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Research Article

An Efficient Method for Numerical Solution of Fractional Pantograph Equations

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Abstract

The pantograph differential equation is one of the most well-known types of delay differential equations. There are many different pantograph equations in quantum mechanics. In this study, we present Legendre - spectral method for solving fractional pantograph equations using Caputo derivative. The primary aim of this approach is to transform such problems into a system of algebraic equation, thereby significantly simplifying their solution. The proposed method is highly successful and effective in solving fractional pantograph equations. All examples are solved with the help of Maple and MATLAB.

Keywords: Shifted Legendre polynomials, spectral method, pantograph equation, Caputo fractional derivative

Kesirli Pantograf Denklemlerinin Nümerik Çözümleri için Etkili Yöntemler

Öz

Pantograf diferansiyel denklemi iyi bilinen gecikmeli diferansiyel denklemlerden biridir. Kuantum mekaniğinde birçok farklı pantograf denklemleri kullanılmaktadır. Bu çalışmada, Caputo kesirli türevi yardımıyla kesirli pantograf diferansiyel denklemlerini çözmek için Legendre - spektral yöntemi uygulanmıştır. Bu yöntem ile, denklemi çözmek için cebirsel bir denklem sistemi oluşturularak, problemi büyük ölçüde basitleştirmek amaçlanmaktadır. Spectral yöntemler, kesirli pantograf denklemlerini çözmede son derece başarılı ve etkilidir. Bu çalışmadaki problemler Maple ve Matlab programları yardımıyla çözülmüştür.

Anahtar Kelimeler: Ötelenmiş Legendre polinomları, spektral metot, pantograf denklemi, Caputo kesirli türevi

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1. Introduction

In the 1960s, there was a desire to develop a new type of electric locomotive to increase train speeds. The most important component of this high-speed train was the pantograph. The purpose of the pantograph is to provide current through an overhead wire that is important for the movement of the locomotive [1]. When the pantograph passes the supports of the overhead wire, it must remain in contact with the overhead wire for the entire time, which is a critical transition. To find the movement of the pantograph, a mathematical model was created and thus the pantograph equation was obtained. While finding solution of model, a delay differential equation (DDE) was encountered, and this delay differential equation was known as the pantograph equation.

Fractional Calculus is the generalization of integral and ordinary derivative to non-integer order. Over the years, many mathematicians contributed. The Fractional Calculus has attracted the interest of researchers in several areas, including mathematics, biology, chemistry, physics, engineering and finance [2-10].

Pantograph differential equation is a type of DDE which is studied in fields such as electrodynamics, biology, control systems and economics. The pantograph equations are functional differential equations with a variable or proportional delay. Many different methods are used to solve pantograph equations [11-15].

In this study, Legendre spectral method is applied to solve fractional pantograph differential equation. Close form of the fractional pantograph differential equation can be written as follows [16]

$$D^{\alpha} y(x) = Q\big(x, y(x), y(\lambda x)\big) \tag{1}$$

Q is nonlinear or linear, continuous functions. The general pantograph equation can be given as following

$$D^{\alpha} y(x) = ay(x) + by(\lambda x)$$

with the initial conditions

$$y(x) = \mu_0 \tag{2}$$

where a, b, λ are real constants, $0 \le \lambda \le 1$.

This study is structured as follows: In section 2, we introduce some preliminaries of Legendre polynomials and definitions regarding to Caputo's derivatives and we present the procedure of the method for solving fractional Pantograph differential equation and formulated equations at some suitable points together with initial conditions. In section 3, illustrative examples are given. Lastly, a brief conclusion is presented in Section 4.

2. Main Theorem and Proof

Some essential mathematical preliminaries and definitions of fractional calculus are given, in Section 2.

Definition 2.1: Riemann - Liouville fractional integral of $\alpha \geq 0$ for a continuous function y on [a, b] which is defined by [17],

$$J_a{}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-\tau)^{\alpha-1} y(\tau)\, d\tau \;,\; \alpha \geq 0$$

for $t > 0 \;, \alpha \;, a \;, t \in R$.

Definition 2.2: Riemann - Liouville fractional derivative order α is defined as [17],

$$D_z{}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau \;,\; n-1 < \alpha < n,\quad n \in \mathrm{N}$$

for $\alpha > 0 \;, t > 0 \;, \alpha \;, a \;, t \in R$.

