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RESEARCH ARTICLE / ARAŞTIRMA MAKALESİ

A New Approach for Time Series Prediction: Fuzzy Regression Network Functions

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Abstract
The fuzzy regression functions (FRFs) constructs a comprehensive model by combining a series of linear functions based on the inputs. However, the relationship between input and output is not always purely linear. The approach presents novel FRFs with nonlinear structures based on neural networks, combining the strengths of both computational models and fuzzy logic. The proposed model generates membership values by fuzzifying real-valued time series observations, utilizing the fuzzy C-means clustering algorithm. Inputs are then created from the real-valued lagged observations and transformed membership values. A set of feed-forward neural networks, corresponding to the number of fuzzy sets, produces outputs as nonlinear functions of the inputs. These outputs are combined based on the membership values, representing the degree to which each time point belongs to the respective fuzzy sets, to generate the final predictions. The proposed prediction model is referred to as Fuzzy Regression Network Functions (FRNFs). The prediction performance of FRNFs is investigated across several criteria by implementing it in various real-world time series datasets.
Keywords: Fuzzy regression network functions, Artificial neural networks, Type-1 fuzzy functions, Time series prediction

I. INTRODUCTION

Accurate prediction of any event and/or phenomenon plays an important role in our daily life. Especially under uncertain situations, prediction exact values is quite tough. This is because of the uncertainty and non-linearity that real-world events often involve. The way to obtain satisfactory estimation results is possible with the use of appropriate and competent estimation tools. From this perspective, fuzzy time series methods are capable of giving superior prediction performance owing to their effective approach to the uncertainty contained in the time series. Also, unlike traditional time series methods, fts methods do not require strict assumptions. All these advantageous situations have increased the interest in this method and make them attractive and popular for different areas such as information technologies, medicine, business, finance and different engineering fields. Fuzzy time series, which was first put forward by Song and Chissom [1], is basically based on Zadeh's [2] fuzzy set theory. Fuzzy time series prediction models basically consist of three stages: fuzzification of observations, identifying the fuzzy relationships, and defuzzification. In the fuzzification phase, the relevant time series with real observations is converted into a fuzzy time series with fuzzy observations by using different fuzzification methods. Fuzzy relations between observations of fuzzy time series are modelled at the determination of fuzzy relations stage. Fuzzy predictions obtained from fuzzy relations modelling stage are transformed into real predictions with an approach adopted in the fuzzification phase and presented to the decision maker. When the fuzzy time series literature is examined, it is seen that different approaches have been put forward for all three stages of the analysis process with the aim of improving the prediction performance of the methods. The methods introduced in the early studies, use partition of universe of discourse in the fuzzification phase. Song and Chissom [1, 3, 4] and Chen [5, 6] determined interval lengths based on fixed and subjective decisions. Huarng determined the interval lengths with a mean and distribution-based approach, and the influence of subjective judgments was removed [7]. In addition, Eğrioğlu et al. [8, 9] introduced different approaches in which fixed interval lengths are determined within the optimization process.

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Huarng and Yu used a ratio-based approach that produces varying interval lengths in universe of discourse stage, with the aim of improving prediction performance, especially in predicting time series with trend [10]. On the other hand, Yolcu et al. developed this idea further and suggested the optimization of the ratio [11]. Panigrahi and Behera developed a mean-based model [12]. In addition to these approaches; while Cheng and Chung and Lee et al. [13, 14] used genetic algorithm, Kuo et al. [15, 16], Davari et al. [17], Hsu et al. [18], Aladağ [19], Cagcag Yolcu and Lam [20] performed the fuzzification process with particle swarm optimization algorithm. Moreover, Yolcu et al. [21], Cai et al. [22], and Jiang et al. [23], to transformed real time series into fuzzy time series, utilized artificial bee colony, ant colony, and harmony search algorithms, respectively. In addition, Cheng et al. [24], Li et al. [25], Alpaslan and Cagcag Yolcu [26], Eğrioğlu et al. [27], Wei et al. [28], Cheng et al. [29], Sun et al. [30], Wang and Liu [31], and Cagcag Yolcu and Alpaslan [32] preferred approaches based on fuzzy means clustering instead of universe of discourse in the fuzzification phase.

In the determination of fuzzy relationships stage, which the internal relationship of the fuzzy time series is determined, Song and Chissom [1, 3, 4] used approaches based on matrix operations, while Sullivan and Woodall [33] used transition matrices consisting of Markov chains. On the other hand, Kocak [34], Cheng et al. (2016) [29], and Kocak [35] determined fuzzy relationships with tables. With the widespread use of artificial neural networks (ANNs), Huarng and Yu [36], Aladag et al. [37], Kocak et al. [38], Wei et al. [28], Wang and Xiong [31], Chen and Chen [39], Arslan and Yolcu [40] determined fuzzy relationships with ANN models with different structures. Although the usage of ANN had become a popular tool, excessive hidden layer neuron number was an issue to be solved. In this direction, Cagcag Yolcu [41] determined fuzzy relationships with a PSO-trained single multiplicative neuron model ANN which does not have a structure problem. In addition to all these studies, Yu and Huarng [42], Alpaslan and Cagcag [26], Alpaslan et al. [43] and Yolcu et al. [44] proposed prediction models that consider the membership degrees of the observations in the determination of fuzzy relations with ANNs. Moreover, Arslan [45] proposed a gated recurrent unit network-based fuzzy time series forecasting model. Although centroid method was preferred for the last stages of fuzzy time series analyses process, Cheng et al. [46] and Aladag et al. [47] utilized adaptive expectation method for the defuzzification stage.

All these studies have shown that; the performances of fuzzy time series prediction models are highly dependent on each of these stages and can be improved with some changes in these stages. Apart from fuzzy time series models, inference-based systems which is a kind of nonlinear mapping that derives its output from fuzzy reasoning and a group of fuzzy if-then rules have been also become popular. ANFIS algorithm the fuzzy-logic based model has been widely used for the prediction of time series. In the literature, Sarıca et al. [48], Catalao et al. [49], Chang [50], Cheng et al. [51, 52], Ho and Tsai [53], and Pousinho et al. [54] utilized ANFIS in prediction problems.

