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Research Article

Interval-Valued Fuzzy Sets on Proximal Relator Spaces

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Abstract

Interval-valued fuzzy sets are a broader form of classical fuzzy sets where membership values are expressed as intervals. In this framework, each element has a real-valued membership degree that falls within the given range of possible values. By helping of this fuzzy relations, it is built the concept of interval-valued fuzzy relations on proximal relator spaces. In the paper, the interval-valued fuzzy proximity axioms are rigorously examined, and corresponding examples are provided to demonstrate their applicability. Also, this paper also considers spatial Lodato and descriptive Lodato proximity relations.

Keywords: Proximity space, fuzzy relation, fuzzy proximity, interval-valued fuzzy sets.

Aralık Değerli Fuzzy Kümeleri Kullanarak Fuzzy Proksimiti Yaklaşımı

Öz

Aralık-değerli fuzzy kümeler, üyelik değerlerinin aralıklar olduğu klasik fuzzy kümelerin bir genellemesidir. Aralık-değerli bulanık kümelerde, muhtemel üyelik derecelerinin üyelik aralığı içinde bir elemanın bir reel değerli üyelik derecesi vardır. Aralık değerli fuzzy bağıntıların yardımıyla, proksimal relator uzayları üzerinde aralık-değerli fuzzy bağıntılar kavramını tanımlıyoruz. Bu çalışmamızda, aralık-değerli fuzzy proksimiti aksiyomları incelenmiş ve bazı örnekler verilmiştir. Ayrıca, uzaysal Lodato ve tanımsal Lodato proksimiti bağıntıları tanımlanmıştır.

Anahtar Kelimeler: Proksimiti uzayı, fuzzy bağıntısı, fuzzy proksimiti, aralık-değerli fuzzy kümeler.

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1. Introduction

Proximity spaces were defined by Efremovič [7]. The set U, together with the proximity relation that M is near (proximal) N for subsets M and N of any set U, is known as proximity space. Readers can easily access many resources about proximity spaces. Naimpally and Warrack wrote a wonderful book about proximity space and presented it to their readers [14]. Descriptive proximity, a more visual representation of proximity, has been utilized in various applications. A general descriptive proximity is an Efremovič proximity. Peters defined relator space $(U, \mathcal{R}_\delta)$ [19]. To understand easily, he used two relations namely EF-proximity and the Lodato proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [17, 18, 20].

$\mathcal{R}$ on U is known as *a relator* on U. $(U, \mathcal{R})$ is called as *a relator space* which is a generalized uniform space lacking all the conditions of uniform space except the reflexivity of the corresponding relations [24]. By using $\mathcal{R}_\delta$ on U, Peters presented a proximal relator space $(U, \mathcal{R}_\delta)$ [20]. For clarity, this space is restricted to three proximity relations only, specifically $(\mathcal{R}_{\tau_{\delta_\Phi}} = \{\delta, \delta_\Phi, \tau_{\delta_\Phi}\})$ [11, 12, 15, 20] .

Fuzzy sets (FS), first introduced by Zadeh, are characterized by a membership function mapping each element to a real value in the range [0,1] [31]. We can see fuzzy set theory various areas such as health science and image processing. However, there may be instabilities in real life, such as hesitation or uncertain decisions get involved and fuzzy set is not available to process some information with fuzziness and uncertainty. Thus, to handle such problems, Atanassov gave an outline of intuitionistic fuzzy sets and their definition (IFS) [1] that can deal with some information with fuzziness and uncertainty in many fields. After fuzzy sets has been caused great achievements and accepted from a mathematical perspective within the scientific domain, researcher proposed a number of theory based on this topic. However, the membership function may contain inaccurate information and might not represent reality precisely in practical applications, such as image processing.

These ideas indicate that in situations where researchers lack precise information about the membership function, constructing fuzzy sets becomes challenging. For this reason, it is proper to represent the membership degree of each element in a fuzzy set using an interval. From these ideas take places the generalization of fuzzy sets known as the framework of interval-valued fuzzy sets, in which the membership degree of each element is represented by a closed subinterval within [0,1].

Interval-valued fuzzy sets were proposed in the seventies. In the same year, Zadeh [32], Grattan-Guiness [9], Jahn [10], and Sambuc [22] each proposed interval-valued fuzzy sets independently, which were subsequently covered in multiple publications. However, the importance of interval-valued fuzzy sets was firmly established by the investigations carried out by Gorzalczany [8] and Türksen [29]. An interval-valued fuzzy set (IVFS) is characterized by a membership function that assigns interval values. Recently, many paper have been published related to IVFS such as approximate reasoning [2], expert systems [4], image processing [3], mobile networks [30], pattern recognition [5], genetic algorithms [13]. Roy and Biswas investigated the subject of interval-valued fuzzy relations. All of these studies demonstrate that the results obtained using IVFSs are superior to those obtained with fuzzy sets. After defining fuzzy proximal relator spaces [15], L-fuzzy proximity [25] and complex fuzzy proximity [26] is presented by researchers. Thereafter, it is defined spherical fuzzy proximities and Pythagorean fuzzy proximities by using fuzzy relations via relator spaces [27, 28]. Besides, Öztürk carried fuzzy proximities to intuitionistic fuzzy environments [16].

In the context of interval-valued fuzzy sets, each interval reflects the expert's knowledge of well-defined lower and upper bounds for the membership degree of an element in the fuzzy set, despite the exact value within this range remaining unspecified. From a theoretical standpoint, this interpretation is the most appropriate when addressing uncertainty or imprecision in the assessment of membership degrees within fuzzy set theory. When classical fuzzy proximity relations are defined from this perspective, an important advantage emerges: interval-valued fuzzy proximity relations enable the expression of vagueness in specifying a precise membership function. This inherent vagueness, introduced by the applying of interval-valued fuzzy sets, gives rise to outcomes that are less specific but potentially more credible and realistic.

