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COMMUTATIVITY CRITERIA FOR PRIME AND SEMI-PRIME RINGS WITH SEMI-INVOLUTION

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Abstract. In this article, we study the concept of semi-involution on rings, which in general, is neither a usual involution nor a Jordan involution. Furthermore, we describe the structure of rings that satisfy certain differential identities on rings with semi-involution.

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1. Introduction

In the early stages of general ring theory, striking successes of that theory were theorems that asserted the commutativity of the ring when the elements of a ring were subjected to certain types of algebraic conditions. Much of the initial thrust of the work in this area was either authored by Herstein or inspired by his work (viz. [3]). In this aspect of investigations, many algebraists have been investigated various results in the literature that indicate how the global structure of a ring is often tightly connected to the behavior of additive maps like involutions, automorphisms, derivations etc.

Herstein [3] showed that a prime ring R with nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be a commutative integral domain if its characteristic is not two, and, if the characteristic equals two, must be commutative or an order in a simple algebra which is 4-dimensional over its center. Several authors have proved commutativity theorems for prime rings admitting derivations which are centralizing on R. We first recall that a mapping $f : R \to R$ is called *centralizing* (on R) if $[f(x), x] \in Z(R)$ for all $x \in R$; in the special case where $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be *commuting* on R. In [5], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative.

Throughout, R denotes an associative ring with center $Z(R)$. The *Lie product* of two elements x and y of R is $[x, y] = xy - yx$; while the symbol $x \circ y$ will stand for the *anti-commutator* $xy + yx$. Recall that R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$. A map $d : R \to R$ is a *derivation* of a ring R if d is additive and satisfies the Leibnitz' rule; $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

An additive mapping $* : R \to R$ is called an *involution* if $*$ is an anti-automorphism of order 2; that is, $x^{**} = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be *hermitian* if $x^* = x$ and *skew-hermitian* if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. Note that $H(R) = S(R)$ if $char(R) = 2$. Therefore, we consider R to be a ring with involution such that $char(R) \neq 2$. The involution is said to be of the *first kind* if $Z(R) \subseteq H(R)$, otherwise it is said to be of the *second kind*. In the later case $S(R) \cap Z(R) \neq \{0\}$.

In the spirit of the definition of involution, Yood [6] introduced Jordan involution. An additive map $\sharp : R \to R$ is called *Jordan involution* if for any $x, y \in R$, $x^{\sharp\sharp} = x$ and $(xy + yx)^\sharp = x^\sharp y^\sharp + y^\sharp x^\sharp$. Obviously, every involution is a Jordan involution but the converse need not be true. For example:

Example 1.1. Let us consider

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sharp = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}.$$

It is straightforward to check that $\sharp$ is a Jordan involution but not an involution.

Our purpose here is to continue this line of investigation by considering a class of special additive mappings, which are particular cases of involution and Jordan involution. More precisely, motivated by the notion of involution and Jordan involution, we introduce the concepts of semi-involution as follows.

Definition 1.2. Let R be a ring with a mapping $\natural$. An element of the ring is called $\natural$-symmetric (resp. $\natural$-skew symmetric), if $x^\natural = x$ (resp. $x^\natural = -x$). The set of all $\natural$-symmetric and $\natural$-skew symmetric elements are denoted by $H_\natural$ and $S_\natural$, respectively.

Definition 1.3. An additive map $\natural : R \to R$ is called *semi-involution* if

- for any $x \in R$, $x^{\natural\natural} = x$,
- for any $x \in R$, $(xz)^\natural = x^\natural z^\natural$, where z is a $\natural$-symmetric or $\natural$-skew symmetric central element of R.

Obviously, if R is 2-torsion free, then every Jordan involution is a semi-involution but the converse need not be true. For example:

Example 1.4. (Semi-involution of the first kind)
Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\natural} = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$.
It is straightforward to check that $\natural$ is a semi-involution of the first kind but not a Jordan involution.

Example 1.5. (Semi-involution of the second kind)
Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{C} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} \bar{a} & -\bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$.
It is straightforward to check that $\natural$ is a semi-involution of the second kind but not a Jordan involution.

Now, we will make some use of the following results, which are essential to prove our main theorems:
Fact 1: Let $(R,\natural)$ be a ring with semi-involution. If R is prime and $d(h) = 0$ for all $h \in H_\natural(R) \cap Z(R)$, then $d(s) = 0$ for all $s \in S_\natural(R) \cap Z(R)$.
Fact 2: Let $(R,\natural)$ be a prime ring of characteristic not 2 with semi-involution of the second kind. Then the following assertions are equivalent:

(1) $[x, x^\natural] \in Z(R)$ for all $x \in R$;
(2) $x \circ x^\natural \in Z(R)$ for all $x \in R$;
(3) R is a commutative integral domain.

