, No.1, 1973, pp.270-283.
(61) Gross, D., R.M. Soland, and C.E. Pinkus, "Designing a Multi-Echelon /Inventory System", in Schwarz L.B. (ed.), Multi-Level Production/Invetory Control System: Theory and Practice, Studies in the Management Science, Vol.16, North-Holland: Amsterdam 1981, pp.11-49.

Subject to

$$\sum^{m}_{i=1}(x_{ij})=1 \qquad j=1,\ldots\ldots,n$$
$$\sum^{m}_{i=1}\sum^{n}_{j=1}(d_{ijk}x_{ij}-r_k y_k)\leq 0 \qquad k=1,\ldots\ldots,p$$
$$x_{ij},y_k=0 \text{ or } 1 \qquad \text{for all } i, j, k$$

It should be noted that the parameters a_{ij} are determined using the Clark-Scarf approach.

Another paper on system design was by Eppen(62). This paper concerns a multi-location newsboy problem with normal demand at each location and identical linear holding and penalty cost functions at each location. Consolidation of demand from several facility is considered, and expression is derived for the result expected holding and penalty costs as a function of the demand parameters for each location (means, variances, and correlation coefficients). The expression is used to demonstrate that *(1)* the expected holding and penalty costs in a decentralised system exceed those in a centralised system, *(2)* the magnitude of the saving depends on the correlation of demands; and *(3)* if demands are identical and uncorrelated, the costs increase as the square root of the number of consolidated demands. The general approach and some expressions are useful in investigating various questions concerning the design and operation of inventory systems.

III. RECENT WORK ON MULTI-ECHELON INVENTORY SYSTEM

Clark (Ref.3) in 1971 has said:

> "...It is probable that research in multi-activity inventory theory has reached a point where highest returns have already been achieved (the easy problems have been solved) and, therefore, marginal returns from further work, are likely to diminish. The principle opportunities for further work both by individual researchers and research teams, probably lie in refinements and extensions of previous results and in the reduction of currently available theory to practice in actual inventory situations. In addition there may still be a small probability that a new basic theory can be developed which would supersede much of the previous results..."

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(62) Eppen, G.D., "Effect of Contralization on Expected Costs in a Multi-Echelon Newsboy Problem", Management Science, Vol.25, No.5, 1979, pp.498-501.

Unfortunately Clark is right; a new basic theory has not been developed. In this section most of the recent work on multi-echelon inventory control theory is reviewed and marginal contributions are discussed.

Recent developments on information processing techniques and computer science are leading to implementation of more effective decision support systems. Some of these support systems are developed for multi-echelon logistics. The most significant system (named Optimiser) is announced at the paper of Cohen et al.(63) Details for the model, its properties, and the effectiveness of the solution algorithm are reported in Cohen et al.(64), Cohen et al.(65) , Cohen et al.(66) , and Cohen et al.(67). Optimiser, a system for flexible and optimal control of service levels and spare parts inventory, was implemented by IBM in its US network for service support. The time-averaged value of inventory recommended by the stocking policies of Optimiser was 20 to 25 percent below that of the existing system. This difference was obtained along with a 10 percent improvement in the parts availability at the lower echelons while maintaining the parts availability levels at the higher echelons. These strategic changes have yielded operational efficiency on the order of 20 million dollars a year.

Mentzer et al.(68) presented a personal computer (PC)-based, multi-echelon, stochastic, dynamic simulator, termed the Strategic Planning Model (SPM). The model is intended as a strategic decision support system generator, which can be configured to represent the

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(63) Cohen, M. , p.V. Kamesam, P. Kleindorfer, H. Lee, A.Tekerian "Optimizer IBM's Multi-Echelon Inventory System for Managing Service Logistics", Interfaces, Vol.20, No. 1, 1990, pp.65-82.
(64) Cohen, M.A., P.R. Kleindorfer, and H.L. Lee, "Optimal Stocking Policies for Low Usage Items in MultiEchelon Inventory Systems", Naval Research Logisdcs Quarterrly, Vol.33, No.l, pp.17-38.
(65) Cohen M.A., P.R. Kleindorfer H.L. Lee,A.P. Tekerian "Excess-Demand Distributions for, MEeS Stocking Policies in Multi-Echelon Logistics Systems", in Chikan A. (ed.), Inventories in Theory and Practice, Elsevier Science Publishers: Amsterdam, 1986, pp-655-667.
(66) Cohen, M.A., P.R. Kleindorfer, and H.L. Lee, "Service Constrained (s,S) Inventory Systems With Priority Demand Classes and Lost Sales", Management Science, Vol.34, No.4,1989, pp.482-499
(67) Cohen, M.A., P.R. Kleindorfer, H.L. Lee, and D.F. Pyke, "Multi-Item Service Constrained (s, S) Policies for Spare Parts Logistics Systems", Working Paper, 1989..
(68) Mentzer, J.T., and R. Gomes, "The Strategic Planning Model: APC- Based Dynamic, Stochastic, Simulation DSS Generator for Managerial Planning", Journal of Business Logistics, Vol.12, No.4, 1991, pp.193-219.

detailed functioning of operating systems, production, or distribution facilities. With system cost operating data entered into the model the effects of varied conditions of market and company operation can be tested.