The interval-valued fuzzy proximity relations are an effective tool to determine proximity of different groups that have similar or different by using an interval with uncertainty.

In this paper, we investigate a interval-valued fuzzy spatial proximity relation and a interval-valued fuzzy spatial Lodato proximity relation on the basis of the proposed interval-valued fuzzy proximity relation. The interval-valued fuzzy proximity relations are useful tools to find out the level of connection and difference between three or more sets. Therefore, our main aim in this paper is to create a theoretical infrastructure for subsequent studies.

1. Preliminaries

This section presents a brief overview of the fundamental definitions related to the topic to enhance understanding.

Definition 1 Let U be a nonempty set. A basic proximity δ is a relation on the power set of a nonempty set $U,$ which holds the following axioms for every subset $M,N,Q\ of\ U$:

(I_0) $M\ \delta\ N \Longrightarrow M \neq \emptyset, N \neq \emptyset$
(I_1) $M\ \delta\ N$ implies $N\ \delta\ M$.
(I_2) $M \cap N \neq \emptyset$ implies $M\ \delta\ N$.
(I_3) $(M \cup N)\ \delta\ P \Leftrightarrow M\ \delta\ P$ or $N\ \delta\ P$.

Furthermore, if δ holds (I_0)-(I_3) and the following axioms, then it is known to be Lodato proximity:

(I_4) ($M\delta N$ and $n\delta P$ for each $n \in N$) $\Longrightarrow M\delta P$.

If δ fulfills a basic proximity conditions and (I_6) below, then it is said to be an Efremovič proximity (EF-proximity) on U.

(I_5) If, for any $M, N \subset U$, $M\ \not\delta\ N$, subsets $P, Q \subset U\ exists,$ satisfying $P \cup Q = U$ with $M\ \not\delta\ P$ and $N\ \not\delta\ Q$.

(U, δ) is known as a basic (Lodato, Efremovič) proximity space. The notation $M\ \delta N$ indicates that M is near to N, while $M\ \not\delta\ N$ indicates that M is far from N.

Definition 2 *[21] Let* $U \neq \emptyset$. A descriptive EF-proximity δ_{Φ} is defined on $\mathcal{P}$(U), and it fulfills the conditions below for all subsets M, N, Q *of* $\mathcal{P}(U)$*:*

(DP_0) $M \ \delta_\Phi \ N$ implies $M \neq \emptyset, N \neq \emptyset$

(DP_1) $M \ \delta_\Phi \ N$ implies $N \ \delta_\Phi \ M$ (descriptive symmetry).

(DP_2) $M \cap N \neq \emptyset$ implies $M \ \delta_\Phi \ N$.

(DP_3) $(M \cup N) \ \delta_\Phi \ Q \Leftrightarrow M \ \delta_\Phi \ Q$ or $N \ \delta_\Phi \ Q$.

(DP_4) If, for any $M, N \subset U$, $M \ \not\delta_\Phi \ N$, subsets P,Q⊂U exists, satisfying P∪Q=U with $M \ \not\delta_\Phi \ P$ and $N \ \not\delta_\Phi \ Q$. (U, δ_Φ) is a descriptive EF-proximity space.

Definition 3 *[15]* For the relator space $(U, \mathcal{R})$,

$$\mu_{\mathcal{R}}: \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow [0,1]$$
$$(M, N) \longmapsto \mu_{\mathcal{R}}(M, N)$$

be a fuzzy relation and $M, N \subset U$; then

$$\mathcal{R} = \{((M, N), \mu_{\mathcal{R}}(M, N)) | (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$$

is known as a fuzzy proximity relation for all $M, N, P \in P(X)$, if it fulfills conditions below:

1) $\mu_{\mathcal{R}}(M, \emptyset) = 0$.

2) $\mu_{\mathcal{R}}(M, N) = \mu_{\mathcal{R}}(N, M)$.

3) $\mu_{\mathcal{R}}(M, N) \neq 0$ implies $M \ \mathcal{R} \ N$.

4) $\mu_{\mathcal{R}}(M, (N \cup Q)) \neq 0$ implies $\mu_{\mathcal{R}}(M, N) \neq 0$ and $M \ \mathcal{R} \ N$ or $\mu_{\mathcal{R}}(M, Q) \neq 0$ and $M \ \mathcal{R} \ Q$.

Definition 4 *[6] Suppose that U be a set that is not empty,* $D([0,1])$ *denote the collection of each closed subinterval of the interval of* $[0,1]$. $N = [N^L, N^U]: U \rightarrow D([0,1])$ *is known as an interval-valued fuzzy set, in U.* N^L *and* N^U *are fuzzy sets in U satisfying* $N^L(t) \leq N^U(t)$ *and* $N(t) = [N^L(t), N^U(t)]$ *for each* $t \in U$. N^L *and* N^U *are the lower and upper points of* $D([0,1])$*, respectively.*
It is evident that every fuzzy set N in U is also an interval-valued fuzzy set represented as $N = [N, N]$.