On the other hand, T1FFs, which form the basis idea of this paper, have recently become more popular as a FIS approach due to their simplicity and rule-free structure. Turksen [55] first introduced a fuzzy functions approach called T1FF, for the solution of regression and clustering problems, and this approach was also used for time series prediction with the simultaneous use of time series within the framework of a regression logic. To get better prediction performance, Aladag et al. [56] took into consideration time series' lagged variables as covariates. After that Aladag et al. [57] proposed another prediction method which the lagged variables of theT1FFs' inputs were determined through binary particle swarm optimization. Tak et al. [58] used T1FFs, taking into account the moving average (MA) model, in their paper. Moreover, although Tak [59] proposed the meta-probabilistic fuzzy function approach, in this approach Tak did not essentially build a prediction model but instead provides a perspective based on the construction of fuzzy functions to combine predictions. In addition to these, several studies using T1FFs have been suggested by Goudarzi [60] and Zarandi [61]. On the other hand, Yalaz and Atay [62] used fuzzy linear regression based on simple membership function and fuzzy rule generation technique for time series data.

When the literature examined in detailed, prediction tools based on artificial intelligence and fuzzy logic are widely used to provide a service for time series prediction. However, accurately predicting time series is a challenging task due to the complex and often chaotic relationships they may contain. FRFs approach, by using some transformation of the memberships as well as the real values of series, create a model with more information. It is important to note that FRFs construct a holistic model by combining a set of linear functions based on the inputs. The relationship between input and output is not always simply linear. This situation is one of the gaps in the fuzzy regression functions literature that needs to be filled. In addition, although each fuzzy function was formed from a multiple linear regression model in the FRF studies in the literature, the basic assumptions of this model were not examined and the validity of the model was not checked. So, the validity of the presented model is an important point and needs to be checked. This study aims to address these gaps by developing fuzzy regression functions with a nonlinear structure, built on feed-forward neural networks. In this paper, as a new approach, Fuzzy Regression Network Functions which capable of modelling nonlinear relationships between inputs and outputs is introduced. In the proposed approach, each fuzzy function is constructed with FFNN instead of a linear regression model, and nonlinear matching is achieved. Moreover, due to the structure of the proposed FNF approach and used FFNN as a fuzzy function, strict assumptions and their examination are not required. In addition, it has been shown over many real time series that the proposed approach has superior prediction performance due to its ability to provide nonlinear matching.

The rest of the paper is fictionalised as follows: The second section outlines the motivation behind this study and highlights its key contributions. With the third section FCM and Type 1 Fuzzy Regression Functions are given. The fourth chapter presents the methodology, including the proposed new approach with its detailed features. The comprehensive results of implementations are investigated in section five with comparatively comments and evaluations. The findings and conclusions are presented and discussed in the sixth section. Finally, future work and limitations are presented in the last section.

II. MOTIVATION & CONTRIBUTIONS

Time series prediction methods aim to accurately forecast events and/or extend identified patterns from historical data. The processes that generate time series data and their underlying system models are often complex, making it difficult to predict precise future values. Accurate and unbiased predictions of data sets, including almost all-time series, generated by such systems cannot always be obtained using well-known linear techniques. Therefore, more advanced algorithms developed for time series prediction are often required. Recently, alternative methods have been utilized for time series prediction. And these models generally based on fuzzy set theory or computational based approaches. uncertainty Although computational models perform exceptionally well in capturing nonlinear patterns due to their adaptability, they do not incorporate measures for uncertainty.

From this perspective, an approach has been proposed in this study that will both shed light on uncertainty and preserve the high adaptability of computational based methods. The main contributions of the proposed Fuzzy Regression Network Functions to the literature can be summarized item by item as follows;

- The proposed Fuzzy Regression Network Functions (FRNF) offer a more realistic approach, by using some transformation of the memberships as well as the real values of series, and also create a model with more information (to compare fuzzy-based FTS models).
- Fuzzy inference systems are generally rule-based and often need expert opinion. Although T1FRF's approach has handled this problem, the generated functions are linear in terms of parameters. However, many real-life problems are not linear. So, the proposed FRNFs use FFNNs to model nonlinear relationships contained in the time series (to compare fuzzy-based FTS and T1FRF models).
- Although each fuzzy function is composed of a multiple linear regression model in T1FRF studies in the literature, the basic assumptions of this model have been ignored. The model's validity has not been assessed. From this point of view, proposed FRNFs do not need strict assumptions existing in linear regression model to satisfy (to compare FRF models).
- FRNFs combine the benefits of both computational-based and fuzzy-based models. (to compare single fuzzy-based and neural network-based models).
- Moreover, the proposed FRNFs perform exceptionally well, furthering the prediction accuracy for almost all time series datasets used in this study.

III. PRELIMINARIES

3.1. Fuzzy C-means

Fuzzy C-means algorithm is one of the fuzzy partitional clustering algorithm which was proposed by Bezdek [63]. The FCM algorithm employs fuzzy membership to assign a degree of belonging to each class. And, the data is divided into fuzzy sets by minimizing the sum of squared errors within the groups. The FCM algorithm uses the membership values and Euclidean distance to compute the objective function. And the form of the objective function tried to be minimized is as follows:

$$J_m(X,V,U)=\sum_{i=1}^{c}\sum_{j=1}^{N}u_{ij}^{\beta}\,d^2(x_j,v_i) \quad \textbf{(1)}$$

Here, u_{ij} represents membership value, v_i shows the cluster centres n symbolizes the number of variables. On the other hand β is weighting exponent (β > 1) which defines the fuzziness of the resulting clusters and $d(x_j,v_i)$ is the distance measure between the observation and the cluster center. J_β is tried to be minimized under the constraints given below.

$$\begin{gathered} 0 \le u_{ij} \le 1\,, \forall i,j \\ 0 \le \sum_{j=1}^{n} u_{ij} \le n\,, \quad \forall i \\ \sum_{i=1}^{c} u_{ij} = 1\,, \qquad \forall j \end{gathered} \quad \textbf{(2)}$$

The minimization [2] of the objective function process of J_β is performed iteratively. In this process the values of u_{ij} and v_i are updated with the given formulas.

$$v_i = \frac{\sum_{j=1}^{n} u_{ij}^{\beta} X_j}{\sum_{j=1}^{n} u_{ij}^{\beta}} \quad \textbf{(3)}$$

$$u_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{d(X_j, v_i)}{d(X_j, v_k)}\right)^{2/(\beta-1)}} \quad \textbf{(4)}$$

3.2. Type 1 Fuzzy Regressions

Type-1 fuzzy regression functions, proposed by Turksen [64], works based on a combination of a multiple linear regression model and FCM. T1FRFs use as many functions as the number of fuzzy sets. In the regression models created for each fuzzy set, in addition to the original time series observations, inputs consist of the membership values of the relevant fuzzy set and some functions of these memberships. The outputs/predictions produced by each regression equation are weighted by their corresponding membership values and converted into final predictions. Let c represents the number of fuzzy sets. Thus, the regression function for each fuzzy set is given in equation (5).

