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Periodic Solutions of Some Higher Order Rational Difference Equations

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Bazı Yüksek Mertebeden Rasyonel Fark Denklemlerinin Periyodik Çözümleri

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Abstract

In this paper, the periodicity of the solutions of some higher order difference equations

$$w_{n+1}=\frac{w_n w_{n-k-1}}{w_{n-k}}, \quad n,k\in\mathbb{N}_0$$

are investigated where the initial conditions are arbitrary real numbers such that the denominator is always nonzero. While investigating the periodicity of this higher order equation, first the special cases of the equation were considered then a generalization was given for the periodicity of the solution of this higher order difference equation. Moreover, some numerical examples are given to illustrate our theoretical results.

Keywords *Periodicity; Difference equations; Positive solutions; P-cycle.*

Öz

Bu makalede, başlangıç koşulları paydanın her zaman sıfırdan farklı olacağı şekilde keyfi reel sayılar olmak üzere, bazı yüksek mertebeden

$$w_{n+1}=\frac{w_n w_{n-k-1}}{w_{n-k}}, \quad n,k\in\mathbb{N}_0$$

fark denklemlerinin çözümlerinin periyodikliği araştırılmıştır. Bu yüksek mertebeden denklemin periyodikliği araştırılırken önce denklemin özel durumları ele alınmış, daha sonra bu yüksek mertebeden fark denkleminin çözümünün periyodikliği için bir genelleme yapılmıştır. Ayrıca teorik sonuçlarımızı göstermek için bazı sayısal örnekler verilmiştir.

Anahtar Kelimeler *Periyodiklik; Fark denklemleri; Pozitif çözümler; P-devir.*

1. Introduction and Preliminaries

Recently, there has been a significant increase and interest in studies on the theory and applications of difference equations. Difference equations are widely used in a variety of mathematical models related to biology, physics, economics, genetics, population dynamics, medicine, physiology, and ecology. So, there are many articles and investigations in this field (Taşdemir et al. 2022, Okumuş and Soykan 2019, Taşdemir 2021, Göcen and Güneysu 2018). Obviously, a great deal of research has also been done on the dynamical behaviors of rational difference equations. For instance, the periodicity of difference equations is a kind of this behavior. Studies concerning to this subject are numerous (Balibrea and Linero 2007, Göcen and Cebeci 2018, Göcen and Cebeci 2022). Especially, studying higher order difference equations is even more interesting. Now, we can give some basic definitions and informations needed throughout this article. For more information, see Camouzis and Ladas 2007, Kulenović and Ladas 2001.

Let J be some interval of real numbers and let $g: J^{k+1}\to J$ be a continuously differentiable function.

A $(k+1).th$ order difference equation is defined as

$$w_{n+1}=g(w_n,w_{n-1},\dots,w_{n-k}),\ \text{n = 0, 1,....} \quad (1)$$

A solution of Eq.(1) is a sequence $\{w_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1) for all $n \geq 0$.

Definition 1. A solution of Eq.(1) that is constant for all n ≥ −k is called an equilibrium solution of Eq.(1). If

$$w_n=\overline{w}, \quad for\ all\ n\geq -k$$

is an equilibrium solution of Eq.(1) then $\overline{w}$ is called an equilibrium point, or simply an equilibrium of Eq.(1).

Definition 2. (Periodicity) A solution $\{w_n\}_{n=-k}^{\infty}$ of Eq.(1) is said to periodic with periodic p if there exists an integer $p\geq 1$ such that

$$w_{n+p}=w_n\ for\ all\ n\geq -k. \quad (2)$$

The solution is having a prime period p if p is the smallest positive integer for which Eq.(2) holds. If there is an integer $N\geq -k$ such that $\{w_n\}_{n=N}^{\infty}$ exhibits a periodicity with period p, then a solution $\{w_n\}_{n=-k}^{\infty}$ of Eq.(1) is defined as eventually periodic with period p, that is

$$w_{n+p}=w_n\ for\ all\ n\geq \text{N}.$$

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2. Literature Review

In this section, firstly we consider the periodicities of the following rational difference equations, see Grove and Ladas (2005). The periodicity of some equations is shown below, see Table 1.