2. Main Theorem and Proof

Definition 5 *Suppose that* $(U, \mathcal{R})$ *be a proximal relator space,* $D(I)$ *be the set of all closed sub-intervals of the unit interval I, and* $\tau_{\mathcal{R}}$ *be a fuzzy relation which is given below.*

$$\tau_{\mathcal{R}}: \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow D(I)$$
$$(M, N) \longmapsto \tau_{\mathcal{R}}(M, N)$$

$\mathcal{R}_\tau$ is named an interval-valued fuzzy proximiy relation, if it holds the following conditions:

$IV1)$ $\tau_{\mathcal{R}}(M, \varnothing) = [0,0] = 0.$

$IV2)$ $\tau_{\mathcal{R}}(M, N) = \tau_{\mathcal{R}}(N, M).$
$IV3)$ $\tau_{\mathcal{R}}(M, N) \neq 0 \Rightarrow M\mathcal{R}_{\tau}N.$
$IV4)$ $\tau_{\mathcal{R}}(M, N \cup K) \neq 0 \Rightarrow \tau_{\mathcal{R}}(M, N) \neq 0$ and $M\mathcal{R}_{\tau}N$ or $\tau_{\mathcal{R}}(M, K) \neq 0$ and $M\mathcal{R}_{\tau}K$.

For each $(M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)$, $\tau_{\mathcal{R}}(M, N)$ gives the interval of the link between the sets M and N. In other words, $\tau_{\mathcal{R}}(M, N)$ means in which interval the sets M and N are close to each other. Furthermore, $\tau_{\mathcal{R}}(M, N)$ is known as interval-valued fuzzy proximity measure. Also, the collection of all interval-valued fuzzy proximity relations on $\mathcal{P}(U) \times \mathcal{P}(U)$ is symbolized by $IVFR(\mathcal{P}(U))$.
$\tau_{\mathcal{R}}(M, N) = [\tau_{\mathcal{R}}^{L}(M, N), \tau_{\mathcal{R}}^{U}(M, N)]$, where $\tau_{\mathcal{R}}^{L}(M, N)$ and $\tau_{\mathcal{R}}^{U}(M, N)$ are the lower and upper points for the link between the sets M and N. Interval-valued fuzzy proximity relations are presented with the relational matrix:

$$\mathcal{R}_{\tau} = \begin{bmatrix} \tau_{\mathcal{R}}(M_1, N_1) & \cdots & \tau_{\mathcal{R}}(M_1, N_k) \\ \vdots & \ddots & \vdots \\ \tau_{\mathcal{R}}(M_k, N_1) & \cdots & \tau_{\mathcal{R}}(M_k, N_k) \end{bmatrix} = \begin{bmatrix} [a_1, b_1] & \cdots & [a_1, b_k] \\ \vdots & \ddots & \vdots \\ [a_k, b_1] & \cdots & [a_k, b_k] \end{bmatrix}.$$

Symbol	Interpretation
U	*set that is not empty,*
$\mathcal{R}$ on U	*relator* on U,
$(U, \mathcal{R})$	*proximal relator space,*
$\tau_{\mathcal{R}}$	*fuzzy relation,*
I	*the unit interval,*
$D(I)$	*the set of all closed sub-intervals of the unit interva,*
$\mathcal{P}(U)$	*the power set of* U,
$IVFR(\mathcal{P}(U))$	the set of all interval-valued fuzzy proximity relations

Table 1 : Symbols of Interval-valued fuzzy proximity relations

Definition 6 *Supoose that $\mathcal{R}_{\tau}$ be an interval-valued fuzzy proximity relations on $\mathcal{P}(U)$. The complement of an interval-valued fuzzy proximity relation $\mathcal{R}_{\tau}^{C}(M, N)$ presented as $\boldsymbol{\tau}_{\mathcal{R}}(M, N)$ determined by*

$$\boldsymbol{\tau}_{\mathcal{R}}(M, N) = [1 - \tau_{\mathcal{R}}^{U}(M, N), 1 - \tau_{\mathcal{R}}^{L}(M, N)]$$

$$(M, N) \in \mathcal{P}(U) \times \mathcal{P}(U).$$

Definition 7 *Let $\tau_{\mathcal{R}_1}(M, N), \tau_{\mathcal{R}_2}(M, N) \in D(I)$. In this case,*

i) $\tau_{\mathcal{R}_1}(M, N) = \tau_{\mathcal{R}_2}(M, N) \Rightarrow \tau_{\mathcal{R}_1}^L(M, N) = \tau_{\mathcal{R}_2}^L(M, N)$ and $\tau_{\mathcal{R}_1}^U(M, N) = \tau_{\mathcal{R}_2}^U(M, N)$.
ii) $\tau_{\mathcal{R}_1}(M, N) \leq \tau_{\mathcal{R}_2}(M, N) \Rightarrow \tau_{\mathcal{R}_1}^L(M, N) \geq \tau_{\mathcal{R}_2}^L(M, N)$ and $\tau_{\mathcal{R}_1}^U(M, N) \leq \tau_{\mathcal{R}_2}^U(M, N)$.

Definition 8 *Suppose that $U \neq \emptyset$ and $\mathcal{R}_\tau$ be an interval-valued fuzzy proximity relations on $\mathcal{P}(U)$. $(U, \mathcal{R}_\tau)$ is known as an interval-valued fuzzy proximal space.*

Definition 9 *Suppose that $(U, \mathcal{R})$ be an interval-valued fuzzy proximal space. If $\mathcal{R}_\tau$ is an interval-valued fuzzy proximity relations on $\mathcal{P}(U)$, in this case $(U, \mathcal{R}, \mathcal{R}_\tau)$ is called an interval-valued fuzzy proximal relator space.*

Example 1 *Let $U = \{a, b, c, d, e, f, g, h, i, j, k\}$. Also, $M = \{m, n, p, r, s, q\}$, $N = \{o, n, p, r, s, w\}$, $P = \{o, n, p, r, s, q\}$ and $Q = \{m, n, p, r, s, y\}$ are subsets of U. Now, we give interval-valued fuzzy proximity relation by using basic proximity. (U, δ) represents basic proximity, and δ is determined as $M \ \delta \ N :\Leftrightarrow M \cap N \neq \emptyset$.*