$$Y^{(i)} = X^{(i)} \beta^{(i)} + \varepsilon^{(i)}, \qquad i = 1,2,\cdots,c \quad \textbf{(5)}$$

In that case, the explanatory (inputs) and dependent (targets) variables are given in equations (6) and (7).

$$X^{(i)} = \begin{bmatrix} u_{i1} & u_{i1}^2 & exp(u_{i1}) & log\left(\frac{1-u_{i1}}{u_{i1}}\right) & x_{11} & x_{21} & \cdots \\ u_{i2} & u_{i2}^2 & exp(u_{i2}) & log\left(\frac{1-u_{i2}}{u_{i2}}\right) & x_{12} & x_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ u_{iN} & u_{iN}^2 & exp(u_{iN}) & log\left(\frac{1-u_{iN}}{u_{iN}}\right) & x_{1N} & x_{2N} & \cdots \end{bmatrix} = 1,2,\cdots,c \quad \textbf{(6)}$$

$$Y^{(i)} = [y_1 \quad y_2 \quad \cdots \quad y_N]^T \quad \textbf{(7)}$$

OLS (ordinary least squares) is used to estimate each T1FRF. Thus, the estimated function becomes as in equation (8).

$$\hat{Y}^{(i)} = X^{(i)} \widehat{\boldsymbol{\beta}}^{(i)}, \quad i = 1,2,\cdots,c \quad \textbf{(8)}$$

Here, $\widehat{\boldsymbol{\beta}}^{(i)}$ is obtained via OLS with the following matrix operations.

$$\widehat{\boldsymbol{\beta}}^{(i)} = \left(X^{(i)\prime} X^{(i)}\right)^{-1} X^{(i)\prime Y^{(i)}}, i = 1,2,\cdots,c \quad \textbf{(9)}$$

The outputs produced by each fuzzy function for each data point are weighted with their corresponding membership values and converted into final predictions as in equation (10).

$$\hat{y}_k = \frac{\sum_{i=1}^{c} \hat{y}_{ik} u_{ik}}{\sum_{i=1}^{c} u_{ik}}, \qquad k = 1,2,\cdots,N \quad \textbf{(10)}$$

IV. THE PROPOSED FRNFs

Although ANNs are commonly utilized for time series prediction, when determining relationships, the inevitable, uncertainty in the data structure is not taken into account in a meaningful way. However, this uncertainty reveals both the existence of fuzzy relations and the necessity of modelling them. At this point, even though T1FRFs have significantly advanced the literature in this sense, it produces outputs as a linear function of the inputs, since each fuzzy function consists of a multiple linear regression model. In short, it contains a linear model in terms of parameters. However, many time series contain nonlinear relationships. From this context, in this study, owing to the superior adaptability of artificial neural networks to data and their capacity to model nonlinear relationships, a superior prediction tool has been introduced for time series prediction. In the study, the introduced new approach' inputs are consisted of not only the memberships produced by FCM but also some functions of these memberships and crisp lagged variables as well. In this approach, each regression function consists of an FFNN. In this study a prediction tool that takes the advantage of FRF by converting more information, obtained from membership transformations, into inputs and has the ability to model nonlinear relationships using FFNNs as a polyaromatic tool is proposed.

In the light of this information, the algorithm of the proposed approach is summarized with some steps:

Step 1 *All the necessity parameters of process are determined*

$maxitr$: Maximum iteration number
p : # lagged crisp variables
K : # hidden layer neuron
t_{tr} : Length of the training set
t_{val} : Length of the validation set
t_{test} : Length of the test set
c : # fuzzy sets
β_f : fuzziness index

Step 2 *The dataset fragmentation is performed*

The data set is divided into three parts as training, validation, and testing subsets A block partitioning strategy is employed to partition the dataset to preserve its time-dependent structure.

$$X = [x_{it}], i = 1,2,\cdots,p\,; t = 1,2,\cdots,T-p \quad \textbf{(11)}$$

$$X_{tr} = [x_{it}], i = 1,2,\cdots,p\,; t = 1,2,\cdots,t_{tr} \quad \textbf{(12)}$$

$$X_{val} = [x_{it}], i = 1,2,\cdots,p\,; t = t_{tr}+1, t_{tr}+2,\cdots,t_{tr}+t_{val} \quad \textbf{(13)}$$

$$X_{test} = [x_{it}], i = 1,2,\cdots,p\,; t = t_{tr} + t_{val} + 1, t_{tr} + t_{val} + 2,\cdots,T \quad \textbf{(14)}$$

Here, T represents the number of observations in time series.

Step 3 *A standardization process is applied to the data*
In this step to be able to converts the structure of different datasets into one common format of data, data standardization is performed as below;

$$X^{st} = \frac{{}^{tr}x^{max} - x_t}{{}^{tr}x^{max} - {}^{tr}x^{min}} \quad t = 1,2,\cdots,T \quad \textbf{(15)}$$

Step 4 *FCM clustering algorithm is performed*
FCM is applied over the training set based on c, the number of fuzzy sets. The input matrix for FCM is given in equation (16)

$$X_{tr} = [\boldsymbol{X}_{t-1}, \boldsymbol{X}_{t-2}, \cdots, \boldsymbol{X}_{t-p}] \quad \textbf{(16)}$$

It means that;

$$X_{tr} = \begin{bmatrix} x_p & x_{p-1} & \cdots & x_1 \\ x_{p+1} & x_p & \cdots & x_2 \\ \vdots & \vdots & \cdots & \vdots \\ x_{t_{tr}-1} & x_{t_{tr}-2} & \cdots & x_{t_{tr}-p} \end{bmatrix} \quad \textbf{(17)}$$

Thus, μ_{tk}, $t = 1,2,\cdots,t_{tr} - p; k = 1,2,\cdots,c$ membership values and v_k cluster centers are obtained. During the operation of FCM, the alpha-cut value, which refers to the threshold that defines the degree of membership of data points in clusters, was taken as zero to operate the prediction system without any loss of information by using all the information.

