Celal Bayar University Journal of Science
Volume 21, Issue 3, 2025, p 137-144
Doi: 10.18466/cbayarfbe.1575598 

**Celal Bayar University Journal of Science**

# The modified sub equation method to Kolmogorov-Petrovskii-Piskunov equation

Hülya Durur[1]*

[1] Department of Computer Engineering, Faculty of Engineering, Ardahan University, Ardahan, Türkiye
* hulyadurur@ardahan.edu.tr
* Orcid: 0000-0002-9297-6873

Received: October 29, 2024
Accepted: April 14, 2025
DOI: 10.18466/cbayarfbe.1575598

**Abstract**

The Kolmogorov-Petrovskii-Piskunov (KPP) equation (eq.) can be considered a generalized form of the Fisher, Huxley and Fitzhugh-Nagumo Eqs., which have applications in chemistry, biology and physics. In this article, the nonlinear KPP eq. is discussed with the modified sub equation method, one of the analytical methods. With the successfully implemented method, trigonometric and hyperbolic solutions of the KPP eq. are presented. 3 D, 2 D and contour graphics are presented by giving arbitrary values to the parameters in the solutions produced. Also, the attained results are compared with the existing solutions in the literature. The effectiveness and applicability of the applied method to nonlinear differential eqs. (NPDEs) are examined in this paper.

**Keywords:** Kolmogorov–Petrovskii–Piskunov equation, the modified sub equation method, traveling wave solutions.

## 1. Introduction

Applied mathematics plays an effective role in modeling many situations encountered in real life, especially for fluid dynamics, plasma physics and virus spreading populations. NPDEs are critical tools in understanding and simulating such real-life phenomena. The scientific community has shown increasing interest in solving these equations. Especially lately, numerical and analytical methods have been used to solve NPDEs. In this context, applied mathematicians are working intensively to reach solutions to these equations and numerous new analytical methods have been introduced to the literature in this field. There are a variety of methods such as Hirota bilinear method [1], *(G′/G)*-expansion method [2], $\left(G'/G^2\right)$ method [3], the modified Kudryashov method [4], the Clarkson–Kruskal (CK) direct method [5], sumudu transform method [6] and so on [7-14] to attain exact solutions of NPDEs.

It is known that the classical KPP eq. explains such phenomena as the propagation of nerve impulses and the evolution of dominant genes (biology), combustion (physics), propagation of concentration waves (chemical kinetics) and many others [15].

In this study, we will consider the KPP eq., a quasi-linear parabolic eq. seen in mathematical biology, combustion theory, and modeling of some reaction-diffusion processes [16].

In order to attain exact solutions of KPP eq. in the form mentioned below, we will employ the modified sub equation method [17]

$$u_t - u_{xx} + \beta u + v u^2 + \delta u^3 = 0, \tag{1}$$

here $\delta, \beta, v$ are real numbers.

KPP eq., includes the Huxley, Burgers-Huxley, Fisher, Fitzhugh-Nagumo and Chaffee-Infanfe eqs., is of great importance in the field of physics [17]. These models are important in biology. The KPP eq. represents the genetic model for the spread of the dominant gene throughout the population. Many scientists have used various techniques to solve the KPP eq. Some of these techniques are *(G′/G)*-expansion method [17], modified simple equation method [18], homotopy analysis method [19], the first integral method [20, 21], the modified trial equation method [22], *(1/G′)* -expansion method [23].

In this study, an effective technique called the modified sub equation method [24], is adopted to attain the exact solutions of the KPP eq. This study is new in the literature because the exact solutions of the model evaluated using modified sub equation method, have not been presented before.

## 2. Modified sub-equation method

We evaluate this method to arrive at solutions for NPDEs [24]. Let's think of NPDEs as

$$W\left(u,u_t,u_x,u_{tt},u_{xx},\ldots\right)=0. \quad \textbf{(2)}$$

Implementation of wave transformation

$$U=u\left(x,t\right)=U\left(\xi\right),\quad \xi=x-wt,\quad w\neq 0, \quad \textbf{(3)}$$

here *w* is speed of wave. Eq. (2) converts into ODE

$$T\left(U,U',U'',\ldots\right)=0. \quad \textbf{(4)}$$

Eq. (4) is thought to have a solution

$$U\left(\xi\right)=a_0+\sum_{i=1}^{N}\left(a_i\psi^i\left(\xi\right)+a_{-i}\psi^{-i}\left(\xi\right)\right). \quad \textbf{(5)}$$