Definition 2.3: The Caputo fractional derivative of order α is defined as

$$D^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n}{dt^n} y(t) dt \tag{3}$$

for $n-1 < \alpha \leq n \;,\; n \in N \;,\; t > 0 \;,\; y \in C_{-1}^n$.

Operator of fractional derivative in the Caputo sense is a linear operation [18],

$$D^{\alpha}(\sigma f(x) + \psi\, g(x)) = \sigma D^{\alpha} f(x) + \psi\, D^{\alpha} g(x)\,. \tag{4}$$

Formula for the Caputo fractional derivative of a power function as follows,

$$D^{\alpha} x^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, n-1 < \alpha < n \;, & \beta > n-1, \quad \beta \in R. \\ \\ 0 \qquad\qquad , n-1 < \alpha < n \;, & \beta \leq n-1, \quad \beta \in \mathrm{N}. \end{cases} \tag{5}$$

The Caputo derivative provides the following

$$D^{\alpha} C = 0 \,,\; \text{C a constant.} \tag{6}$$

2.1 The Shifted Legendre Polynomials

The Legendre polynomials are orthogonal polynomials on the interval $[-1,1]$ and are written with the following recurrence relation [19],

$$L_{i+1}(z)=\frac{(2i+1)}{(i+1)}zL_i(z)-\frac{i}{(i+1)}L_{i-1}(z),\quad i=1,2,...$$

where $L_0(z)=1$ and $L_1(z)=z$. Change of variable $z=2x-1$ is performed and interval $[-1,1]$ is converted to interval $[0,1]$. Shifted Legendre polynomials are denoted by $P_i(x)$. The shifted Legendre polynomials are defined as follows:

$$P_{i+1}(x)=\frac{(2i+1)(2x-1)}{(i+1)}P_i(z)-\frac{i}{(i+1)}P_{i-1}(x)$$

$P_0(x)=1$ and $P_1(x)=2x-1$. $P_i(x)$ is given in analytical form as following

$$P_i(x)=\sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2}x^k \tag{7}$$

Note that $P_i(0)=(-1)^i$ and $P_i(1)=1$. From orthogonality condition

$$\int_0^1 P_i(x)P_j(x)dx=\begin{cases}\dfrac{1}{2i+1}, & for \quad i=j\\ 0\ , & \quad\quad\ \ i\neq j\end{cases}$$

The function $y(x)$, integrable in interval $[0,1]$. $y(x)$ may be expressed as

$$y(x)=\sum_{j=0}^{\infty}c_jP_j(x)$$

where coefficients c_j can be given as:

$$c_j=(2j+1)\int_0^1 y(x)P_j(x)dx,\qquad j=1,2,...$$

We consider the first $m+1$ terms of the shifted Legendre polynomials [19]. We can write as

$$y_m(x)=\sum_{j=0}^{m}c_jP_j(x).$$

Theorem 1: [20] Suppose that $y(x)$ is approximated by shifted Legendre polynomials as $y_m(x)=\sum_{j=0}^{m}c_jP_j(x)$ for $\alpha>0$, then

$$D^{\alpha}(y_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i \Omega_{i,k}{}^{(\alpha)} x^{k-\alpha}$$

$\Omega_{i,k}{}^{(\alpha)}$ can be written as follows

$$\Omega_{i,k}{}^{(\alpha)} = \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\,\Gamma(k+1-\alpha)}$$

Proof: Fractional derivative in Caputo sense is a linear operation. We have

$$D^{\alpha}(y_m(x)) = \sum_{i=0}^{m} c_i D^{(\alpha)}(P_i(x))$$

Substituting equations (4), (5) and (6) in equation (7); we obtain

$$D^{\alpha}(P_i(x)) = 0\,,\quad i = 0,1,\ldots,\lceil\alpha\rceil-1,\quad \alpha > 0$$

$$D^{\alpha}P_i(x) = \sum_{k=0}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{\alpha}x^k = \sum_{k=\lceil\alpha\rceil}^{i}\frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\,\Gamma(k+1-\alpha)} x^{k-\alpha}$$

So, Theorem 1 is proved.