2. Identities with commutator

In 1997, M. Hongan [4] established that if a 2-torsion free prime ring admits a derivation d such that $d[x,y] + [x,y] \in Z(R)$ (or $d[x,y] - [x,y] \in Z(R)$) for all $x, y \in R$, then R is commutative. Motivated by the above results, in this section we explore the commutativity of a prime ring R in which the derivation d satisfies a more general identities.

Theorem 2.1. *Let $(R,\natural)$ be a prime ring of characteristic not 2 with semi-involution of the second kind. If R admits a derivation d, then the following assertions are equivalent:*

(1) $d[x, x^\natural] \pm [x, x^\natural] \in Z(R)$ *for all* $x \in R$;
(2) R *is a commutative integral domain.*

Proof. It is obvious that (2) implies (1). So we need to prove that (1) $\Rightarrow$ (2).
(1) $\Rightarrow$ (2) We are given that:

$$d[x, x^\natural] + [x, x^\natural] \in Z(R) \quad \text{for all } x \in R. \tag{1}$$

If $d = 0$, then (1) reduces to $[x, x^\natural] \in Z(R)$ for all $x \in R$. So the commutativity of R follows from Fact 2. Hence we may suppose that $d \neq 0$.
A linearization of (1) yields that $d[x, y^\natural] + d[y, x^\natural] + [x, y^\natural] + [y, x^\natural] \in Z(R)$ and writing $y^\natural$ instead of y we get

$$d[x, y] + d[y^\natural, x^\natural] + [x, y] + [y^\natural, x^\natural] \in Z(R) \quad \text{for all } x, y \in R. \tag{2}$$

Replacing y by yh in (2), where $h \in Z(R) \cap H_\natural(R)$ and using it, we obtain

$$([x, y] + [y^\natural, x^\natural])d(h) \in Z(R) \quad \text{for all } x, y \in R. \tag{3}$$

Since R is prime, either $d(h) = 0$ or $[x, y] + [y^\natural, x^\natural] \in Z(R)$.
If $d(h) = 0$ for all $h \in Z(R) \cap H_\natural(R)$, by Fact 1, we conclude that

$$d(s) = 0 \quad \text{for all } s \in Z(R) \cap S(R). \tag{4}$$

Replacing y by ys in (2) where $s \in Z(R) \cap S_\natural(R) \backslash \{0\}$

$$(d[x, y] - d[y^\natural, x^\natural] + [x, y] - [y^\natural, x^\natural])s \in Z(R) \quad \text{for all } x, y \in R. \tag{5}$$

Accordingly, we get

$$d[x, y] - d[y^\natural, x^\natural] + [x, y] - [y^\natural, x^\natural] \in Z(R) \quad \text{for all } x, y \in R. \tag{6}$$

Combining (2) with (6), we find that $d[x, y] + [x, y] \in Z(R)$ for all $x, y \in R$. By [4, Theorem 1], we conclude that R is commutative.
If $[x, y]+[y^\natural, x^\natural] \in Z(R)$ for all $x, y \in R$, then replacing y by $x^\natural$, we get $[x, x^\natural] \in Z(R)$ for all $x \in R$, proving that R is commutative.
The case for "-" follows from the first implication with a slight modification. □

Motivated by the notion of the SCP derivation, the authors in [2] initiated the study of a more general concept by considering the identity $[d(x), d(x^*)] = [x, x^*]$. More precisely, they proved in [2, Theorem 1] that a prime ring $(R, *)$ with involution of the second kind must be commutative if it admits a nonzero derivation d which satisfies $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$.

In the following theorem, we study a more general class of derivations by considering the identity $[d(x), d(x^\natural)] - [x, x^\natural] \in Z(R)$ for all $x \in R$. Our next result is a generalization of [2, Theorem 1], and [1, Theorem 2.6].

