



On finite groups in which every maximal subgroup of order divisible by p is p -decomposable

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Abstract

We obtain a complete classification of a finite group G in which every maximal subgroup of order divisible by p is p -decomposable for a given prime divisor p of $|G|$ and our results generalize a recent result of Shi and Tian.

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1. Introduction

In this paper, all groups are assumed to be finite. It is known that if a group G can be written as the direct product of a Sylow p -subgroup of G and a Hall p' -subgroup of G , then G is called a p -decomposable group. A group G is not p -decomposable but all of its proper subgroups are p -decomposable, then group G is called inner p -decomposable. A group G is called inner-nilpotent group, if G is non-nilpotent but all of its proper subgroups are nilpotent. A group G is called p -closed, if its Sylow p -subgroup is normal in G . Specially, if $p \nmid |G|$, G is p -closed. Inner-nilpotent group G is a group whose order is $p^\alpha q^\beta$, where p, q are distinct prime number. There is a normal Sylow subgroup and a non-normal Sylow subgroup in G , the non-normal Sylow subgroup is a cyclic group. If the Sylow q -subgroup of G is normal, we call inner-nilpotent group G as a q -fundamental group [2]. In this paper, the symbol $P : Q$ represents the semidirect product of P and Q , where P is normal in G . Shi and Tian [5, Theorem 1.1], characterized the structure of a group in which every maximal subgroup of order divisible by p is nilpotent (or abelian). In this paper, considering any fixed prime divisor p of the order of a group G , we obtain the following result in Theorem 1.1 whose proof is given in Section 3. The symbols appearing in this paper can be found in [3]. In this paper, P_i stands for the Sylow- i subgroup of G .

Theorem 1.1 Suppose that G is a group and p is a given prime divisor of $|G|$. Then every maximal subgroup of G of order divisible by p is p -decomposable if and only if one of the following statements holds:

- (1) G is a p -decomposable group.
- (2) G is not a p -decomposable group, and the following statements are true.

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(2.1) $G = P : Q$ is an inner p -decomposable group, where P is a Sylow p -subgroup of G and Q is a cyclic Sylow q -subgroup, $p \neq q$, and Q has only one maximal subgroup;

(2.2) $G = Q : P$ is an inner nilpotent group, where P is a Sylow p -subgroup of G and where Q is a Sylow q -subgroup of G .

By Theorem 1.1, the following corollary emerges.

Corollary 1.2 ([5, Theorem 1.1]) Suppose that G is a group and p is any fixed prime divisor of $|G|$, then every maximal subgroup of G of order divisible by p is nilpotent if and only if one of the following statements holds:

- (a) G is a nilpotent group;
- (b) $G = P : Q$ is an inner-nilpotent group, where $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$, P is normal in G , $p \neq q$;
- (c) $G = Q : P$ is an inner-nilpotent group, where $Q \in \text{Syl}_q(G)$ and $P \in \text{Syl}_p(G)$, Q is normal in G , $q \neq p$;
- (d) $G = Z_p : K$, where K is an inner-nilpotent group and $(p, |K|) = 1$.

2. A lemma

Lemma 2.1 ([2, Theorem 1]) Inner- p -closed group has the following two forms (1) $G/\Phi(G)$ is a simple group of complex order; (2) G is a q -fundamental group whose order is $p^\alpha q^\beta$.

3. Proof of Theorem 1.1

Proof. The sufficiency part is evident, we only need to prove the necessity part. For a finite group G , it is either p -decomposable or non- p -decomposable. If G is p -decomposable, the conclusion (1) is obviously correct. In the following discussion we suppose G is not p -decomposable.

Now we choose any maximal subgroup H of G . If $p \mid |H|$, then H is p -decomposable; if $p \nmid |H|$, then H is also evidently p -decomposable. Hence, every maximal subgroup of G is p -decomposable. So, G is inner p -decomposable.

In the following we divide our arguments into two cases.