2.2. Legendre spectral method

Let us handle the pantograph differential equation given in equation (1). In this section, we will use the Legendre spectral method to solve this pantograph differential equation with the initial conditions. If the proposed method is applied to fractional pantograph differential equation, approximate solution can be written as follows,

$$y_m(x) = \sum_{j=0}^{m} c_j P_j(x) \tag{8}$$

Substituting equation (8) and Theorem 1 into equation (1), the following expression is obtained,

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i \Omega_{i,k}{}^{(\alpha)} x^{k-\alpha} = Q\left(x, \sum_{j=0}^{m} c_j P_j(x), \sum_{j=0}^{m} c_j P_j(\lambda x)\right). \tag{9}$$

We can collocate equation (9) at $(m+1-\lceil\alpha\rceil)$ points x_p and rearranging this expression as,

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i \Omega_{i,k}{}^{(\alpha)} x_p{}^{k-\alpha} = Q\left(x_p, \sum_{j=0}^{m} c_j P_j(x_p), \sum_{j=0}^{m} c_j P_j(\lambda x_p)\right),\ p = 0,1,\ldots,\ m-\lceil\alpha\rceil \tag{10}$$

We will use roots of the shifted Legendre polynomial $P_{m+1-\lceil\alpha\rceil}(x)$ for proper collocation points.

Also, if equation (8) is written for the initial condition, we can write the following expression,

$$\sum_{i=0}^{m}(-1)^i c_i = \mu_0 \tag{11}$$

With equation (10) and the equation of the initial condition, $(m+1)$ linear or nonlinear equations are obtained, and these equations form a system of equations. The system of equation obtained is solved by using Newton iteration method for the unknown coefficients c_i. As a result, approximate solution of $y(x)$ given in equation (1) may be found [20].

3. Results and Discussion

Legendre spectral method is applied to solve the fractional pantograph equation, in this section. All computations, figures and table are generated using the Maple and MATLAB.

Example 1: Consider the pantograph equation [21],

$$D^{\alpha} y(x) = -\frac{5}{4} e^{\frac{-x}{4}} y\left(\frac{4}{5}x\right) \tag{12}$$

with the initial conditions

$$y(0) = 1 \tag{13}$$

The exact solution is $y(x) = e^{-1.25x}$ for this problem. The polynomial solution for $m = 3$ of Example 1 can be written as,

$$y_3(x) = \sum_{j=0}^{3} c_j P_j(x) \tag{14}$$

Using equation (10) we get

$$\sum_{i=\lceil\alpha\rceil}^{3} \sum_{k=\lceil\alpha\rceil}^{i} c_i \Omega_{i,k}^{(1.0)} x_p^{\,k-1.0} \approx Q\left(x_p, \sum_{j=0}^{3} c_j P_j(x_p), \sum_{j=0}^{3} c_j P_j(\lambda x_p) \right), \quad p = 0, 1, 2. \tag{15}$$

If equation (11) is rewritten for the initial conditions of Example 1, we get

$$c_0 - c_1 + c_2 - c_3 = 1 \tag{16}$$

By using equations (15) and (16), we obtain $(m+1)$ equations. Solving equations (15) and (16) together, the coefficients of the polynomial solution are found as follows,

$$c_0 = 0.5502578, c_1 = -0.4126084, c_2 = -0.04079138, c_3 = 0.0046574$$

Thus, the approximate solution of this problem can be obtained as

$$y(x) = -0.1799140x^3 + 0.7049723x^2 - 1.238428x + 1.$$

The graph of exact solution and approximate solutions at different values of α for $m = 3$ is shown in Figure 1 to make it easier to compare with exact solution. In Figure 2, approximate solutions at different values of α are compared and it can be seen, approximate solution approaches to exact solution when α is close to 1. In Figure 3, we gave the absolute error graph of the obtained numerical results for Example 1.

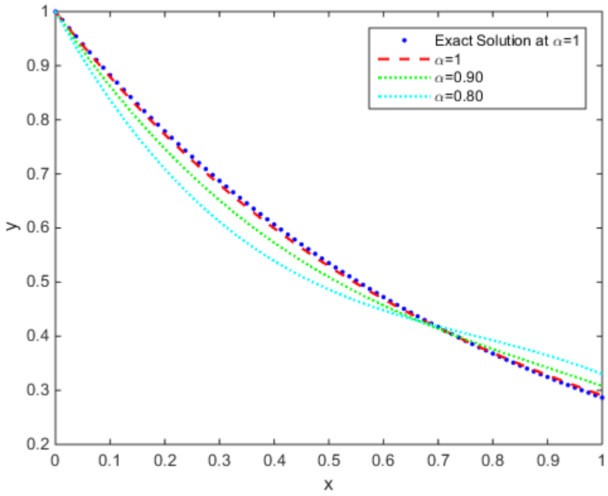

Figure 1. Exact solution and approximate solutions for $\alpha = 1, 0.90, 0.80$ for Example 1.