Theorem 2.2. *Let $(R, \natural)$ be a prime ring of characteristic not 2 with semi-involution of the second kind. If R admits a derivation d, then the following assertions are equivalent:*

(1) $[d(x), d(x^\natural)] \pm [x, x^\natural] \in Z(R)$ *for all* $x \in R$;
(2) *R is a commutative integral domain.*

Proof. If $d = 0$, then our conditions reduce to $[x, x^\natural] \in Z(R)$ for all $x \in R$. So the commutativity of R follows from Fact 2. Hence we may suppose that $d \neq 0$. First we handle the case for "+" and linearize it, we get

$$[d(x), d(y^\natural)] + [d(y), d(x^\natural)] + [x, y^\natural] + [y, x^\natural] \in Z(R) \quad \text{for all } x, y \in R. \tag{7}$$

For any $h \in Z(R) \cap H_\natural(R)\backslash\{0\}$, replace y by $y^\natural h$ in (7), gives

$$([d(x), y] + [y^\natural, d(x^\natural)])d(h) \in Z(R) \quad \text{for all } x, y \in R. \tag{8}$$

This implies either $[d(x), y] + [y^\natural, d(x^\natural)] \in Z(R)$ or $d(h) = 0$ for all $x, y \in R$ and $h \in Z(R) \cap H_\natural(R)$. Assume that $d(h) = 0$ for all $h \in Z(R) \cap H_\natural(R)$. Keep in mind the last argument, put y as ys where $s \in Z(R) \cap S_\natural(R)\backslash\{0\}$ in (7), we obtain

$$-[d(x), d(y^\natural)] + [d(y), d(x^\natural)] - [x, y^\natural] + [y, x^\natural] \in Z(R) \quad \text{for all } x, y \in R. \tag{9}$$

Combination of (7) and (9) yields

$$[d(x), d(y)] + [x, y] \in Z(R) \quad \text{for all } x, y \in R. \tag{10}$$

Hence, we get the required result.

Now consider $[d(x), y] + [y^\natural, d(x^\natural)] \in Z(R)$. One can easily deduce from the last relation that $[d(x), y] \in Z(R)$ for all $x, y \in R$. Thus R must be a commutative integral domain from [5].

Similarly, we can prove the case of $[d(x), d(x^\natural)] - [x, x^\natural] \in Z(R)$ for all $x \in R$. □

3. Identities with anti-commutator

Our aim in the next theorems is to give conditions with the anti-commutator that assure the commutativity of R.

Theorem 3.1. *Let $(R, \natural)$ be a prime ring of characteristic not 2 with semi-involution of the second kind. If R admits a derivation d, then the following assertions are equivalent:*

1. *$d(x \circ x^\natural) \pm (x \circ x^\natural) \in Z(R)$ for all $x \in R$;*
2. *R is a commutative integral domain.*

Proof. If $d = 0$, then our conditions reduce to $(x \circ x^\natural) \in Z(R)$ for all $x \in R$. So the commutativity of R follows from Fact 2. Hence we may suppose that $d \neq 0$ and

$$d(x \circ x^\natural) + x \circ x^\natural \in Z(R) \quad \text{for all } x \in R. \tag{11}$$

A linearization of (11) yields

$$d(x \circ y^\natural) + d(y \circ x^\natural) + x \circ y^\natural + y \circ x^\natural \in Z(R) \quad \text{for all } x, y \in R \tag{12}$$

so that

$$d(x\circ y)+d(y^{\natural}\circ x^{\natural})+x\circ y+y^{\natural}\circ x^{\natural}\in Z(R) \quad \text{for all } x,y\in R. \tag{13}$$

Replacing y by yh where $h\in Z(R)\cap H_{\natural}(R)\backslash\{0\}$, and using (13), we obtain

$$(x\circ y+y^{\natural}\circ x^{\natural})d(h)\in Z(R) \quad \text{for all } x,y\in R. \tag{14}$$

Since R is prime, either $d(h)=0$ or $x\circ y+y^{\natural}\circ x^{\natural}\in Z(R)$. For the last case, substituting $x^{\natural}$ for y and using Fact 2, we get $R=Z(R)$. Then $d(h)=0$ for all $h\in Z(R)\cap H_{\natural}(R)$. By Fact 1, we get $d(s)=0$ for all $s\in Z(R)\cap S_{\natural}(R)$. Replacing y by ys in (13), where $s\in Z(R)\cap S_{\natural}(R)\backslash\{0\}$, we have

$$(d(x\circ y)-d(y^{\natural}\circ x^{\natural})+x\circ y-y^{\natural}\circ x^{\natural})s\in Z(R) \quad \text{for all } x,y\in R, \tag{15}$$

which leads to

$$d(x\circ y)-d(y^{\natural}\circ x^{\natural})+x\circ y-y^{\natural}\circ x^{\natural}\in Z(R) \quad \text{for all } x,y\in R. \tag{16}$$