Therefore, we easily see $M \ \delta \ N$, $M \ \delta \ P$, $M \ \delta \ Q$, $N \ \delta \ P$, $N \ \delta \ Q$ and $P \ \delta \ Q$ for $M \cap N \neq \emptyset$, $M \cap P \neq \emptyset$, $M \cap Q \neq \emptyset$, $N \cap P \neq \emptyset$, $N \cap Q \neq \emptyset$ and $P \cap Q \neq \emptyset$.

$$\tau_{\mathcal{R}}: \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow D(I)$$
$$(M, N) \longmapsto \tau_{\mathcal{R}}(M, N)$$

Afterward, it can be seen that $\tau_{\mathcal{R}}$ is shown below.

$$\tau_\delta(M, N) = [0.2, 0.5]$$
$$\tau_\delta(M, P) = [0.3, 0.7]$$
$$\tau_\delta(M, Q) = [0.11, 0.3]$$
$$\tau_\delta(N, P) = [0.13, 0.6]$$
$$\tau_\delta(N, Q) = [0.21, 0.4]$$
$$\tau_\delta(P, Q) = [0.18, 0.5]$$

In this case, we have that

$\tau_\delta^U(M, N) = 0.2$ gives upper bound of the interval [0.2,0.5] and relationship for M and N,

$\tau_\delta^L(M, N) = 0.5$ gives lower bound of the interval [0.2,0.5] and relationship for M and N.

In a similar way, we determine the maximum and minimum strength of the connection for the other sets. Thus,

M and N, $(M \ \mathcal{R}_{([0.2,0.5])} N)$ are proximal to each other,
M and P, $(M \ \mathcal{R}_{([0.3,0.7])} P)$ are proximal to each other.

The interval-valued fuzzy proximity can be represented using a relational matrix:

$$\mathcal{R}_\tau = \begin{bmatrix} [1,1] & [0.2,0.5] & [0.7,0.3] & [0.11,0.3] \\ [0.2,0.5] & [1,1] & [0.13,0.6] & [0.21,0.4] \\ [0.3,0.7] & [0.13,0.6] & [1,1] & [0.18,0.5] \\ [0.11,0.3] & [0.21,0.4] & [0.18,0.5] & [1,1] \end{bmatrix}.$$

From here, $\mathcal{R}_\tau$ satisfies the axioms $(IV1)-(IV4)$ and so $\mathcal{R}_\tau$ is an interval-valued fuzzy proximity relation.

In this example, we determine the proximity of the sets M, N, P and Q by employing an interval to quantify their similarity or difference. Utilizing interval-valued fuzzy proximity relations, we assess the degrees of connection and dissimilarity among M, N, P and Q For instance, it is observed that the maximum strength of the relation between P and Q is 0.5, whereas the minimum strength is 0.18.

Example 2 *By using Example 1,* $(U, \mathcal{R}_\tau)$ *is an interval-valued fuzzy proximal space. With the basic proximity* $\mathcal{R} = \{\delta\}$*,* $(U, \mathcal{R}, \mathcal{R}_\tau)$ *is an interval-valued fuzzy proximal relator space.*

Example 3 *By using Example 1, we can easily find* $\boldsymbol{x}_{\mathcal{R}}(M, N) = [1 - \tau_{\mathcal{R}}^{U}(M, N), 1 - \tau_{\mathcal{R}}^{L}(M, N)]$

$$\mathcal{R}_\tau^C = \begin{bmatrix} [0,0] & [0.8,0.5] & [0.3,0.7] & [0.99,0.7] \\ [0.8,0.5] & [0,0] & [0.87,0.4] & [0.79,0.6] \\ [0.3,0.7] & [0.87,0.4] & [0,0] & [0.82,0.5] \\ [0.99,0.7] & [0.79,0.6] & [0.82,0.5] & [0,0] \end{bmatrix}.$$

Therefore, $\mathcal{R}_\tau^C$ satisfies the conditions $(IV1)-(IV4)$ and so $\mathcal{R}_{\mathcal{P}}^C$is an interval-valued fuzzy proximity relation.

Definition 10 *Suppose that* (U, δ) *be a proximity space and* δ_τ *be an interval-valued fuzzy proximity relation.* δ_τ *is named an interval-valued fuzzy spatial proximity relation if* δ_τ *meets the followings* $(IV\delta1)$-$(IV\delta4)$*:*

$IV\delta1)$ $\tau_\delta(M, \varnothing) = [0,0]$.
$IV\delta2)$ $\tau_\delta(M, N) = \tau_\delta(N, M)$.
$IV\delta3)$ $\tau_\delta(M, N) \neq 0 \Rightarrow M\delta_\tau N$.
$IV\delta4)$ $\tau_\delta(M, N \cup K) \neq 0$ implies $\tau_\delta(M, N) \neq 0$; $M\delta_\tau N$ or $\tau_\delta(M, K) \neq 0$; $M\delta_\tau K$.

Also, (U, δ, δ_τ) is known as an interval-valued fuzzy spatial proximity space.

Definition 11 *Suppose that* δ_τ *be an interval-valued fuzzy proximity relation. If* δ_τ *meets the axioms* $(IV\delta1)$-$(IV\delta4)$ *in Definition 10 and the following axiom* $(IV\delta5)$*, it is known as an interval-valued fuzzy spatial Lodato proximity relation. Furthermore,* (U, δ, δ_τ) *is called an interval-valued fuzzy spatial Lodato proximity space.*

$IV\delta5)$ $\tau_\delta(M, N) \neq 0$ and $\tau_\delta(n, K) \neq 0$ for all $n \in N$ implies $\tau_\delta(M, K) \neq 0$ and $M\delta_\tau K$.