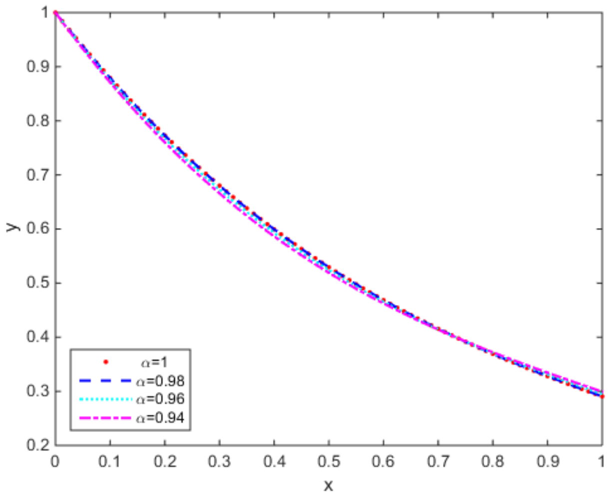

Figure 2. Approximate solutions at $\alpha = 1, 0.98, 0.96, 0.94$ for Example 1.

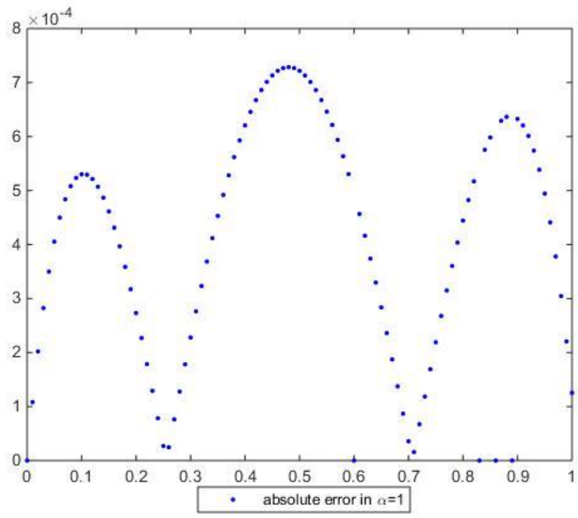

Figure 3. The absolute error at $\alpha = 1$ for Example 1.

Example 2: Consider the pantograph equation [22],

$$D^{\alpha} y(x) = -y(x) + \frac{1}{2} y\left(\frac{1}{2}x\right) + \cos(t) + \sin(t) - \sin\left(\frac{1}{2}x\right)$$

with the initial conditions

$$y(0) = 0$$

Exact solution of this example at $\alpha = 1$ is $y(x) = \sin(x)$.

In Figure 4, the exact solution of Example 2 for $\alpha = 1$ and the obtained numerical results according to the Legendre spectral method are compared for different values of α. From this figure, the proposed method gives quite good results.

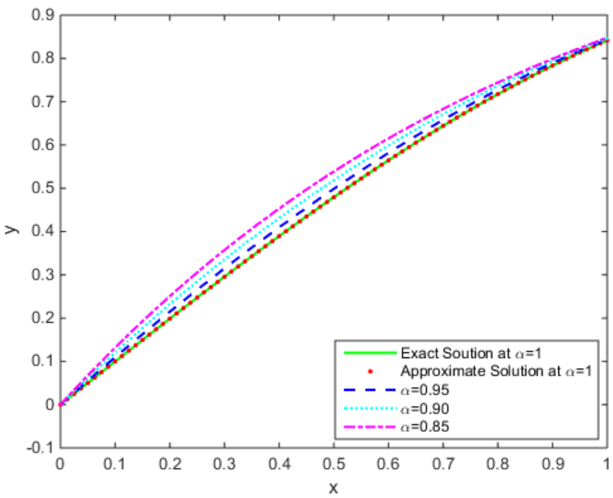

Figure 4. Comparison of exact and approximate solutions in different values of α for Example 2.