At least one of the "$a_N$" coefficients must be different from zero. Constants to be specified inhere are $a_i,\ \left(0\leq i\leq N\right)$, and according to the principle of balance, $N\in\left\{1,2,3,\ldots\right\}$ is attained by balancing the term in (4) and, solution of the Riccati eq. is $\psi\left(\xi\right)$

$$\psi'\left(\xi\right)=\mu+\left(\psi\left(\xi\right)\right)^2, \quad \textbf{(6)}$$

here $\mu$ can be any kind of constant. Some special solutions of Riccati eq. in (6) are given below.

$$\psi\left(\xi\right)=\begin{cases}-\sqrt{-\mu}\tanh\left(\sqrt{-\mu}\xi\right), & \mu<0\\ -\sqrt{-\mu}\coth\left(\sqrt{-\mu}\xi\right), & \mu<0\\ \sqrt{\mu}\tan\left(\sqrt{\mu}\xi\right), & \mu>0\\ -\sqrt{\mu}\cot\left(\sqrt{\mu}\xi\right), & \mu>0\\ -\dfrac{1}{\xi+M}, & \mu=0\quad (M\ is\ a\ const.)\end{cases} \quad \textbf{(7)}$$

In eq. (4), by applying eqs. (5, 6), equating all coefficients in $a_i,\left(i=0,1,\ldots,N\right)$ to zero, new polynomial with respect to $\psi\left(\xi\right)$ is attained according to a non-linear system of algebraic eqs. resulting in $\psi^i\left(\xi\right),\left(i=0,1,\ldots,N\right)$. To find solutions to nonlinear algebraic eqs., we decide on $M,w,\mu,a_i,\ \left(i=0,1,\ldots,N\right)$ constants. The solutions of eq. (6) are substituted into the constants generated from this system and placed inside eq. (5) by aid of formula (7). In this way, exact solutions for eq. (2) are attained.

## 3. Application of the method

If the transformation in eq. (3) is applied, taking into account eq. (1), the following nODE is obtained.

$$-U''-wU'+\beta U+vU^2+\delta U^3=0. \quad \textbf{(8)}$$

In the eq. (8), we consider the non-linear term and the highest-order linear term. These terms are $U''$ and $U^3$. Applying the balancing principle in eq. (8), we attain $N=1$ and in eq. (5) the following situation is reached:

$$U\left(\xi\right)=a_1\psi\left(\xi\right)+a_2\frac{1}{\psi\left(\xi\right)}+a_0, \quad \textbf{(9)}$$

inhere $a_0,a_1,a_2$ are constants to be determined.

If eq. (9) is written in eq. (8) and following systems of eqs. are written with the necessary corrections:

$$\begin{aligned}
&\left(\psi\left(\xi\right)\right)^0: && \beta a_0+va_0^2+\delta a_0^3-w\mu a_1+2va_1a_2+6\delta a_0a_1a_2+wa_2=0,\\
&\left(\psi\left(\xi\right)\right)^1: && \beta a_1-2\mu a_1+2va_0a_1+3\delta a_0^2a_1+3\delta a_1^2a_2=0,\\
&\left(\psi\left(\xi\right)\right)^2: && -wa_1+va_1^2+3\delta a_0a_1^2=0,\\
&\left(\psi\left(\xi\right)\right)^3: && -2a_1+\delta a_1^3=0,\\
&\frac{1}{\left(\psi\left(\xi\right)\right)^1}: && \beta a_2-2\mu a_2+2va_0a_2+3\delta a_0^2a_2+3\delta a_1a_2^2=0,\\
&\frac{1}{\left(\psi\left(\xi\right)\right)^2}: && w\mu a_2+va_2^2+3\delta a_0a_2^2=0,\\
&\frac{1}{\left(\psi\left(\xi\right)\right)^3}: && -2\mu^2a_2+\delta a_2^3=0.
\end{aligned} \quad \textbf{(10)}$$

$\delta,\beta,\mu,v,w$ and $a_0,a_1,a_2$ constants are reached from eq. (10) the system employing a program.