Definition 12 *Suppose that δ_{Φ_τ} be an interval valued fuzzy proximity relation. $(U, \delta_\Phi, \delta_{\Phi_\tau})$ is said to be an interval valued fuzzy descriptive Lodato proximity space, if δ_{Φ_τ} meets the followings*

$IV\delta_\Phi 1)$ $\tau_{\delta_\Phi}(M, \varnothing) = [0,0]$.
$IV\delta_\Phi 2)$ $\tau_{\delta_\Phi}(M, N) = \tau_{\delta_\Phi}(N, M)$.
$IV\delta_\Phi 3)$ $\tau_{\delta_\Phi}(M, N) \neq 0 \Rightarrow M\delta_{\Phi_\tau}N$.
$IV\delta_\Phi 4)$ $\tau_{\delta_\Phi}(M, N \cup K) \neq 0$ implies $\tau_{\delta_\Phi}(M, N) \neq 0$; $M\delta_{\Phi_\tau}N$ or $\tau_{\delta_\Phi}(M, K) \neq 0$; $M\delta_{\Phi_\tau}K$.
$IV\delta_\Phi 5)$ $\tau_{\delta_\Phi}(M, N) \neq 0$ and $\tau_{\delta_\Phi}(k, K) \neq 0$ for all $n \in N$ implies $\tau_{\delta_\Phi}(M, K) \neq 0$ and $M\delta_{\Phi_\tau}K$.

Remark 1 The theory of descriptive nearness is typically employed when handling subsets that share certain common properties, even if they are not spatially proximate. We consider non-abstract points as those with specific locations and measurable attributes. The descriptive theory is especially applicable when emphasizing the distinguishing features of sets of non-abstract points. As an illustration, for a picture element k in a digital image, its gray-level intensity or color can be considered. Nearness or apartness primarily relies on the selected characteristics under comparison. Although the sets are not physically close, they can be regarded as descriptively near according to their features.

As we said before, the uncertainty that comes from using interval-valued fuzzy sets results in outcomes that are not very precise but may be more reliable and realistic. Interval-valued fuzzy proximity relations are a useful method for measuring how close different groups are to each other, even when there is uncertainty, by using value ranges instead of exact numbers. When comparing interval-valued fuzzy descriptive Lodato proximity relations with interval-valued fuzzy spatial Lodato proximity relations, using interval-valued fuzzy proximity relations, one can determine the degree of nearness between distinct groups, even when the relations may not be spatially close but can still be considered descriptively close.

Definition 13 *Suppose that $\mathcal{R}_{\tau_1}$ and $\mathcal{R}_{\tau_2}$ be interval-valued fuzzy proximity relations on $\mathcal{P}(U)$. In this case,*

1) $\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_2} = \{((M, N), \tau_{\mathcal{R}_1}(M, N) \wedge \tau_{\mathcal{R}_2}(M, N) | (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$.
Namely, $\wedge\, \tau_{\mathcal{R}_i}(M, N) = [\wedge \tau^L_{\mathcal{R}_i}(M, N),\ \wedge \tau^U_{\mathcal{R}_i}(M, N)]$.
2) $\mathcal{R}_{\tau_1} \vee \mathcal{R}_{\tau_2} = \{((M, N), \tau_{\mathcal{R}_1}(M, N) \vee \tau_{\mathcal{R}_2}(M, N) | (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$.
Namely, $\vee \tau_{\mathcal{R}_i}(M, N) = [\vee \tau^L_{\mathcal{R}_i}(M, N),\ \vee \tau^U_{\mathcal{R}_i}(M, N)]$.

Theorem 1 *Let $\mathcal{R}_{\tau_1}$, $\mathcal{R}_{\tau_2}$ and $\mathcal{R}_{\tau_3}$ be interval-valued fuzzy proximity relations on $\mathcal{P}(U)$. The following properties hold:*

1) $\mathcal{R}_{\tau_1} \wedge (\mathcal{R}_{\tau_2} \vee \mathcal{R}_{\tau_3}) = (\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_2}) \vee (\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_3})$.
2) $\mathcal{R}_{\tau_1} \vee (\mathcal{R}_{\tau_2} \wedge \mathcal{R}_{\tau_3}) = (\mathcal{R}_{\tau_1} \vee \mathcal{R}_{\tau_2}) \wedge (\mathcal{R}_{\tau_1} \vee \mathcal{R}_{\tau_3})$.
3) $\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_2} \leq \mathcal{R}_{\tau_1}$.

4) $\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_2} \leq \mathcal{R}_{\tau_2}$.

Proof. Let $\mathcal{R}_{\tau_1}$, $\mathcal{R}_{\tau_2}$ and $\mathcal{R}_{\tau_3}$ be interval-valued fuzzy proximity relations.