Combining (13) with (16), we find that

$$d(x\circ y)+x\circ y\in Z(R) \quad \text{for all } x,y\in R. \tag{17}$$

Substituting h for y in (17), where $h\in Z(R)\cap H_{\natural}(R)\backslash\{0\}$, we arrive at

$$d(x)+x\in Z(R) \quad \text{for all } x,y\in R. \tag{18}$$

Thus R must be a commutative ring from the celebrated theorem of Posner [5]. If $d(x\circ x^{\natural})-x\circ x^{\natural}\in Z(R)$ for all $x\in R$, then grammar similar arguments as above with a slight modification. □

Theorem 3.2. *Let $(R,\natural)$ be a prime ring of characteristic not 2 with semi-involution of the second kind. If R admits a derivation d, then the following assertions are equivalent:*

(1) $d(x)\circ d(x^{\natural})\pm(x\circ x^{\natural})\in Z(R)$ *for all $x\in R$;*
(2) *R is a commutative integral domain.*

Proof. For the nontrivial implication, assume that

$$d(x)\circ d(x^{\natural})\pm(x\circ x^{\natural})\in Z(R) \quad \text{for all } x\in R. \tag{19}$$

If $d=0$, then our conditions reduce to $(x\circ x^{\natural})\in Z(R)$ for all $x\in R$. So the commutativity of R follows from Fact 2. Hence we may suppose that $d\neq 0$.
A linearization of (19) yields that

$$d(x)\circ d(y^{\natural})+d(y)\circ d(x^{\natural})\pm(x\circ y^{\natural})\pm(y\circ x^{\natural})\in Z(R) \quad \text{for all } x,y\in R \tag{20}$$

and hence

$$d(x) \circ d(y) + d(y^\natural) \circ d(x^\natural) \pm (x \circ y) \pm (y^\natural \circ x^\natural) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{21}$$

Replacing y by yh in (21) where $h \in Z(R) \cap H_\natural(R) \backslash \{0\}$, and using (21), we obtain

$$(d(x) \circ y + y^\natural \circ d(x^\natural))d(h) \in Z(R) \quad \text{for all } \ x, y \in R. \tag{22}$$

Since R is prime, either $d(h) = 0$ or $d(x) \circ y + y^\natural \circ d(x^\natural) \in Z(R)$.
Suppose that $d(x) \circ y + y^\natural \circ d(x^\natural) \in Z(R)$ for all $x, y \in R$. If we put $y = s$ and $y = h$ in the last equation, we get $d(x) \in Z(R)$ for all $x \in R$. Consequently, R is a commutative integral domain. Hence $d(h) = 0$ for all $h \in Z(R) \cap H_\natural(R)$. Now replacing y by s and h in (21), where $s \in Z(R) \cap S_\natural(R) \backslash \{0\}$, $h \in Z(R) \cap H_\natural(R) \backslash \{0\}$, we arrive at

$$x - x^\natural \in Z(R) \quad \text{for all } \ x \in R \tag{23}$$

$$x + x^\natural \in Z(R) \quad \text{for all } \ x \in R. \tag{24}$$

Combining (23) with (24), we get the required result. □

4. Examples proving the necessity of our conditions

The following example proves the condition "$\natural$ is of the second kind" is necessary in our theorems.

Example 4.1. Let us consider

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\natural = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It is straightforward to check that R is a prime ring of characteristic not 2 and $\natural$ is a semi-involution of the first kind. Since $[x, x^\natural] = 0$ for all $x \in R$, any derivation satisfies the conditions of Theorem 2.1 but R is not commutative. Furthermore, for all $x \in R$, we have

$$x \circ x^\natural = 2(ad - bc)I_2 \in Z(R),$$

then the zero derivation satisfies the conditions of Theorems 2.2, 3.1 and 3.2 but R is not commutative.

The following example proves the condition "$\natural$ is of the second kind" is necessary in our theorems.

Example 4.2. Let us consider $(R, \natural)$ as in the preceding example and (S, σ) be a commutative ring with involution of the second kind (for example the field of complex numbers with the conjugation involution). If we set $\mathcal{R} = R \times S$, then it is obvious to verify that $(\mathcal{R}, \tau)$ is a semi-prime ring with semi-involution of the second kind where

$$\tau(r, s) = (r^{\natural}, \sigma(s)) \quad \text{for all} \quad (r, s) \in R \times S.$$

Then any derivation satisfies the conditions of Theorem 2.1 but R is not commutative. Furthermore, the zero derivation satisfies the conditions of Theorems 2.2, 3.1 and 3.2 but R is not commutative.

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