Celal Bayar University Journal of Science
Volume 21, Issue 3, 2025, p 137-144
Doi: 10.18466/cbayarfbe.1575598 

**Case 1:**

$$\beta=-\frac{2\left(va_0^3+4a_2^2\right)}{a_0^2},\quad \mu=-\frac{a_2^2}{a_0^2},\quad a_1=0,\quad \delta=\frac{2a_2^2}{a_0^4},\quad w=\frac{va_0^3+6a_2^2}{a_0a_2}. \tag{11}$$

The hyperbolic solution to eq. (1) can be determined by writing the values of eq. (11) in eq. (9).

$$u_1(x,t)=a_0-\frac{coth\left[\sqrt{\frac{a_2^2}{a_0^2}}\left(x-\frac{t\left(va_0^3+6a_2^2\right)}{a_0a_2}\right)\right]a_2}{\sqrt{\frac{a_2^2}{a_0^2}}}. \tag{12}$$

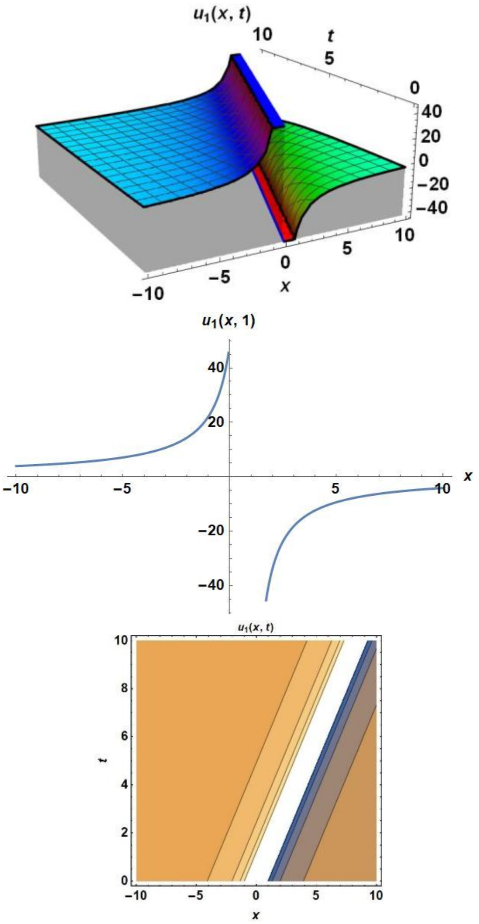


**Fig.1:** Graphs of the eq. (12) for $a_0=0.2,\quad a_2=0.001,\quad v=0.02$.

**Case 2:**

$$\beta=-\frac{2\left(va_0^3+4a_2^2\right)}{a_0^2},\quad \mu=-\frac{a_2^2}{a_0^2},\quad a_1=0,\quad \delta=\frac{2a_2^2}{a_0^4},\quad w=\frac{va_0^3+6a_2^2}{a_0a_2}. \tag{13}$$

The hyperbolic solution to eq. (1) can be determined by writing the values of eq. (13) in eq. (9).

$$u_2(x,t)=a_0-\frac{a_2\,tanh\left[\sqrt{\frac{a_2^2}{a_0^2}}\left(x-\frac{t\left(va_0^3+6a_2^2\right)}{a_0a_2}\right)\right]}{\sqrt{\frac{a_2^2}{a_0^2}}}. \tag{14}$$

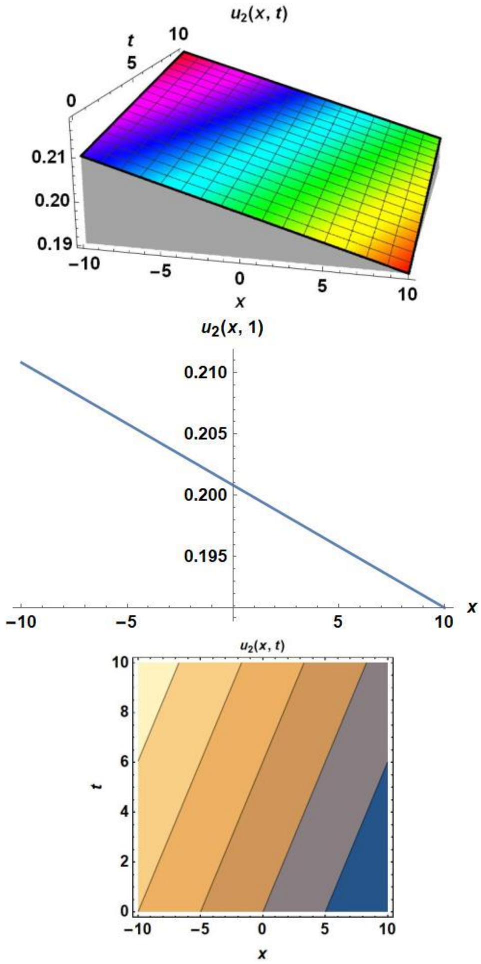


**Fig.2:** Graphs of the eq. (14) for $a_0=0.2,\quad a_2=0.001,\quad v=0.02$.

**Case 3:**

$$\beta=\frac{4a_2^2}{a_0^2},\quad \mu=\frac{a_2^2}{a_0^2},\quad a_1=0,\quad v=-\frac{4a_2^2}{a_0^3},\quad \delta=\frac{2a_2^2}{a_0^4},\quad w=-\frac{2a_2}{a_0}. \tag{15}$$