Step 5 *The input matrix is created*
In this step, an input matrix is created for each fuzzy regression network. The input matrix consists of lagged variables, memberships and some transformations of these memberships. In this study, three different transformations are used: quadratic, exponential, and logarithmic. These transformations form the 2nd, 3rd, and 4th columns in the input matrix given as ${}^{(k)}X$ in equation (18) respectively. For the k^{th}, $k = 1,2,\cdots,c$ fuzzy regression network, the input matrix can be given as below.

$${}^{(k)}X = \begin{bmatrix} {}^{(k)}\mu_1 & {}^{(k)}\mu_1^2 & \exp\left({}^{(k)}\mu_1\right) & \log\left(\frac{1 - {}^{(k)}\mu_1}{{}^{(k)}\mu_1}\right) & x_p & x_{p-1} & \cdots & x_1 \\ {}^{(k)}\mu_2 & {}^{(k)}\mu_2^2 & \exp\left({}^{(k)}\mu_2\right) & \log\left(\frac{1 - {}^{(k)}\mu_2}{{}^{(k)}\mu_2}\right) & x_{p+1} & x_p & \cdots & x_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \cdots & \vdots \\ {}^{(k)}\mu_{t_{tr}-p} & {}^{(k)}\mu_{t_{tr}-p}^2 & \exp\left({}^{(k)}\mu_{t_{tr}-p}\right) & \log\left(\frac{1 - {}^{(k)}\mu_{t_{tr}-p}}{{}^{(k)}\mu_{t_{tr}-p}}\right) & x_{t_{tr}-1} & x_{t_{tr}-2} & \cdots & x_{t_{tr}-p} \end{bmatrix} \quad \textbf{(18)}$$

The target values corresponding to these inputs are as in equation (19).

$${}^{(k)}Y = [x_{p+1}, x_{p+2}, \cdots, x_{t_{tr}}]^T \quad \textbf{(19)}$$

The basic structure of the proposed approach is given with Figure 1 as visual.

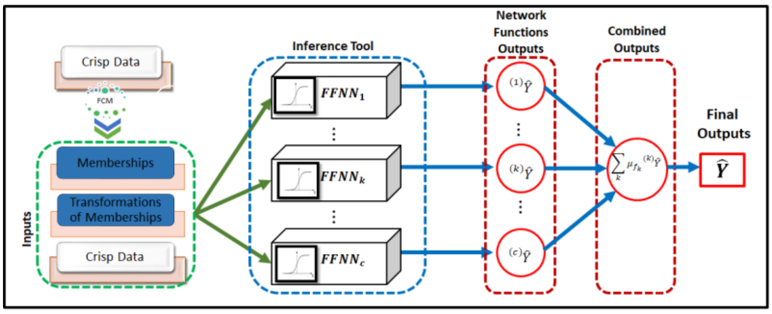


Figure1. The architecture of proposed Fuzzy Regression Network Functions

Step 6 *Training of networks is carried out*
Depending on the potential parameter values, networks are trained for all possible combinations. The training of neural networks is performed with the Levenberg Marquardt backpropagation algorithm. The number of network functions trained for each combination is equal to the fuzzy set number. Fuzzy networks produce outputs (predictions) as a nonlinear combination of their inputs. These outputs are weighted with the membership values of the relevant observation set and thus converted to final outputs.

Step 7 *The best combination of parameters are determined*
Data points of validation sets are transformed into inputs as an out of sample for the trained networks. For

this purpose, firstly, lagged variables are established for the validation set as in equation (20). A vector of target values given in equation (21) is also created for the validation set.

$$X_{val} = \begin{bmatrix} x_{t_{tr}} & x_{t_{tr}-1} & \cdots & x_{t_{tr}-p} \\ x_{t_{tr}+1} & x_{t_{tr}} & \cdots & x_{t_{tr}-p+1} \\ \vdots & \vdots & \cdots & \vdots \\ x_{t_{tr}+t_{val}-1} & x_{t_{tr}+t_{val}-2} & \cdots & x_{t_{tr}+t_{val}-p} \end{bmatrix} \quad \textbf{(20)}$$

$${}^{(k)}_{val}Y = [x_{t_{tr}+1}, x_{t_{tr}+2}, \cdots, x_{t_{tr}+t_{val}}]^T \quad \textbf{(21)}$$

Secondly, membership values are calculated based on cluster centers determined by FCM as in equation (22).

$$\mu_{tk} = \frac{1}{\sum_{l=1}^{c}\left(\frac{d(X_t, v_k)}{d(X_t, v_l)}\right)^{2/(\beta-1)}}, \quad k = 1,2,\cdots,c \quad \textbf{(22)}$$

Moreover, for validation set, memberships, some transformations of memberships, and lagged variables with real values constitute the inputs of the trained neural networks. The fitness function values for the validation set are calculated over the final outputs of the trained neural networks. In this process the mean square error (MSE) criterion has been taken as fitness function. Thus, the combination that gives the lowest RMSE given in equation (23) value for the validation set is determined as the best parameter values.

$$MSE = \sqrt{\frac{1}{t_{val}}\sum_{t=1}^{t_{val}}(Target_t - Forecast_t)^2} \quad \textbf{(23)}$$

Step 8 *With the best combination of parameters, predictions for the test set are obtained.*

Predictions are obtained for the out-of-sample dataset to evaluate the performance of the proposed FNFs. Memberships and input matrix can be created for the test set as similar to the validation set. Predictions of the test set are generated by FRNFs using the best combination of parameter values, and converted into final f predictions via membership degrees.

V. EXPERIMENTAL RESULTS AND DISCUSSION

5.1. Data preparation

The prediction capacity and performance of the proposed FRNFs have been examined with various financial time series applications. These time series are Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX), and Istanbul Stock Exchange (IEX). Table 1 provides a summary of the datasets along with their relevant application features.

The data sets were divided into three parts as raining, validation, and test set. Depending on the potential parameter values, networks were trained for all possible combinations. The best combination of parameters was determined on validation sets. And the performance evaluation was discussed on test sets. The size of the validation and test sets was chosen equally to each other in all implementations. In total, analyses were fulfilled for 26 different data sets.

Tablo 1. The application features and the data sets.

Series No	Time Series	Year	# of Observations	Size of Training Set	Size of Validation & Testing Sets
1		2000	271	177	47
2		2001	244	158	43
3	TAIEX	2002	248	162	43
4		2003	249	163	43
5		2004	250	160	45
6 / 17 / 28		2008	249	163	43
7 / 18 / 29		2009	247	159	44
8 / 19 / 30		2010	250	160	45
9 / 20 / 31		2011	247	159	44
10 / 21 / 32		2012	246	162	42
11 / 22 / 33	TAIEX	2013	244	158	43
12 / 23 / 34		2014	248	162	43
13 / 24 / 35		2015	244	156	44
14 / 25 / 36		2016	242	154	44
15 / 26 / 37		2017	243	157	43
16 / 27 / 38		2018	245	161	42
39 / 40		2009	103 / 103	89 / 73	7 / 15
41 / 42		2010	104 / 104	90 / 74	7 / 15
42 / 44	IEX	2011	106 / 106	92 / 76	7 / 15
45 / 46		2012	106 / 106	92 / 76	7 / 15
47 / 48		2013	106 / 106	92 / 76	7 / 15

Moreover, the grid-search algorithm has been used to determine the best hyper-parameters. The basic hyper-parameters and potential values of them which produced the points of search space in grid-search are given in Table 2.