1) We utilize the fact that the operators $\wedge$ and $\vee$ satisfy the distributive law when applied to elements of $[0,1]$.

$$\begin{aligned}\tau_{\mathcal{R}_1\wedge(\mathcal{R}_2\vee\mathcal{R}_3)}(M,N) &= \tau_{\mathcal{R}_1}(M,N) \wedge \{\tau_{\mathcal{R}_2}(M,N) \vee \mu_{\mathcal{R}_3}(M,N)\}\\ &= \{\tau_{\mathcal{R}_1}(M,N) \wedge \tau_{\mathcal{R}_2}(M,N)\} \vee \{\tau_{\mathcal{R}_1}(M,N) \wedge \tau_{\mathcal{R}_3}(M,N)\}\\ &= \tau_{\mathcal{R}_1\wedge\mathcal{R}_2}(M,N) \vee \tau_{\mathcal{R}_1\wedge\mathcal{R}_3}(M,N)\\ &= \tau_{(\mathcal{R}_1\wedge\mathcal{R}_2)\vee(\mathcal{R}_1\wedge\mathcal{R}_3)}(M,N).\end{aligned}$$

2) The proof can be carried out in a manner similar to the previous one.

$$\begin{aligned}\tau_{\mathcal{R}_1\vee(\mathcal{R}_2\wedge\mathcal{R}_3)}(M,N) &= \tau_{\mathcal{R}_1}(M,N) \vee \{\tau_{\mathcal{R}_2}(M,N) \wedge \mu_{\mathcal{R}_3}(M,N)\}\\ &= \{\tau_{\mathcal{R}_1}(M,N) \vee \tau_{\mathcal{R}_2}(M,N)\} \wedge \{\tau_{\mathcal{R}_1}(M,N) \vee \tau_{\mathcal{R}_3}(M,N)\}\\ &= \tau_{\mathcal{R}_1\vee\mathcal{R}_2}(M,N) \wedge \tau_{\mathcal{R}_1\vee\mathcal{R}_3}(M,N)\\ &= \tau_{(\mathcal{R}_1\vee\mathcal{R}_2)\wedge(\mathcal{R}_1\vee\mathcal{R}_3)}(M,N).\end{aligned}$$

$3-4)$ $\tau_{\mathcal{R}_1\wedge\mathcal{R}_3}(M,N) = \tau_{\mathcal{R}_1}(M,N) \wedge \tau_{\mathcal{R}_2}(M,N) \leq \tau_{\mathcal{R}_1}(M,N)$ (or $\tau_{\mathcal{R}_2}(M,N)$) the property of the the operator $\wedge$.

Thus, $\mathcal{R}_{\tau_1} \wedge \mathcal{R}_{\tau_2} \leq \mathcal{R}_{\tau_1}$(or $\mathcal{R}_{\tau_2}$).

Definition 14 *Let* $\mathcal{R}_{\tau_1} \in IVFR(\mathcal{P}(U) \times \mathcal{P}(V))$ *and* $\mathcal{R}_{\tau_2} \in IVFR(\mathcal{P}(V) \times \mathcal{P}(W))$ *be two interval-valued fuzzy proximity relations:*

$$\mathcal{R}_{\tau_1} = \{\big((M,N), \tau_{\mathcal{R}_1}(M,N)\big) \,|\, (M,N) \in \mathcal{P}(U) \times \mathcal{P}(V)\}$$

and
$$\mathcal{R}_{\tau_2} = \{\big((N,K), \tau_{\mathcal{R}_2}(N,K)\big) \,|\, (N,K) \in \mathcal{P}(V) \times \mathcal{P}(W)\}.$$

$\mathcal{R}_{\tau_1} \circ \mathcal{R}_{\tau_2} \in IVFR(\mathcal{P}(U) \times \mathcal{P}(W))$ is determined as

$\mathcal{R}_{\tau_1} \circ \mathcal{R}_{\tau_2} = \{\big((M,K), \tau_{\mathcal{R}_1\circ\mathcal{R}_2}(M,K)\big) \,|\, (M,K) \in \mathcal{P}(U) \times \mathcal{P}(W)\}$ where

$$\mathcal{R}_{\tau_1} \circ \mathcal{R}_{\tau_2} = [\bigvee_{N\in P(V)} \{{\mathcal{R}_{\tau_1}}^L(M,N) \wedge \mathcal{R}^L_{\tau_2}(N,K)\}, \bigvee_{N\in P(V)} \{{\mathcal{R}_{\tau_1}}^U(M,N) \wedge \mathcal{R}^U_{\tau_2}(N,K)\}].$$

Example 4 *Let* $\mathcal{R}_{\tau_1}$ *and* $\mathcal{R}_{\tau_2}$ *be two interval-valued fuzzy proximity relations on* $\mathcal{P}(U)$.

$$\mathcal{R}_{\tau_1} = \begin{bmatrix} [1,1] & [0.4,0.1] & [0.6,0.8] & [0.2,0.5] \\ [0.4,0.1] & [1,1] & [0.16,0.42] & [0.6,0.17] \\ [0.6,0.8] & [0.16,0.42] & [1,1] & [0.3,0.25] \\ [0.2,0.5] & [0.6,0.17] & [0.3,0.25] & [1,1] \end{bmatrix}$$

and

$$\mathcal{R}_{\tau_2} = \begin{bmatrix} [1,1] & [0.7,0.23] & [0.4,0.42] & [0.11,0.5] \\ [0.7,0.23] & [1,1] & [0.75,0.3] & [0.66,0.15] \\ [0.4,0.42] & [0.75,0.3] & [1,1] & [0.8,0.9] \\ [0.11,0.5] & [0.66,0.15] & [0.8,0.9] & [1,1] \end{bmatrix}.$$

We can be found the max–min composition of them.

$$\mathcal{R}_{\tau_1} \circ \mathcal{R}_{\tau_2} = \begin{bmatrix} [1,1] & [0.7,0.3] & [0.6,0.8] & [0.6,0.8] \\ [0.7,0.42] & [1,1] & [0.75,0.42] & [0.66,0.42] \\ [0.6,0.8] & [0.75,0.42] & [1,1] & [0.8,0.9] \\ [0.6,0.5] & [0.66,0.25] & [0.8,0.9] & [1,1] \end{bmatrix}.$$