Another decision support system was designed by Tripp et al.[69] to help logistics managers assess wartime readiness and to identify resource and policy changes that could improve it. The decision support system known as the weapon system management information system (WSMIS), detects situations where theatre wartime sortie capabilty might be jeopardised and by what resource shortages or logistics processing bottlenecks. The WSMIS is designed to distinguish between planned and actual logistics support capabilities.

One of the recent papers on design of multi-echelon systems is published by Gray et al.[70]. In this paper the composite design and operating problems for a typical order-consolidation warehouse are described and modelled. These problems include warehouse layout equipment and technology selection, item location, zoning, picker routing, pick list generation and order batching. The complexity of the overall problem mandates developing a new multi-stage hierarchical decision approach. The hierarchical approach utilities a sequence of coordinated mathematical models to evaluate the major economic trade-offs and to prune the decision space to a few superior alternatives. Detailed simulation employing actual warehousing data is then used for validation and fine-tuning of the resulting design and operating policies.

A framework for the planning and control of the materials flow in a multi-item production system is presented by Zijm.[71] The prime objective is to meet a prespecified customer service level at minimum overall costs. The system incorporates several new concepts, in particular multi-echelon structures and hierarchical planning procedures, based on a product family structure. The basic algorithm

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(69) Tripp, R.S., I.k. Cohen, R.J. Hillestad, R.W. Clark, S.B. Limpert, and S.K. Kassicieh, "A Decision Support System for Assessing and Controlling the Effectiveness of Multi-Echelon Logistics Actions", Interfaces, Vol.21, No. 4,1991, pp. 11-25.

(70) Gray, A.E., U.S. Karmarkar A. Seidmann, "Design and Operation of an Order-Consolidation Warehouse: Models and Application", European Journal of Operational Research, Vol.58, No.1, 1992, pp.14-36.

(71) Zijm, W.H.M., "Hierarclical on International Journal of Production Economics, Vol.26, No.1-3, 1992, pp. 257-264.

framework that is needed to turn conceptual ideas into operational procedures is described.

Friedman[72] makes some inroads into the problem of determining the optimal number of echelons. An extended lot-size model with no shortages, the usual array of underlying regularity assumption but with the added option of storing inventory in several vertical echelons form the modelling framework. The trade-off between stocking two different echelons on the inventory ladder is that on the higher levels the carrying charges are lower but the handling charges are higher. The initial objective is to minimise the total cost per unit time via the determination of the lot-size, or alternatively the cycle's length, and simultaneously its distrubution among the available echelons, or alternatively again the proration of the inventory cycle into subperiods in which demand is met by different echelons. After establishing this the final objective of finding the number of echelons for which aforementioned cost is smallest is being taken up.

An inventory system with one warehouse and N retailers where lead times are constant and the retailer face independent Poisson demand is considered by Axsater[73]. Simple recursive procedures for determining the holding and shortage costs of different control policies are provided. Svonoros et al.[74] present a model which assumes that the exogenous demands are independent Poisson processes and each location follows a one-for-one replenishment policy. Transit times are modelled in a way that closely follows the standard treatment of stochastic lead-times in single-location models. Simple methods are described for computing or approximating the steady-state behaviour of the system. The results show that, in sharp contrast to prior multi-echelon models, transit-time variances play an important role in system performance. Iterative computational formulas are developed by Daryanani et al.[75] for the steady-state probabilities of an exponential single-

(72) Friedman, M.F., "A Distribution Multi-Echelon Lot-Size Model", European Journal of Operational Research, Vol.57,1992, pp.54-70.
(73) Axsater, S., "Simple Solution Procedures for a Class of Two-Echelon Inventory Problems", Operations Research. Vol.38, No. 1, 1990, pp.64-69.
(74) Svonoros, A., and P. Zipkin, "Evaluation of One-for-One Replenishment Policies for Multiechelon Inventorv Systems, Management Science Vol.37, No.1,1991, pp.68-83.
(75) Daryanani S., and D.R. Miller, "Calculation of Steady-State Probabilities for- Repair Facilities with Retum Priorites", Operations Research, Supplement, May/June 1992, pp.S248-S256.

channel repair facility with multiple Poisson sources and a dynamic return policy. Such facilities occur as part of multi-echelon repairable item pro-visioning systems in which backorders are filled according to need instead of FIFO or SIRO policies. The solution technique is based on the taboo structure.