Tablo 2. The possible values of hyper-parameters.

# fuzzy sets	**# lagged crisp variable**	**# hidden layer neuron**	**Fuzziness Index**	**# model's target**
from 3 to 7	from 2 to 5	from 2 to 5	(1.8, 1.9, 2.0, 2.1, 2.2)	1

5.2. Performance Measure

The prediction performance of the introduced approach, FRNF, have been examined with different evaluation metrics. As in many studies, Root Mean Square Error (RMSE) and Average Absolute Percent Error (MAPE) have been used as performance measures. The formulation of RMSE, the standard deviation of the residuals is commonly used in prediction problems to verify experimental results, is given in equation (24).

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(Target_t - Forecast_t)^2} \quad \textbf{(24)}$$

Although the RMSE criterion is accepted as an effective evaluation criterion in comparing models, it cannot give clear information about the level of accuracy of the predictions produced by the model alone, i.e. whether it produces satisfactory predictive results. From this point of view, in this study, to evaluate the proposed model from different aspects, we also choose the MAPE criterion.

MAPE produces a scale-independent metric that can be used to understand and compare model performance. Also, it measures this accuracy as a percentage regardless of the size of the data. The formulation of MAPE is given in equation (25).

$$MAPE = \frac{1}{T}\sum_{t=1}^{T}\left|\frac{Target_t - Forecast_t}{Target_t}\right| \times 100\% \quad \textbf{(25)}$$

Another method, which shows how well the obtained prediction results are compatible with real observations, is designing a linear regression model to be designed between the predictions and targets. Some properties of the linear regression model can be examined to prove the superior prediction performance of the proposed model. Two key parameters (β and R^2) of a regression model given by equation (26) were used as other metrics to measure the performance of the proposed model.

$$Y_t = \beta\hat{Y}_t + \varepsilon_t \quad \textbf{(26)}$$

Apart from all these metrics, different graphics that present the harmony between the predicted and the target values are also used as visual metrics.

5.3. Implementation Results

The performance of proposed FRNFs has been evaluated on the results obtained from some different financial time series analysis.

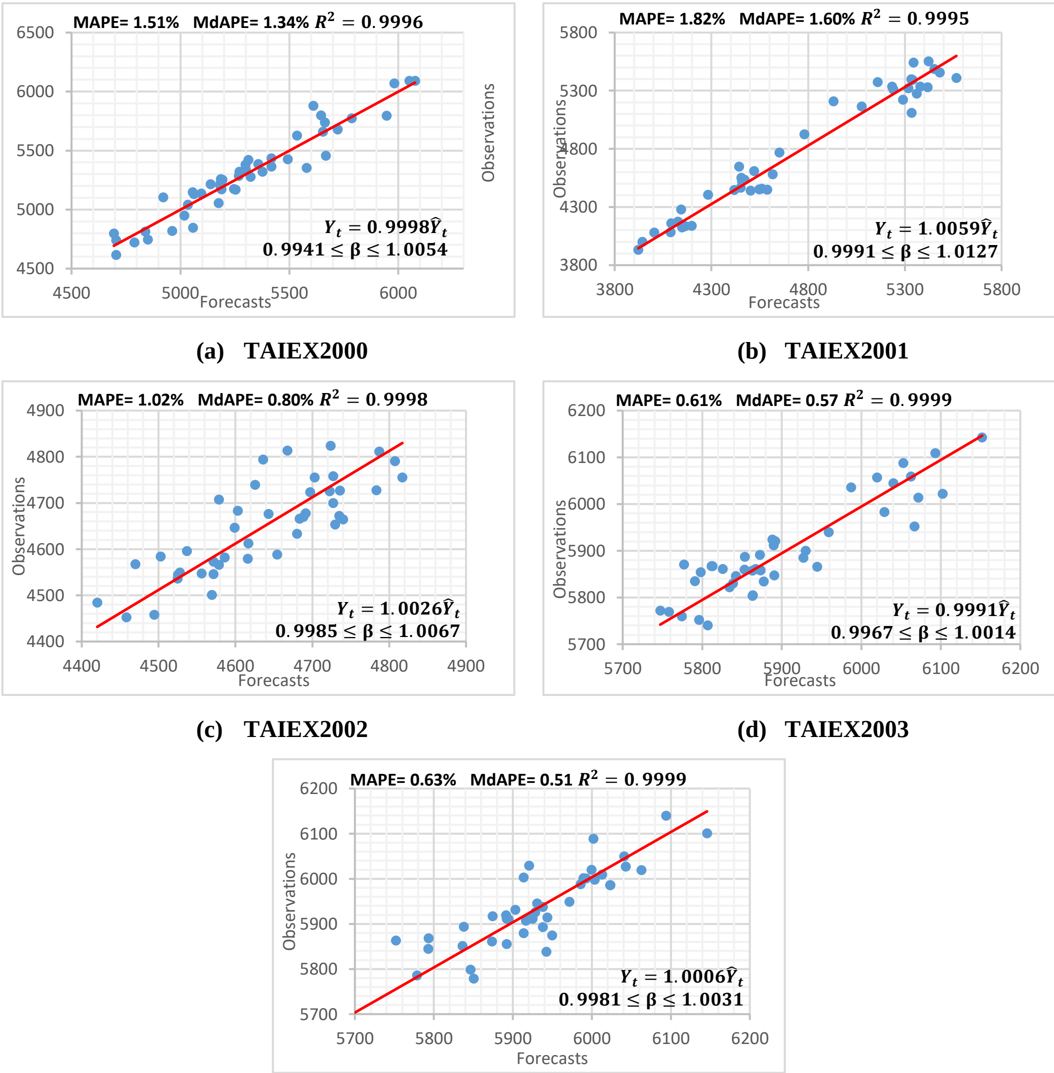
5.3.1. TAIEX Implementations - the daily data sets between 2000-2004 years

First, daily observed TAIEX datasets recorded from 2000-2004 were analysed using the proposed FRNFs. The produced prediction results, in terms of RMSE metric, are summarized in Table 3 together with the results of some other state-of-the-art models in order to provide a comparative perspective. When these results given in Table 3 are evaluated, it is seen that the predictions produced by the proposed FRFNs for TAIEX 2000-2004 datasets have the highest accuracy. The proposed model has superior performance over many fuzzy time series methods. Moreover, the FRNFs method seems to be superior, especially when compared to both some ANN ((CNN and LSTM [65]) and some Type-1 fuzzy regression function approaches ([58, 66, 67]). When the datasets are approached from a holistic point of view, it is seen that the proposed FRFNs perform the best in terms of average and median statistics. When the results presented in Table 3 are evaluated from another perspective, it is seen that the proposed FRNFs improved the prediction performance by approximately 5% to 10% when each TAIEX time series is evaluated separately, compared to the best of the state-of-the-art model available in the literature. Moreover, when the 5 TAIEX time series are considered as a whole, it made approximately 9% progress on average.