The trigonometric solution to eq. (1) can be determined by writing the values of eq. (15) in eq. (9).

$$u_3\left(x,t\right)=a_0+\frac{cot\left[\sqrt{\frac{a_2^2}{a_0^2}}\left(x+\frac{2ta_2}{a_0}\right)\right]a_2}{\sqrt{\frac{a_2^2}{a_0^2}}}. \quad \mathbf{(16)}$$

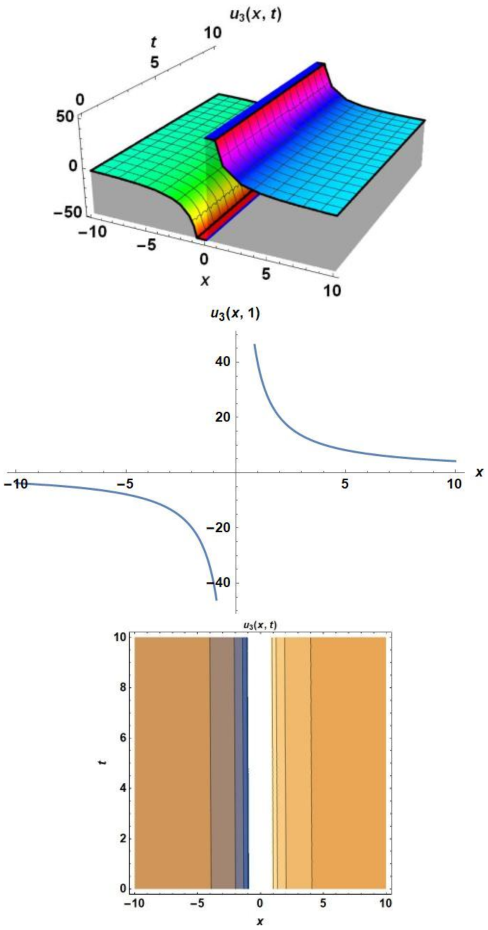


**Fig.3:** Graphs of the eq. (16) for $a_0=0.2, \quad a_2=0.001$.

**Case 4:**

$$\beta=\frac{4a_2^2}{a_0^2},\quad \mu=\frac{a_2^2}{a_0^2},\quad a_1=0,\quad v=-\frac{4a_2^2}{a_0^3},\quad \delta=\frac{2a_2^2}{a_0^4},\quad w=-\frac{2a_2}{a_0}. \quad \mathbf{(17)}$$

The trigonometric solution to eq. (1) can be determined by writing the values of eq. (17) in eq. (9).

$$u_4\left(x,t\right)=a_0-\frac{a_2\,tan\left[\sqrt{\frac{a_2^2}{a_0^2}}\left(x+\frac{2ta_2}{a_0}\right)\right]}{\sqrt{\frac{a_2^2}{a_0^2}}}. \quad \mathbf{(18)}$$

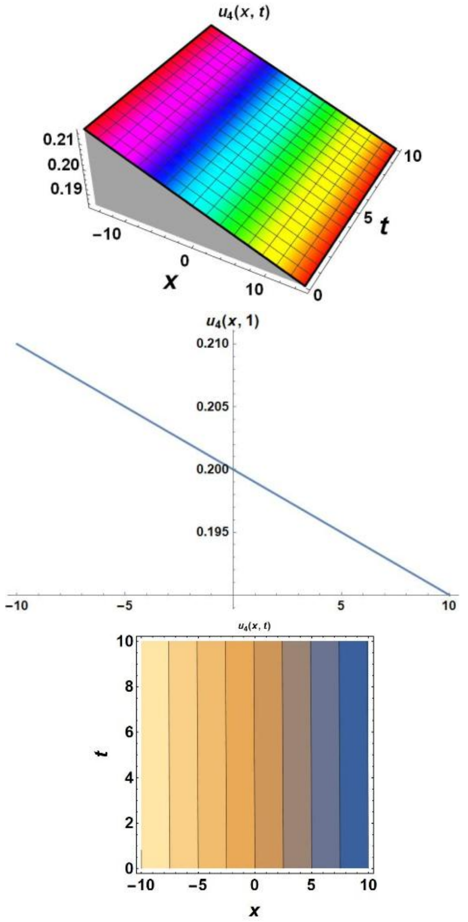


**Fig.4:** Graphs of the eq. (18) for $a_0=0.2, \quad a_2=0.001$.

**Case 5:**

$$\beta=\frac{2\left(a_0^2+\mu a_1^2\right)}{a_1^2},\quad a_2=0,\quad v=-\frac{4a_0}{a_1^2},\quad \delta=\frac{2}{a_1^2},\quad w=\frac{2a_0}{a_1}. \quad \mathbf{(19)}$$