Definition 15 *Let $\mathcal{R}_{\tau_1}, \mathcal{R}_{\tau_2}$ present two interval-valued fuzzy proximity relations, and $\tau_{\mathcal{R}_1}, \tau_{\mathcal{R}_2}$ present membership functions of $\mathcal{R}_{\tau_1}, \mathcal{R}_{\tau_2}$. The interval-valued fuzzy proximity intersection of $\tau_{\mathcal{R}_1}$ and $\tau_{\mathcal{R}_2}$ is expressed by $\tau_{\mathcal{R}_1} \otimes \tau_{\mathcal{R}_2}$, and described as below.*

$$\begin{aligned} (\tau_{\mathcal{R}_1} \otimes \tau_{\mathcal{R}_2})(M,N) &= \tau_{\mathcal{R}_1}(M,N) \wedge \tau_{\mathcal{R}_2}(M,N) \\ &= [\tau^L_{\mathcal{R}_1}(M,N) \wedge \tau^L_{\mathcal{R}_2}(M,N), \tau^U_{\mathcal{R}_1}(M,N) \wedge \tau^U_{\mathcal{R}_2}(M,N)]. \end{aligned}$$

Example 5 *Let $\mathcal{R}_{\tau_1}$ and $\mathcal{R}_{\tau_2}$ be two interval-valued fuzzy proximity relations on $\mathcal{P}(U)$.*

$$\mathcal{R}_{\tau_1} = \begin{bmatrix} [1,1] & [0.4,0.1] & [0.6,0.8] & [0.2,0.5] \\ [0.4,0.1] & [1,1] & [0.16,0.42] & [0.6,0.17] \\ [0.6,0.8] & [0.16,0.42] & [1,1] & [0.3,0.25] \\ [0.2,0.5] & [0.6,0.17] & [0.3,0.25] & [1,1] \end{bmatrix}$$

and

$$\mathcal{R}_{\tau_2} = \begin{bmatrix} [1,1] & [0.7,0.23] & [0.4,0.42] & [0.11,0.5] \\ [0.7,0.23] & [1,1] & [0.75,0.3] & [0.66,0.15] \\ [0.4,0.42] & [0.75,0.3] & [1,1] & [0.8,0.9] \\ [0.11,0.5] & [0.66,0.15] & [0.8,0.9] & [1,1] \end{bmatrix}.$$

We can be find the intersection of them.

$$\mathcal{R}_{\tau_1 \otimes \tau_2} = \begin{bmatrix} [1,1] & [0.4,0.1] & [0.4,0.42] & [0.11,0.5] \\ [0.4,0.1] & [1,1] & [0.16,0.3] & [0.6,0.15] \\ [0.4,0.42] & [0.16,0.3] & [1,1] & [0.3,0.25] \\ [0.11,0.5] & [0.6,0.15] & [0.3,0.25] & [1,1] \end{bmatrix}.$$

Definition 16 *Let $\mathcal{R}_{\tau_1}, \mathcal{R}_{\tau_2}$ present two interval-valued fuzzy proximity relations, and $\tau_{\mathcal{R}_1}, \tau_{\mathcal{R}_2}$ present membership functions of $\mathcal{R}_{\tau_1}, \mathcal{R}_{\tau_2}$. The interval-valued fuzzy proximity union of $\tau_{\mathcal{R}_1}$ and $\tau_{\mathcal{R}_2}$ is expressed by $\tau_{\mathcal{R}_1} \oplus \tau_{\mathcal{R}_2}$, and described as below.*

$$\begin{aligned} (\tau_{\mathcal{R}_1} \oplus \tau_{\mathcal{R}_2})(M,N) &= \tau_{\mathcal{R}_1}(M,N) \vee \tau_{\mathcal{R}_2}(M,N) \\ &= [\tau^L_{\mathcal{R}_1}(M,N) \vee \tau^L_{\mathcal{R}_2}(M,N), \tau^U_{\mathcal{R}_1}(M,N) \vee \tau^U_{\mathcal{R}_2}(M,N)]. \end{aligned}$$

Example 6 *Let $\mathcal{R}_{\tau_1}$ and $\mathcal{R}_{\tau_2}$ be two interval-valued fuzzy proximity relations on $\mathcal{P}(U)$.*

$$\mathcal{R}_{\tau_1} = \begin{bmatrix} [1,1] & [0.4,0.1] & [0.6,0.8] & [0.2,0.5] \\ [0.4,0.1] & [1,1] & [0.16,0.42] & [0.6,0.17] \\ [0.6,0.8] & [0.16,0.42] & [1,1] & [0.3,0.25] \\ [0.2,0.5] & [0.6,0.17] & [0.3,0.25] & [1,1] \end{bmatrix}$$

and

$$\mathcal{R}_{\tau_2} = \begin{bmatrix} [1,1] & [0.7,0.23] & [0.4,0.42] & [0.11,0.5] \\ [0.7,0.23] & [1,1] & [0.75,0.3] & [0.66,0.15] \\ [0.4,0.42] & [0.75,0.3] & [1,1] & [0.8,0.9] \\ [0.11,0.5] & [0.66,0.15] & [0.8,0.9] & [1,1] \end{bmatrix}.$$

We can find the union of them.

$$\mathcal{R}_{\tau_1 \oplus \tau_2} = \begin{bmatrix} [1,1] & [0.7,0.23] & [0.6,0.8] & [0.2,0.5] \\ [0.7,0.23] & [1,1] & [0.75,0.42] & [0.66,0.17] \\ [0.6,0.8] & [0.75,0.42] & [1,1] & [0.8,0.9] \\ [0.2,0.5] & [0.66,0.17] & [0.8,0.9] & [1,1] \end{bmatrix}.$$

4. Conclusion

The interval-valued fuzzy set is an extended concept in which the membership of each element is represented by a closed interval. Clearly, according to this explanation, it entails two functions N^U and N^L. To explore the new approach using proximal spaces for interval-valued fuzzy sets, we introduce the definition of interval-valued fuzzy proximal spaces. After investigating some results concerning interval-valued fuzzy proximity relations, our study has focused on the relationship between Lodato proximity and interval-valued fuzzy proximity. Future work may focus on expanding interval-valued fuzzy sets to encompass other forms, for instance, neutrosophic, bipolar fuzzy sets, characteristically near sets.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

References

[1] Atanassov K.T., (1986) Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20, 87–96.