The superior performance of the proposed FRNFs, which is numerically proven with the RMSE metric, can also be supported by some visuals presented in Figure 2. For all datasets, as other evidence of the special performance of the proposed prediction tool, the visuals in Figure 2 indicate a strong consistency between the predictions produced by FRNFs and the observed time series points. Again, the parameter values of regression models, which are given with these figures and established between the observed and predicted values of the data points, point to the results in the same direction. The estimation of the regression coefficient ($\hat{\beta}$) and the determination coefficient (R^2) for all TAIEX time series are very close to 1. Also, at the 95% level established for β, all confidence intervals cover 1. These findings can be seen as proof that the proposed FRFNs approach is a prediction tool with a high capability.

Tablo 3. The possible values of hyper-parameters.

Models	Time Series / TAIEX Data Sets					RMSE's	
	2000	**2001**	**2002**	**2003**	**2004**	**Average**	**Median**
[1]	293	116	76	77	82	129	82
[5]	225	116	76	77	82	115	82
[7][1]	473	359	234	247	384	339	359
[7][2]	473	810	116	308	384	418	384
[36]	133	124	82	62	85	97	85
[18]	152	130	84	56	116	108	116
[68]	154	124	93	66	72	102	93
[42]	131	130	80	58	67	93	80
[37]	168	120	76	58	63	97	76
[69]	129	113	67	54	60	85	67
[70]	124	115	71	58	58	85	71
[71]	131	113	66	52	54	83	66
[8]	255	130	84	56	116	128	116
[44]	227	102	66	51	55	100	66
[72]	126	114	65	54	53	82	65
[39]	125	115	65	53	53	82	65
[22]	132	113	60	52	50	81	60
[73]	140	120	77	60	59	91	77
[29]	126	113	63	51	54	81	63
[74]	180	134	81	77	55	105	81
[75]	129	110	60	51	53	81	60
[76]	127	110	62	53	53	81	62
[58]	128	106	65	52	54	81	65
[77]	137	115	66	57	61	87	66
[78]	124	112	63	52	54	81	63
[48]	123	111	66	52	54	81	66
[66]	120	113	63	49	52	79	63
[79]	122	110	54	51	50	77	54
[80]	122	107	64	52	53	80	64
[65]	105	110	60	51	50	75	60
LSTM from [65]	136	101	89	92	70	98	92
[67]	118	104	64	51	52	78	64
[45]	189	104	156	69	53	114	104
The Proposed FRNFs	**100**	**96**	**51**	**44**	**48**	**68**	**51**
Progress Rate (%)	***5***	***5***	***6***	***10***	***4***	***9***	***6***



(a) TAIEX2000 **(b) TAIEX2001**

(c) TAIEX2002 **(d) TAIEX2003**

(e) TAIEX2004

Figure 2. The harmony between the predictions and observations

Moreover, MAPE and MdAPE criteria are also presented in Figure 2 as a measure of performance. The obtained predictions for the TAIEX2000 and TAIEX2001 datasets were produced with MAPE and MdAPE values below 2%. Moreover, for the other 3 TAIEX time series, these values were obtained around 1% or even below 1%. The best values obtained for the parameters are presented in Table 4.

Table 4. The best values obtained for the parameters for TAIEX2000-2004

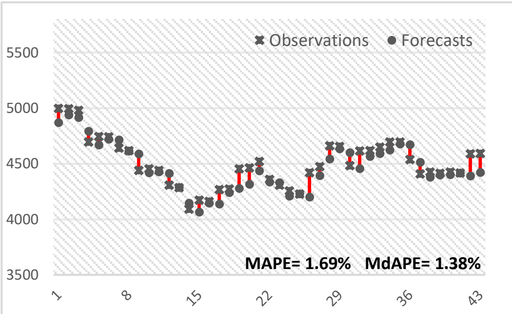
Data Sets	Parameters			
TAIEX	***p***	***K***	***c***	$\boldsymbol{\beta_f}$
2000	5	4	3	1.9
2001	4	2	6	1.8
2002	4	2	5	1.8
2003	3	2	3	2.2
2004	3	3	5	1.8

5.3.2. TAIEX Implementations - the daily data sets between 2008-2018 years

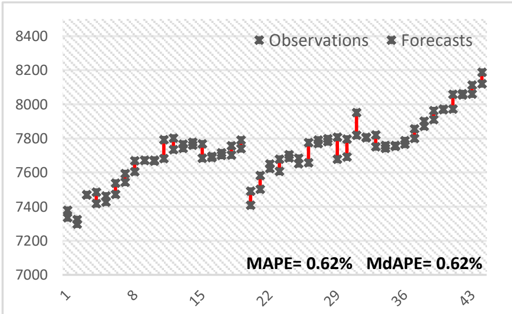
Second, eleven TAIEX datasets observed daily from 2008 to 2018 were analysed. The obtained results are discussed by comparing them with the results produced in some other studies. The results in terms of the RMSE metric are given in Table 5. As can be seen from Table 5, the proposed method exhibits the best prediction performance for the time series of these 11-year TAIEX datasets. In addition, when the average of the RMSE values obtained for the 11 different time series is considered, it is seen that there has been progress of about 7% ((74.32-67.87)/ 74.32). As seen from Table 5, the proposed FRNFs improve the prediction performance by approximately up to 13% when each dataset is evaluated separately, compared to the best model available in the literature. In addition, when the 11 data sets are considered an entire, it made approximately 9% progress on average.

Some visual graphs can also be created showing that the proposed FRFNs approach produces very successful prediction results for these TAIEX data sets. These visuals are graphs showing observed and predicted values and residuals between them. One such bundle of visuals is given in Figure 3.

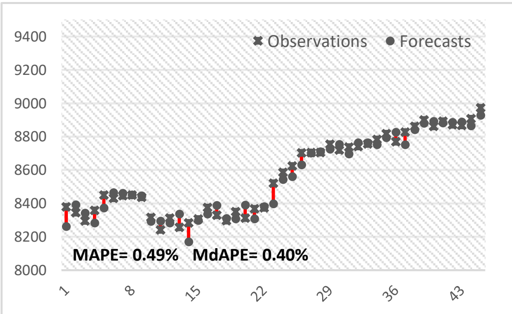
When Figure 3 is investigated detailed, it is clearly seen that the predicted values are in harmony with the observed data points. The red lines representing the residuals are quite short in almost all data points, with the exception of the 3-year dataset specifically. Even for these 3 datasets (2011, 2016, and 2018 years), while some residuals may seem relatively large, they still indicate fairly reasonable error levels. While the MAPE value, which is a relative error measure for 2011, indicates an extraordinary prediction performance of approximately 1%, these values were even below 1% for 2016 and 2018. Considering that the MAPE values are even below 1% for most of the other years, the superior performance of the proposed FRNFs approach is once again remarkable.