Table 1. Periodicity of some difference equations

Difference Equation		Period
$w_{n+1}=\frac{1}{w_n}$,	$n\in\mathbb{N}_0$	2
$w_{n+1}=\frac{1}{w_n w_{n-1}}$,	$n\in\mathbb{N}_0$	3
$w_{n+1}=\frac{1}{w_{n-1}}$,	$n\in\mathbb{N}_0$	4
$w_{n+1}=\frac{w_n}{w_{n-1}}$,	$n\in\mathbb{N}_0$	6
$w_{n+1}=\frac{w_n w_{n-2}}{w_{n-1}w_{n-3}}$,	$k\in\mathbb{N},\quad n=0,1,\ldots,$	10
$w_{n+1}=\frac{1}{w_n w_{n-1}w_{n-2}\ldots w_{n-k}}$,	$k\in\mathbb{N},\quad n=0,1,\ldots,$	k+2
$w_{n+1}=\frac{w_n w_{n-2}\ldots w_{n-2k}}{w_{n-1}w_{n-3}\ldots w_{n-(2k+1)}}$,	$k\in\mathbb{N},\quad n=0,1,\ldots,$	4k+6

Amleh et al. (1999) investigated the periodic nature of the equation

$$w_{n+1}=\alpha+\frac{w_{n-1}}{w_n}$$

where the initial values are positive constant, and $\alpha\geq 0$.

Devault et al. (2003) studied the dynamic behavior of the equation

$$w_{n+1}=p+\frac{w_{n-k}}{w_n}$$

in which the initial conditions and p are positive values.

For $p\leq 5$, *Balibrea and Linero (2007) found all the p-cycles having the form*

$w_{n+3}=w_i f(w_j,w_k)$, *where* $i,j,k\in\{n,n+1,n+2\}$ *are pairwise distinct and* $f:(0,\infty)^2\to(0,\infty)$ *is continuous.*

Motivated by all above difference equations, our aim in this study is to investigate the periodicity of the solutions of the following higher order rational difference equation.

3. Main Result

In this part, by increasing the order of the difference equation, we showed that the solutions of the following higher order difference equations

$$w_{n+1}=\frac{w_n w_{n-k-1}}{w_{n-k}} \tag{3}$$

are periodic.

Firstly, we assume that the case $k=1$, *in Eq.(3), that is,*

$$w_{n+1}=\frac{w_n w_{n-2}}{w_{n-1}}. \tag{4}$$

Theorem 3.

Every solution of the following third-order rational difference equation

$$w_{n+1}=\frac{w_n w_{n-2}}{w_{n-1}}$$

is periodic with period four.

Proof.

Suppose that the initial conditions are non-zero real numbers denoted by

$$w_{-2}=a,\ w_{-1}=b\ and\ w_0=c.$$

Then, we can easily obtain

$$w_1=\frac{w_0 w_{-2}}{w_{-1}}=\frac{ca}{b},$$
$$w_2=\frac{w_1 w_{-1}}{w_0}=\frac{\frac{ca}{b}b}{c}=a,$$
$$w_3=\frac{w_2 w_0}{w_1}=\frac{ac}{\frac{ca}{b}}=b,$$
$$w_4=\frac{w_3 w_1}{w_2}=\frac{b\frac{ca}{b}}{a}=c=w_0,$$
$$w_5=\frac{w_4 w_2}{w_3}=\frac{\frac{ca}{b}.b}{c}=\frac{ca}{b}=w_1$$
$$w_6=\frac{w_5 w_3}{w_4}=\frac{\frac{ca}{b}b}{\mathrm{c}}=a=w_2,$$
$$\vdots$$

Therefore, the solution of Eq.(4) is period-4 sequence, $\{a,b,c,\frac{ca}{b},a,b,c,\frac{ca}{b},\ldots\}$.

In the following example, let us show that every solution of Eq.(4) is periodic with period four.

Example 4.

Suggest that the difference equation (3) with $\mathrm{k}=1$ and the initial conditions $\mathrm{x}_{-2}=0.2$, $\mathrm{x}_{-1}=0.4$, $\mathrm{x}_0=-0.1$. Then, every solution is periodic with period four.

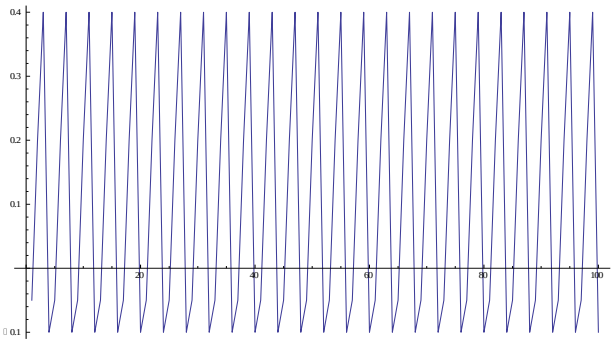


Figure 1. The periodic solutions of Eq.(3) with $k=1$.