[2] Bustince H., (2000) Indicator of inclusion grade for intervalvalued fuzzy sets. Application to approximate reasoning based on interval-valued fuzzy sets., International Journal of Approximate Reasoning, 23(3) 137-209.

[3] Bustince H., Barrenechea, E., Pagola, M., Fernandez, J., (2009) Interval-valued fuzzy sets constructed from matrices: Application to edge detection., Fuzzy Sets and Systems, 60(13), 1819-1840.

[4] Chen S., Wang H., (2009) Evaluating students answer scripts based on interval-valued fuzzy grade sheets., *Expert Systems with Applications*, 36(6), 9839-9846 (2009).

[5] Choi, B., Rhee, F., (2009) Interval type-2 fuzzy membership function generation methods for pattern recognition, Information Sciences, 179(13), 2102-2122.

[6] Concilio A.D., Guadagni C., Peters J.F., Ramanna S., (2018) Descriptive proximities. properties and interplay between classical proximities and overlap, Mat. Comput. Sci. 12: 91–106.

[7] Efremoviĉ V.A., (1952) The geometry of proximity. Mat. Sb. N.S. 31(73), 189–200.

[8] Gorzalzany M.B., (1987) A method of inference inapproximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21, 1-17.

[9] Grattan-Guiness I., (1975) Fuzzy membership mapped onto interval and many-valued quantities. Z. Math. Logik. Grundladen Math. 22, 149-160.

[10] Jahn K.U., (1975) Intervall-wertige Mengen, Math.Nach. 68, 115-132.

[11] Lodato M.W., (1964) On topologically induced generalized proximity relations I. Proc. Amer. Math. Soc. 15(3), 417–422.

[12] Lodato M.W., (1966) On topologically induced generalized proximity relations II. Pacific J. Math. 17(1), 131–135.

[13] Martinez, R., Castillo, O., Aguilar, L.T., (2009) Optimization of interval type-2 fuzzy logic controllers for a perturbed autonomous wheeled mobile robot usinggenetic algorithms., Information Sciences, 179(13), 2158- 2174.

[14] Naimpally S.A., Warrack B.D., (1970) Proximity Spaces. Cambridge Tract. 59, England.

[15] Öztürk M.A., İnan E., Tekin Ö., Peters J.F. (2019) Fuzzy proximal relator spaces. Neural Comput. Appl. 31(7), 3201–3210.

[16] Öztürk M.A., (2025) Intuitionistic fuzzy proximal relator spaces, Southeast Asian Bull. Math., (Accepted).

[17] Peters J.F. (2013) Local near sets: pattern discovery in proximity spaces, Mat. Comput. Sci. 7(1), 87–106.

[18] Peters J.F., Naimpally S.A., (2012) Applications of near sets. Notices of the Amer. Math. Soc. 59(4), 536–542.

[19] Peters J.F.,(2013) Near sets: an introduction. Math. Comput. Sci. 7(1), 3–9.

[20] Peters J.F., (2016) Proximal relator spaces. Filomat 30(2), 469–472.

[21] Peters J.F., (2016) Computational Proximity. Excursions in the Topology of Digital Images, Intelligent Systems Reference Library, Springer.

[22] Sambuc R., (1975) Fonctions ϕ-floues. Application laide au diagnostic en pathologie thyroidienne, PhD Thesis, Univ. Marseille, France.

[23] Roy M.K., Biswas R. (1992) I-v fuzzy relations and Sanchez's approach for medical diagnosis, Fuzzy Sets and Systems 47, 35-38.

[24] Szás À., (1987) Basic tools and mild continuties in relator spaces. Acta Math. Hungar. 50(3-4), 177–201.

[25] Tekin Ö., Öztürk M.A., İnan E., (2021) L-Fuzzy relations via proximal spaces. Thai J. Math. 19(2), 557–570, (2021).

[26] Tekin Ö., Öztürk M.A., (2022) Complex fuzzy proximal relator spaces, J. Mult.-Val. Log. Soft Comput. 38(3-4), 355–385.

[27] Tekin Ö., (2023) An approach for spherical fuzzy relations via relator spaces, J Intell Fuzzy Systems, 45(4), 6875–6886.

[28] Tekin Ö., (2024) A novel approximation of Pythagorean fuzzy sets via proximal relator spaces, J. Mult.-Val. Log. Soft Comput. 44(1-2), 71-93.

[29] Turksen I.B., (1986) Interval valued fuzzy sets based on normal forms. Fuzzy Sets and Systems, 20(2), 191-210.

[30] Xia, X. S., Liang, Q. L., (2008) Crosslayer design for mobile ad hoc networks using interval type-2 fuzzy logic systems., International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 16(3), 391- 408.

[31] Zadeh L.A., (1965) Fuzzy sets, Inform. and Control 8, 338-353.

[32] Zadeh L.A., (1975) The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8, 199-249.