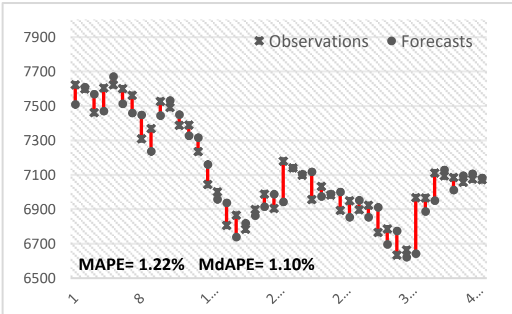
(a) TAIEX2008



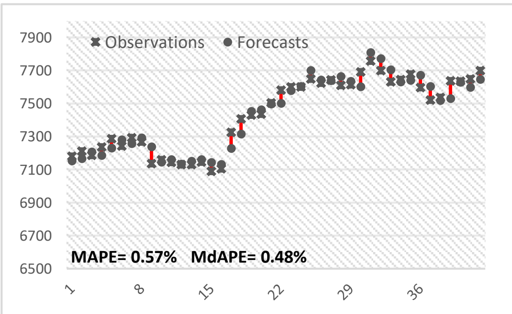
(b) TAIEX2009



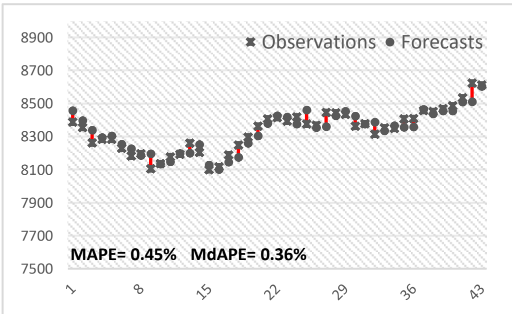
(c) TAIEX2010



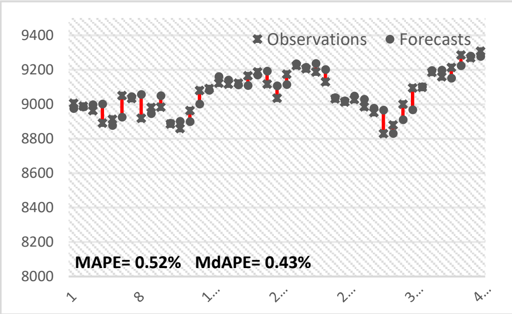
(d) TAIEX2011



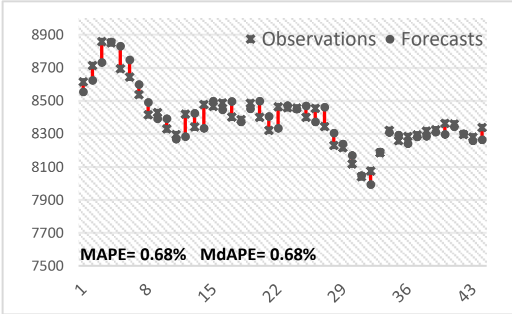
(e) TAIEX2012



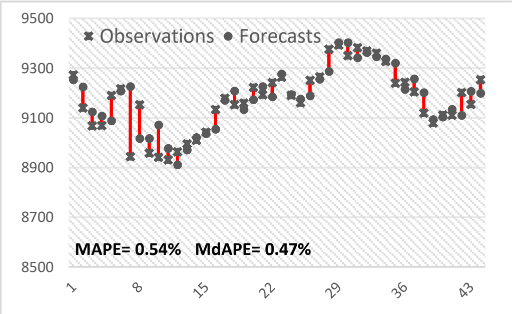
(f) TAIEX2013



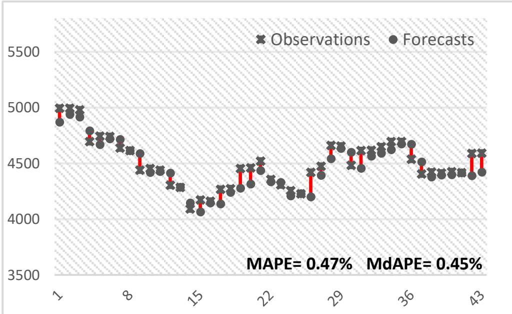
(g) TAIEX2014



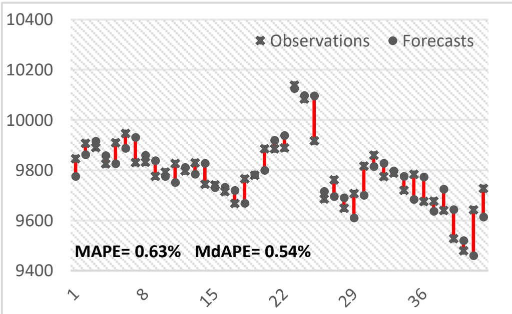
(h) TAIEX2015



(i) TAIEX2016



(j) TAIEX2017



(k) TAIEX2018

Figure 3. The together plot of the observations, predictions, and residuals for TAIEX (2008-2018)

Table 5. The results of current models in terms of RMSE for TAIEX (2008-2018).