The hyperbolic solution to eq. (1) can be determined by writing the values of eq. (19) in eq. (9).

$$u_5\left(x,t\right)=a_0-\sqrt{-\mu}a_1\,tanh\left[\sqrt{-\mu}\left(x-\frac{2ta_0}{a_1}\right)\right]. \quad \mathbf{(20)}$$

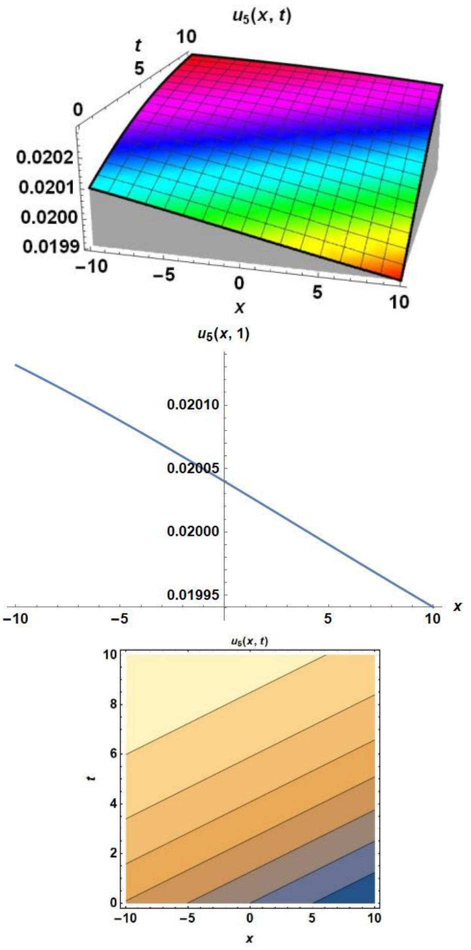


Fig.5: Graphs of the eq. (20) for $a_0 = 0.02, a_1 = 0.01, \mu = -0.001$.

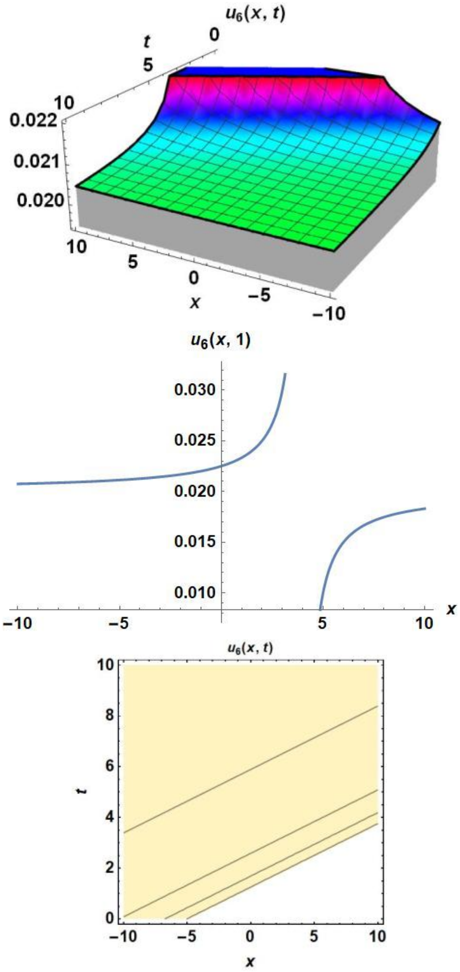


**Fig.6:** Graphs of the eq. (22) for $a_0 = 0.02, a_1 = 0.01, \mu = -0.001$.

**Case 6:**

$$\beta = \frac{2\left(a_0^2 + \mu a_1^2\right)}{a_1^2}, \quad a_2 = 0, \quad v = -\frac{4a_0}{a_1^2}, \quad \delta = \frac{2}{a_1^2}, \quad w = \frac{2a_0}{a_1}. \quad \textbf{(21)}$$