The concept of echelon stock is adopted by Chiu et al.[76] so as to decompose an N-stage lot-sizing problem into N independent single stage subproblems. Each subproblem is represented by the topology structure used by Afentakis et al. (Ref.15). A dynamic programming algorithm is developed to obtain the optimal solution. This algorithm is based on a pointer method and can be easily extended to capacitated problems.

The lot-size models are well known models in operational research. Especially the multiechelon, lot-size inventory problems have received much attention in the literature. Richter and Voros[77] examined the stability of a schedule. The stability region of a schedule means the set of cost inputs having the same production plan for a demand series. For the singlelevel lot-sizing stability problem, it has been pointed out that the stability region is a convex cone. Omitting the need for strong assumptions, it is shown that this convex cone property can be extended to more general multi-level problems with certain cost functions. Analysing the structure of an optimal schedule, it is also shown that this production plan can be expressed by a regeneration matrix. Nine lot-sizing rules are evaluated by Choi et al.[78] using simulation where two sets of demand patterns are used. Lot-sizing rules are: lot-for-lot economic order quantity, periodic order quantity, least unit cost, least total cost, part-period balancing, Silver-Meal, Wagner-Whitin, and economic order quantity-economic production hybrid. The analysis shows that the periodic order quantity rule performed best in the majority of test cases. The part-period balancing, least total cost, and least unit cost rules generally ranked in the upper half while the other rules are generally ranked in the lower half. Hsu and El-

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(76) Chiu, H.N., and T.M. Lin, "An Optimal Model and a Heuristic Technique for Multi-Stage Lot-Sizing Problems: Algorithms and PerformanceTests", Engineering Costs and Production Economics Vol.16 No.2,1989, pp.151 -160.
(77) Richter, K, and J. Voros "On the Stability Region for Multi-Level Inventory Problems", European Journal of Operational Research Vol.41, No.2, pp 169-173.
(78) Choi, R.H., E.M. Malstrom, and R.D. Tsai, "Evaluating Lot-Sizing Methods in Inventory Systems by Simulation", Production and Inventory Management, Vol.29, No.4,1988, pp.4-11.

Najdawi(79) examined four safety stock planning methods and five lotsizing rules under three levels of forecasting error. The total costs of various strategies at each level of forecasting error are compared. Gupta and Keung(80) examined the multi-stage lot-sizing models that assume constant demand, time-varying demand, and a rollinghorizon.

The problem of determining the approximate timing and quantities of shipments in a multiitem, multi-stage inventory distribution system is considered by Roundy(81). Several different items are stocked at each of a number of different locations. External demand for each of the items takes place at a constant, location-dependent rate. All demands must be met without back orders and stockouts. The costs that are to be minimised are linear holding and order costs. The total order cost incurred at any given time is a function of the set of item ordered at that time, but not of the quantities of the items ordered. A heuristic algorithm is presented that is guaranteed to produce policies that are within 2 % of optimal. Joneja(82) presented a simple single-pass approximation algorithm which is proved that, in the worst case, the performance of the algorithm is uniformly bounded where the bounds are independent of the size of the system, the cost parameters, and the demand pattern. Changes in the time horizon or in the demand forecasts far in the future have little effect on the production policy in the current time period. Hence, the algorithm has the advantage of controlling nervousness of the generated policy. Bregman et al.(83) developed a heuristic algorithm for solving the multiechelon inventory control problem that balances transportation costs against the cost of holding inventory in a multi-echelon environment. Same authors(84) developed and tested another heuristic procedure for

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(79) Hsu, J.I., and M.K. El-Najdawi, "Integrating Safety Stock and Lot-Sizing Policies for Multi-Stage Inventory Systems Under Uncertainty", Journal of Business Logistics, Vol.12, No.2,1991, pp.221-238.
(80) Gupta, Y.P., and Y. Keung, "A Review of Multi-Stage Lot-Sizing Models", International Journal of Operations and Production Management, Vol 10, No.9, 1990, pp.57-73.
(81) Roundy, R.O., "Computing Nested Reorder Intervals for Multi-Item Distribution Systems", Operations Research,Vol.38, No.1,1990, pp.37-52.
(82) Joneia, D., "Multi-Echelon Assembly Systems with Nonstationary Demands: Heuristics and Worst Case Performance Bounds", Operations Research, Vol.39, No.3,199 1, pp.512-518.
(83) Bregman, R.L., L.P. Ritzman and L.J. Krajewski, "A Heuristic Algorithm for Managing Inventory Multi-Echelon Environment, Journal of Operations Management Vol.8, No.3,1989, pp. 186-208.
(84) Bregman R.L., L.P.Ritzman, and L.J.Krajewski, "A Heuristic for the Control of Inventory Echelon Environment with Transportation Costs and Capacity

controlling finished goods in a multi-echelon environment with significant transportation costs and capacity limitations on both storage and transportation resources.