Models	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	Average
[4]	225.38	204.11	151.42	237.43	81.08	82.3	98.64	176.6	122.71	91.69	173.36	149.52
[5]	186.48	207.42	211.21	215.13	78.37	184.83	308.72	100.75	316.57	902.18	492.89	291.32
[7]	105.10	78.25	103.84	117.78	59.14	49.59	87.52	90.72	80.98	63.20	186.21	92.94
[81]	138.94	71.30	72.92	115.54	62.32	49.53	69.60	78.73	82.22	62.98	102.19	82.39
[82]	142.38	110.88	100.76	180.79	63.30	63.16	71.62	104.4	88.35	89.02	231.35	113.27
[10]	131.18	71.48	74.07	119.34	59.55	50.09	65.72	79.93	82.10	61.67	101.99	81.56
[68]	144.23	72.16	62.23	114.25	59.94	50.97	64.79	79.27	82.29	62.15	107.08	81.76
[42]	129.48	69.91	67.11	123.03	58.37	50.39	69.15	78.22	80.51	64.25	103.74	81.29
[18]	140.49	70.34	93.08	118.89	62.35	49.63	66.55	78.46	82.73	62.21	109.39	84.92
[83]	142.37	112.45	99.83	164.97	63.31	64.76	74.85	94.33	86.59	88.93	161.78	104.92
[84]	114.03	70.91	68.59	117.02	60.50	49.92	70.44	77.67	80.86	62.61	106.75	79.94
[85]	129.29	70.27	75.46	111.45	63.20	50.84	67.67	78.33	84.39	61.09	103.26	81.39
[86]	137.44	71.76	74.52	117.38	61.59	51.37	69.23	79.89	83.53	62.37	103.24	82.94
[87][1]	108.52	121.46	75.33	128.46	60.43	51.17	79.44	94.92	80.39	78.76	88.23	87.92
[87][2]	108.57	68.57	52.15	113.38	58.84	48.87	65.90	80.22	82.24	64.31	74.52	74.32
[88]	140.19	73.22	80.37	124.38	62.53	52.39	66.26	82.78	80.14	61.98	99.03	83.93
[89]	126.29	73.77	143.53	122.49	60.73	55.38	66.15	79.47	83.18	65.54	203.72	98.20
[90]	140.48	70.63	67.46	121.27	61.10	50.23	67.08	80.65	81.12	66.34	98.13	82.23
[91]	113.03	72.30	62.82	113.69	60.45	50.17	68.28	78.65	82.43	62.49	104.49	78.98
The Proposed FRFNs	**95.81**	**59.85**	**51.38**	**106.07**	**51.68**	**46.32**	**60.04**	**70.98**	**69.59**	**58.00**	**74.31**	**67.87**
Progress Rate (%)	***9***	***13***	***1***	***5***	***11***	***5***	***7***	***9***	***13***	***5***	***1***	***9***

The best values obtained for the parameters are presented in Table 6.

Table 6. The best values obtained for the parameters for TAIEX2008-2018.

Data Sets	Parameters			
TAIEX	p	K	c	β_f
2008	5	3	3	1.9
2009	3	3	6	2.1
2010	4	3	3	1.9
2011	2	5	7	2.2
2012	3	3	5	2.2
2013	2	4	3	1.9
2014	5	5	4	2.1
2015	4	4	6	1.9
2016	2	4	3	2.0
2017	4	3	4	2.0
2018	2	4	7	1.8

5.3.3. IEX Implementations

Finally, the prediction ability of the proposed FRNFs has been revealed by analysing daily IEX datasets observed in 5 different years from 2009 to 2013. A total of 10 different analyses were performed using two different test sets of lengths 7 and 15 for each of the IEX time series. The obtained results were evaluated comparatively with traditional time series estimation methods, and also both fuzzy-based and ANN-based models. For this purpose, RMSE criterion values have been given in Table 7 and Table 7 for all prediction models.

The proposed FRFNs exhibited the best predictive performance for all-time series excluding a single IEX dataset. Especially in the time series in 2010, 2011, and 2013, where the test set length was 7, nearly 40% progress has been achieved in prediction accuracy. Moreover, in the time series in 2011 and 2013, where the test set length was 15, over 10% progress has been achieved in prediction accuracy. In addition, in the time series where the test set length was 7 in 2012, the level of progress was observed to be close to 20%. Considering all 10 data sets, it is seen that an average of 20% performance improvement is achieved.

Table 7. The results in terms of RMSE for IEX data sets.

	Year / Test Set Size										RMSE's Average
	2009/7	2009/15	2010/7	2010/15	2011/7	2011/15	2012/7	2012/15	2013/7	2013/15	
[92]	345	540	1221	1612	1058	1130	651	621	1362	1269	981
[93]	345	540	1208	1612	1057	1130	651	621	1362	1269	980
[94]	325	525	1077	1603	920	1096	775	783	1315	1233	965
[1]	1402	1754	1128	1742	1396	1360	1292	1047	1450	1931	1450
[73]	267	514	1050	1357	765	917	590	**582**	786	1208	804
[77]	405	647	1141	2033	1007	1134	634	938	1447	1413	1080
[78]	261	503	1144	1303	960	1009	634	629	1418	1264	913
[48]	240	467	1136	1451	987	999	631	619	1362	1256	915
[55]	446	534	1180	1852	1083	1146	1034	1038	1512	1279	1110
[66]	319	495	1080	1575	915	1028	720	676	1251	1237	930
[67]	240	500	1045	1300	946	1000	662	762	832	1207	849
Proposed FRFNs	**213**	**466**	**663**	**1247**	**442**	**806**	**477**	716	**477**	**1084**	**649**
Progress Rate (%)	***11***	***1***	***37***	***4***	***42***	***12***	***19***	***NA***	***39***	***10***	***19***

VI. CONCLUSIONS

Fuzzy-based prediction models are able to produce very satisfactory results, as they offer a flexible approach to the uncertainty contained in the time series. Also, T1-FRF approach, by using some transformation of the memberships as well as the real values of series, create a model with more information. However, there are some significant problems, considered as a gap in the literature and worth investigating. The first one is that the T1-FRF forms a holistic model based on the combination of a set of linear functions of the inputs. But the relationship between input and output is not always simply linear. The second gap is, FRF studies in the literature do not take into account the rigid assumptions of the linear regression model and the validity of the model is not checked. Moreover, current studies on time series prediction consider the data sets either from computational based point of view or fuzzy based points. So, this issue can be considered as another gap belongs to current studies.

In this study, a novel prediction approach has been proposed in order to both fill all these important gaps in the literature and improve the prediction performance of the model significantly. The proposed FRNFs, by using some transformation of the memberships as well as the real values of series, create a model with more information and so offer a more realistic approach. The proposed FRNFs are capable of modelling nonlinear relationships in time series, thanks to their properties and structure. The proposed FRNFs do not need strict assumptions existing in linear regression model to satisfy, thanks to the used computational-based approach.

In addition to completing the shortcomings of the current studies in the literature, the proposed method's superior prediction performance has been proven by the analysis of different real-life time series.

It has been clearly demonstrated that the proposed prediction tool has a superior predictive ability as well as overcoming the existing problems in the literature. With these aspects, the proposed FRNF is expected to be an effective prediction tool that researchers and practitioners in many fields can benefit from dealing with the time series problem.

VII. FUTURE WORK AND LIMITATIONS

The present approach is limited to tuning the hyper parameters of FFNN, used as a nonlinear fuzzy function. Although this problem has been tried to be overcome by using the validation set, the search space has been limited. Therefore, in future studies, hyperparameter tuning can be carried out in an optimization process.

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