The hyperbolic solution to eq. (1) can be determined by writing the values of eq. (21) in eq. (9).

$$u_6\left(x,t\right) = a_0 - \sqrt{-\mu}\, coth\left[\sqrt{-\mu}\left(x - \frac{2ta_0}{a_1}\right)\right] a_1. \quad \textbf{(22)}$$

**Case 7:**

$$\beta = \frac{2\left(a_0^2 + \mu a_1^2\right)}{a_1^2}, \quad a_2 = 0, \quad v = -\frac{4a_0}{a_1^2}, \quad \delta = \frac{2}{a_1^2}, \quad w = \frac{2a_0}{a_1}. \quad \textbf{(23)}$$

The trigonometric solution to eq. (1) can be determined by writing the values of eq. (23) in eq. (9).

$$u_7\left(x,t\right) = a_0 + \sqrt{\mu} a_1\, tan\left[\sqrt{\mu}\left(x - \frac{2ta_0}{a_1}\right)\right]. \quad \textbf{(24)}$$

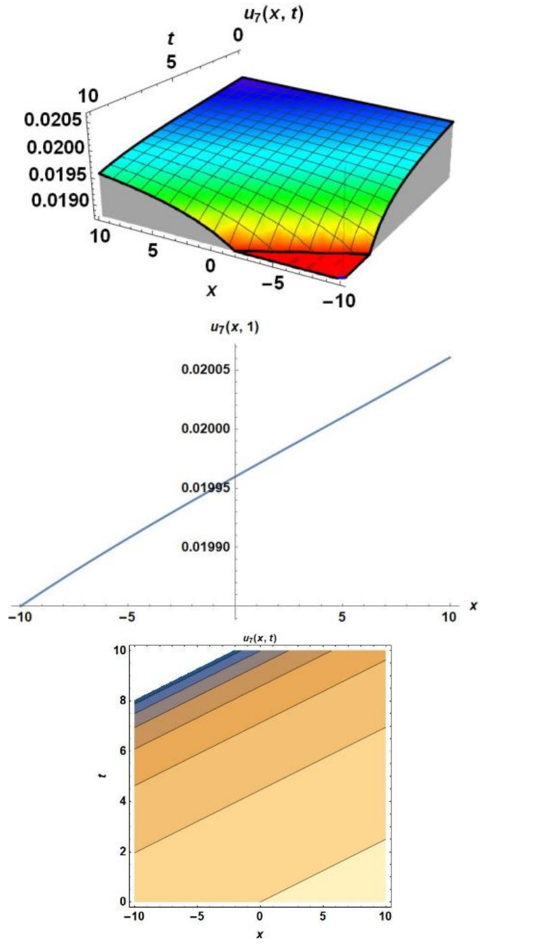


**Fig.7:** Graphs of the eq. (24) for $a_0 = 0.02, a_1 = 0.01, \mu = 0.001$.

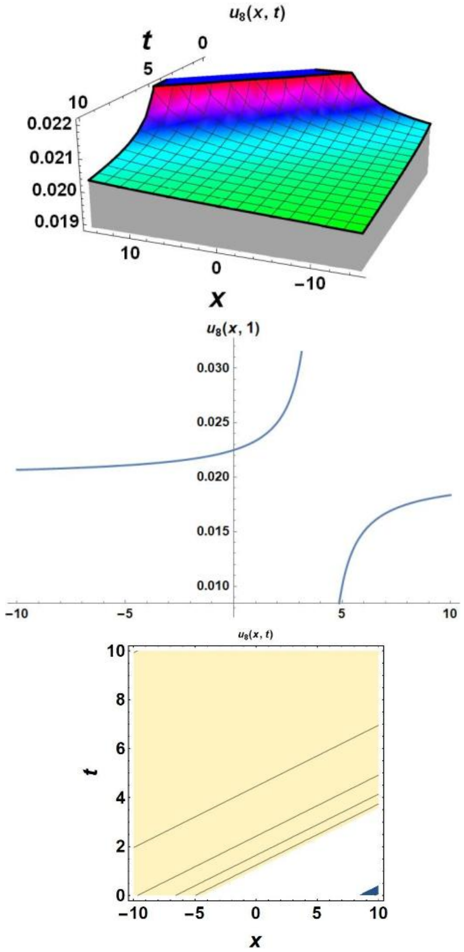


**Fig.8:** Graphs of the eq. (26) for $a_0 = 0.02, a_1 = 0.01, \mu = 0.001$.

**Case 8:**

$$\beta = \frac{2\left(a_0^2 + \mu a_1^2\right)}{a_1^2}, \quad a_2 = 0, \quad v = -\frac{4a_0}{a_1^2}, \quad \delta = \frac{2}{a_1^2}, \quad w = \frac{2a_0}{a_1}. \qquad \textbf{(25)}$$