A model that is a direct generalisation of the initial work of Clark and Scarf (Ref.28) is presented by Rosling(85). In the model, an assembly system that has random demands and proportional costs of production and stock holding activities is considered. Under certain assumptions it is shown that the assembly system can be remodelled as a series system. Inderfurth(86) develops a procedure for determining the optimal size and distribution of safety stocks in a general serial or divergent production and distribution process ruled by a base-stock control policy. A dynamic programming algorithm for solving the safety stock optimisation problem is presented.

Some other specific topics that examined by the researchers are price discounting in multiechelon distribution systems(87), multi-echelon (R, S) inventory models(88), numerical evaluation for multi-echelon systems(89), and correlated demands (both across warehouses and in time) in multi-echelon inventory systems.(90)

IV. EPILOGUE

Looking at the whole work on the multi-echelon inventory control problem, it is observed that the problem as formulated has been solved. Additional research on refinements and extensions of previously developed approaches seems to go on. From our point of view, simulation technique and heuristic approaches are the key

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Limitations", Journal of the Operational Research Society, Vol.41, No.9, 199 pp.809- 820.

(85) Rosling, K., "Optimal Inventory Policies for Assembly Systems Under Random Demands", Operations Research, Vol.37, No.4,1989, pp.565-579.

(86) Inderfurth, K., "Safety Stock Optimization in Multi-Stage Inventory Production Economics, Vol.24, No.1-2,1991, pp.103-113.

(87) Jaikmar, R., and V.K. Rangan, Price Discounting in Multi-Echelonh Distribution Systems", Engineering Costs and Production Economics, Vol 19, No. 1-3,1990, pp. 103-113.

(88) Sinha, D., and K.F. Matta, "Multiechelon (R, S) Inventory Model", Decision Sciences, Vol.22, No.3, 1991, pp.484-499.

(89) Van Houtum, G.J., and W.H.M. Zijm, "Computational Procedures for Stochastic Multi-Echelon Production Systems", International Journal of Production Economics, Vol.23, No.1-3, 1991, pp.223-237.

(90) Nesim, E., H.H. Warren, and S. Nahmias, "Optimal Centralised Ordering Policies in Multi-Echelon Inventory Systems with Correlated Demands", Management Science, Vol. 36, No.3, 1990, pp.381-392.

tools of operational research to investigate complicated multi-echelon system. Wagner(91) has said

> "...at this time I know of no approach, other than computer simulation that give equally reliable estimates of the service and inventory levels for a proposed design for a multi-product, multi-warehouse system."

Peterson and Siver(92) have also said

> "...it will be clear that analytical decision rules for hierarchical inventory systems involving probabilistic demand are out of question, at least for the present. Therefore, simulation models have and will continue to be used to develop control guidelines for such situations."

Especially, decision support systems which exploit the recent developments in the area of artificial intelligence and information technology, are and will be on the rise.

Although there are many important control problems in the multi-echelon inventory area, this has never had any real impact on the development of control theory, which has traditionally been directed towards more technically oriented dynamical systems. However, in the short run results from servomechanism theory can be used for multi-echelon inventory control purposes in the same sense as the other techniques described previously. As Hadley and Whitin(93) assert

> "...most of the analytical work in this area has been done in electrical engineering in the course of studying electromechanical servomechanisms. Little has been done to apply the results to inventory system, or to modify the analytical results already available to make them more useful in the analysis of inventory problems."

This area needs further investigation. The application of servomechanism theory in the analysis and design of multi-echelon inventory systems is on the research agenda of the author.

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(91) Wagner, H.M., "The Design of Production and Inventory Systems for Multiwarehouse Companies", Operations Research, Vol.22, 1974, pp.278-291.
(92) Peterson, R., and E.A. Silver, Decision Systems for Inventory Management and Production Planning, Wiley/Hamilton, 1979.
(93) Hadley, G., and T.M. Whitin, "A Review of Alternative Approaches to Inventory Theory", RM-4185-PR, US Goverment Research Reports, Document No.AD-605843, The RAND Corp., Santa Monica, Calif., 1964.