Example 3: Consider the pantograph equation [23],

$$D^{\alpha} y(x) = \frac{1}{2} y(x) + \frac{1}{2} e^{\frac{x}{2}} y\left(\frac{x}{2}\right)$$

with the initial conditions

$$y(0) = 1$$

Exact solution of this example at $\alpha = 1$ is $y(x) = e^{x}$.

In Table 1, we gave the absolute errors for $\alpha = 1$. From Table 1, it is seen that the proposed method is high accuracy. In Figure 5, the graph of the approximate solution and the exact solution for $\alpha = 1$ are shown. Approximate solutions for variable values of α is shown in Figure 6.

Table 1. Absolute error at $\alpha = 1$ for Example 3.

x	Exact Solution	$\alpha = 1$	Absolute Error
0.0	1.00000000	0.99999995	0.0000000
0.1	1.10517092	1.10586952	6.9861E-4
0.2	1.22140276	1.22193013	5.2737E-4
0.3	1.34985881	1.34985740	0.0000000
0.4	1.49182470	1.49132269	4.9779E-4
0.5	1.64872127	1.64801425	7.0703E-4
0.6	1.82211880	1.82159498	0.0000000
0.7	2.01375271	2.01374469	0.0000000
0.8	2.22554093	2.22613897	5.9804E-4
0.9	2.45960311	2.46045338	8.5037E-4
1.0	2.71828183	2.71836351	8.1681E-5

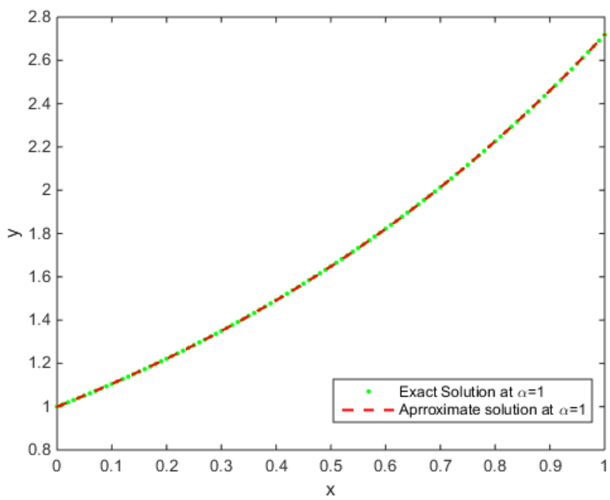

Figure 5. Comparison of exact and approximate solutions at $\alpha = 1$ for Example 3.

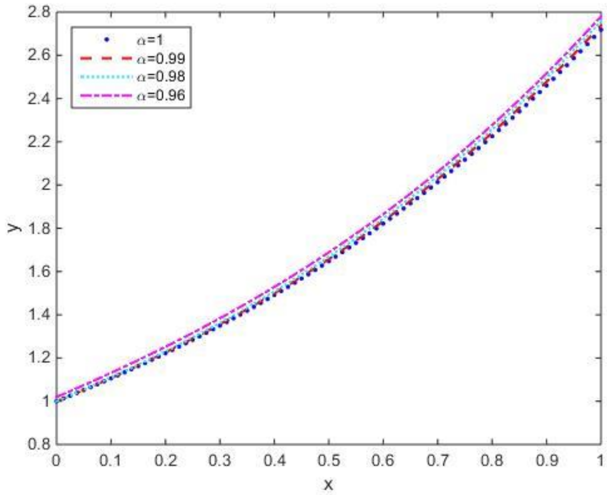

Figure 6. Approximate solutions in different values of α for Example 3.

4. Conclusion

The pantograph differential equation is one of the most known delay differential equations. In this study, the Legendre spectral method is presented to solve fractional pantograph differential equation. The fractional derivatives are described in the Caputo sense. Legendre spectral method is presented in Section 2. The properties of shifted Legendre polynomials are used while creating the method. The method presented in Section 3 has been applied to three examples. The MATLAB and Maple programs have been used to solve these problems. All the given examples show that the Legendre spectral method is highly applicable and effective in solving the fractional pantograph differential equation. If the pantograph equation does not have an exact solution, good approximation solutions can be obtained by the proposed method. The accuracy rate of fractional pantograph equation was investigated with different values m and α values. Better results are obtained while α is close to 1. It has been seen that the with the proposed method has obtained high accurate solutions for the fractional pantograph differential equation.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally to study.

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