The trigonometric solution to eq. (1) can be determined by writing the values of eq. (25) in eq. (9).

$$u_8\left(x,t\right) = a_0 - \sqrt{\mu}\, cot\left[\sqrt{\mu}\left(x - \frac{2ta_0}{a_1}\right)\right] a_1. \qquad \textbf{(26)}$$

## 4. Results and Discussion

In this study, by applying the modified sub equation method to the KPP model, we obtained the hyperbolic type of traveling wave solutions eqs. (12, 14, 20, 22) and the trigonometric type of traveling wave solutions in eqs. (16, 18, 24, 26) that satisfy eq. (1). The 2 D, 3 D and contour graphs of the hyperbolic solutions we attained are presented in Fig.s 1-2 and 5-6. The 2 D, 3 D and contour graphs of the trigonometric solutions we attained are presented in Fig.s 3-4 and 7-8.

If we compare the solutions obtained in the study with the solution in the literature; Wongsaijai et al. obtained similar solutions to the solutions we obtained with the tanh method [25].

However, in our study using modified sub equation method, hyperbolic and trigonometric solutions were attained by taking into account the situations in eqs. (11), (15) and (19).

For the set (1-8); Wongsaijai et al. obtained the hyperbolic solutions in eqs. (10-11) under the restriction number (12); the hyperbolic solutions in eqs. (13-14) under the restriction number (15); the hyperbolic solutions in eqs. (16-17) under the restriction number (18); and the hyperbolic solutions in eqs. (19-20) under the restriction number (21) [25].
As can be seen, the methods in the studies are different and our produced solutions are of trigonometric and hyperbolic type, while the solutions they obtained in the Wongsaijai et al. study are of hyperbolic type and are different solutions [25].

## 5. Conclusion

In this study, we have been reached the exact solutions of the KPP eq. by using the modified sub equation method. Hyperbolic and trigonometric type solutions of the KPP eq. are presented with a powerful, reliable and effective method. Exact solutions are known to have a significant impact on a wide range of physical events.We think that when the constants in the exact solutions generated in this work acquire physical relevance, their value will increase. Contour 2 D and 3 D graphs are presented for appropriate values of the constants in the produced solutions. These solutions were produced using the Mathematica program.The applied method is reliable, easy and effective in finding exact solutions of NPDEs.

## Author's Contributions

**Hülya Durur:** Conducted to the literature review for this study and contributed to the methodology, software, conceptualization, writing–review & editing, investigation, and formal analysis.

## Ethics

Regarding the publication of this manuscript are no ethical issues.

## References

**[1].** Yang, L., & Gao, B. (2024). The nondegenerate solitons solutions for the generalized coupled higher-order nonlinear Schrödinger equations with variable coefficients via the Hirota bilinear method. *Chaos, Solitons & Fractals*; 184: 115009.

**[2].** Zayed, E. M., & Gepreel, K. A. (2009). Some applications of the G′ G-expansion method to non-linear partial differential equations. *Applied Mathematics and Computation;* 212(1): 1-13.

**[3].** Rehman, S. U., Yusuf, A., Bilal, M., Younas, U., Younis, M., & Sulaiman, T. A. (2020). Application of ($G^1/G^2$)-expansion method to microstructured solids, magneto-electro-elastic circular rod and (2+1)-dimensional nonlinear electrical lines. *Mathematics in Engineering, Science & Aerospace (MESA)*; 11(4).

**[4].** Kumar, D., Seadawy, A. R., & Joardar, A. K. (2018). Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. *Chinese journal of physics;* 56(1): 75-85.

**[5].** Su-Ping, Q., & Li-Xin, T. (2007). Modification of the Clarkson–Kruskal direct method for a coupled system. *Chinese Physics Letters;* 24(10): 2720.

**[6].** Darzi, R., Mohammadzade, B., Mousavi, S., & Beheshti, R. (2013). Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation. *J. Math. Comput. Sci*; 6(1): 79-84.

**[7].** Yokus, A., & Isah, M. A. (2022). Investigation of internal dynamics of soliton with the help of traveling wave soliton solution of Hamilton amplitude equation. Optical and Quantum Electronics; 54(8): 528.