Now, let $k=2$, in Eq.(3), that is,

$$w_{n+1}=\frac{w_n w_{n-3}}{w_{n-2}}. \tag{5}$$

Theorem 5.

Every solution of the following rational fourth-order difference equation

$$w_{n+1}=\frac{w_n w_{n-3}}{w_{n-2}}$$

is periodic with period six.

Proof.

Let us prove this theorem differently from the proof of Theorem 3. Form the Eq.(5), we have

$w_{n+6} = \frac{w_{n+5}w_{n+2}}{w_{n+3}}$.

Then, if we substitute w_{n+5} obtained from Eq.(5) in the above equation, it follows that

$w_{n+6} = \frac{w_{n+5}w_{n+2}}{w_{n+3}} = \frac{\frac{w_{n+4}w_{n+1}}{w_{n+2}}\, w_{n+2}}{w_{n+3}} = \frac{w_{n+4}w_{n+1}}{w_{n+3}}$.

Similarly, we write w_{n+4} as derived from Eq.(5), then we obtain

$w_{n+6} = \frac{w_{n+4}w_{n+1}}{w_{n+3}} = \frac{\frac{w_{n+3}w_n}{w_{n+1}}\, w_{n+1}}{w_{n+3}} = w_n$.

So, we can easily see that $w_{n+6} = w_n$.

Hence, the solution of Eq.(5) is periodic with period six.

The proof is complete. In this proof, we use two substitution w_{n+5}, w_{n+4} (because k=2) in the Eq.(5) respectively. This proof is a special case of the proof of main result, i.e., proof of Theorem 6.

The next theorem can be proved using a similar technique.

In conclusion, the following main theorem states that the difference equation (3) is periodic with period (2k+2).

Theorem 6.

Given a non-negative integer $k \geq 0$, every solution of the following higher order rational difference equation

$w_{n+1} = \frac{w_n w_{n-k-1}}{w_{n-k}}$,

is periodic with period $(2k+2)$.

Proof.

Consider the initial conditions of Eq.(3) are non-zero real numbers $w_{-k-1}, w_{-k}, w_{-k+1}, \dots, w_0$, then we have to show that $w_{n+(2k+2)} = w_n$. From the Eq.(3), we have

$w_{n+(2k+2)} = \frac{w_{n+2k+1}w_{n+k}}{w_{n+k+1}}$.

Then, if we substitute w_{n+2k+1} obtained from Eq.(3) in the above equation, we can see that

$= \frac{\frac{w_{n+2k}w_{n+k-1}}{w_{n+k}}\, w_{n+k}}{w_{n+k+1}}$

$= \frac{w_{n+2k}w_{n+k-1}}{w_{n+k+1}}$.

Similarly, if we take the equivalent of w_{n+2k} in Eq.(3) and write it in above equation, we have

$= \frac{\frac{w_{n+2k-1}w_{n+k-2}}{w_{n+k-1}}\, w_{n+k-1}}{w_{n+k+1}}$

$= \frac{w_{n+2k-1}w_{n+k-2}}{w_{n+k+1}}$.

Additionally, we get w_{n+2k-1} using the Eq.(3) we obtain

$= \frac{\frac{w_{n+2k-2}w_{n+k-3}}{w_{n+k-2}}\, w_{n+k-2}}{w_{n+k+1}}$

$= \frac{w_{n+2k-2}w_{n+k-3}}{w_{n+k+1}}$.

Again, if we write the equivalent of w_{n+2k-2} in the Eq.(3), it follows that

$= \frac{\frac{w_{n+2k-3}w_{n+k-4}}{w_{n+k-3}}\, w_{n+k-3}}{w_{n+k+1}}$

$= \frac{w_{n+2k-3}w_{n+k-4}}{w_{n+k+1}}$

$\vdots$

By continuing this operations *k*-times in this way, that is, if we write its equivalent in the main equation instead of the first factor in the numerator and making the necessary simplifications, we have

$w_{n+(2k+2)} = \frac{w_{n+k+1}w_n}{w_{n+k+1}} = w_n$.

Thus, we obtain that every solution of the higher-order difference equation (3) is periodic with period $(2k+2)$.

Hence, the proof is complete.

4. Conclusion

In this work, the periodic solutions of some higher order rational difference equations are investigated. While investigating the periodicity of this higher order rational difference equation, firstly the special cases of the equation were discussed. The periodicity of the equation was examined by increasing the order of the equation step by step. As a result, a generalization was given for the periodicity of this type of difference equations.

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement

Author: Conceptualization, investigation, methodology and software, visualization and writing – original draft.

Declaration of Competing Interest

The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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