**[8].** Ali, K. K., Osman, M. S., & Abdel-Aty, M. (2020). New optical solitary wave solutions of Fokas-Lenells equation in optical fiber via Sine-Gordon expansion method. *Alexandria Engineering Journal;* 59(3): 1191-1196.

**[9].** Malik, S., Kumar, S., Akbulut, A., & Rezazadeh, H. (2023). Some exact solitons to the (2+ 1)-dimensional Broer–Kaup–Kupershmidt system with two different methods. *Optical and Quantum Electronics;* 55(14): 1215.

**[10].** Rabie, W. B., Ahmed, H. M., Hashemi, M. S., Mirzazadeh, M., & Bayram, M. (2024). Generating optical solitons in the extended (3+ 1)-dimensional nonlinear Kudryashov's equation using the extended f-expansion method. *Optical and Quantum Electronics; 56*(5): 894.

**[11].** Subaşı, M., & Durur, H. (2023). Refraction simulation of nonlinear wave for Shallow Water-Like equation. *Celal Bayar University Journal of Science;* 19(1): 47-52.

**[12].** Ali, K. K., Tarla, S., Ali, M. R., Yusuf, A., & Yilmazer, R. (2023). Consistent solitons in the plasma and optical fiber for complex Hirota-dynamical model. *Results in Physics;* 47: 106393.

**[13].** Durur, H., Arslantürk, R., & Aydın, A. (2024). Mathematical analysis of electric signal transmission in semi-conductor materials. *Modern Physics Letters B;* 38(17): 2450132.

**[14].** Kamel, N. M., Ahmed, H. M., & Rabie, W. B. (2024). Retrieval of soliton solutions for 4th-order (2+ 1)-dimensional Schrödinger equation with higher-order odd and even terms by modified Sardar sub-equation method. *Ain Shams Engineering Journal;* 15(7): 102808.

**[15].** Mochizuki, K., & Shishmarev, I. A. (2001). Large Time Asymptotics of Small Solutions to Generalized Kolmogorov-Petrovskii-Piskunov Equation. *Funkcialaj Ekvacioj Serio Internacia*; 44(1): 99-118.

**[16].** Pikulin, S. V. (2018). Traveling-wave solutions of the Kolmogorov–Petrovskii–Piskunov equation. *Computational Mathematics and Mathematical Physics;* 58: 230-237.

**[17].** Feng, J., Li, W. and Wan, Q., Using G′ G-expansion method to seek the traveling wave solution of Kolmogorov–Petrovskii–Piskunov equation. *Applied Mathematics and Computation;* 217(12): 5860-5865, (2011).

**[18].** Zayed, E. M. E. and Ibrahim, S. H., Exact solutions of Kolmogorov-Petrovskii-Piskunov equation using the modified simple equation method. *Acta Mathematicae Applicatae Sinica, English Series;* 30(3): 749-754, (2014).

**[19].** Hariharan, G. (2013). The homotopy analysis method applied to the Kolmogorov–Petrovskii–Piskunov (KPP) and fractional KPP equations. *Journal of Mathematical Chemistry;* 51(3): 992-1000.

**[20].** Unal, A. O. On the Kolmogorov–Petrovskii–Piskunov equation. *Commun. Fac. Sci. Univ. Ank. Series A;* 1, (2013).

**[21].** Rouhparvar, H., Travelling wave solution of the Kolmogorov-Petrovskii-Piskunov equation by the first integral method. *Bulletin of the Malaysian Mathematical Sciences Society;* 37(1), (2014).

**[22].** Demiray, Ş. T., & Duman, S. (2021). The modified trial equation method to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation and Kolmogorov-Petrovskii-Piskunov equation. *Balıkesir University Journal of the Institute of Science*; *23*(2): 673-684.

**[23].** Durur, H., & Yokuş, A. (2020). Analytical solutions of Kolmogorov–Petrovskii–Piskunov equation. *Balıkesir University Journal of the Institute of Science*; 22(2): 628-636.

**[24].** Duran, S., Yokuş, A., Durur, H., & Kaya, D. (2021). Refraction simulation of internal solitary waves for the fractional Benjamin–Ono equation in fluid dynamics. *Modern Physics Letters B*; *35*(26): 2150363.

**[25].** Wongsaijai, B., Aydemir, T., Ak, T., & Dhawan, S. (2024). Analytical and numerical techniques for initial-boundary value problems of Kolmogorov–Petrovsky–Piskunov equation. *Numerical Methods for Partial Differential Equations;* 40(1): e22693.