Case 1. G is a p -closed group.

Let P be a Sylow p -subgroup of G , we have $P \trianglelefteq G$. By Schur-Zassenhaus theorem, G has a p -complement Q . Then $G = P : Q$. Choose any maximal subgroup K_1 of Q , we can get PK_1 is a maximal subgroup of G , then PK_1 is p -decomposable. Thus $PK_1 = P \times K_1$. If Q has at least two different maximal subgroups K_1 and K_2 . Then $PK_i = P \times K_i$, where $i=1, 2$. Since $Q = \langle K_1, K_2 \rangle$, we have $PQ = P \times Q$, a contradiction. Hence, Q has the unique maximal subgroup K_1 , then Q is a cyclic Sylow q -subgroup, where q is a prime number and $p \neq q$. Thus, (2.1) is proved.

Case 2. G is not a p -closed group.

Since G is inner p -decomposable, G is inner p -closed. By Lemma 2.1, we can get (a) $G/\Phi(G)$ is a non-abelian simple group; (b) G is a q -fundamental group.

Assume that the case(a) occurs. Let $\bar{G} = G/\Phi(G)$. By \bar{G} is a non-abelian simple group, we get G is a non-abelian simple group if $\Phi(G) = 1$. By classification of finite simple groups, there are three types of non-abelian simple groups. They are alternating groups, sporadic simple groups and simple groups of Lie type, respectively. In the following we

divide our arguments into three cases.

Case 2.1. G is an alternating group, $G \cong A_n$. Suppose $p \nmid n$ and $n \geq 6$. A_{n-1} is the maximal subgroup of A_n , evidently, $p \mid |A_{n-1}|$. However, A_{n-1} is a simple group and it's not p -decomposable, a contradiction. So $p \mid n$ and $p > n - 1$, thus $p = n$, now we have $A_n = A_p$, and we can select the maximal subgroup $N_G(P)$ of G , where P is a Sylow p -subgroup of G . We can get $N_G(P) = P : C_{\frac{p-1}{2}}$, it's clearly that it is not p -decomposable. Hence it's a contradiction. If $n = 5$, A_5 has three prime factors, they are 2, 3, 5 respectively. We consider the maximal subgroup S_3 if $p = 3$, since $3 \mid |S_3|$, $N_G(P) = P : C_2$ is non-decomposable, a contradiction. We consider the maximal subgroup D_{10} if $p=2$ or 5, $p \mid |D_{10}|$, $D_{10} = C_5 : C_2$ is non-decomposable, a contradiction.

Case 2.2. G is a sporadic simple group.

Suppose $G \cong M_{11}$, then $\pi(G) = \{2, 3, 5, 11\}$. By [3], We get the maximal subgroup $L_2(11)$ of G whose prime factors are also $p = 2, 3, 5$ or 11. Since $p \mid |L_2(11)|$, $L_2(11) = P_2 \times H$, it contradicts that $L_2(11)$ is a simple group.

Suppose $G \cong Suz$, then $\pi(G) = \{2, 3, 5, 7, 11, 13\}$. By [3], if $p = 2, 3, 5, 11$, we consider the maximal subgroup $M_{12} : 2$ of G . $p \mid |M_{12} : 2|$, but $M_{12} : 2$ is not p -decomposable, a contradiction. If $p = 7$, we consider the maximal subgroup A_7 of G . $7 \mid |A_7|$, then $A_7 = P_7 \times H$, it contradicts that A_7 is a simple group. If $p = 13$, we consider the maximal subgroup $L_3(3) : 2$ of G . $13 \mid |L_3(3) : 2|$, but $L_3(3) : 2$ is not p -decomposable, a contradiction.

Suppose $G \cong Fi_{23}$, then $\pi(G) = \{2, 3, 5, 7, 11, 13, 17, 23\}$. By [3], if $p = 2, 3, 5, 7, 17$, we consider the maximal subgroup $S_8(2)$ of G . Since $p \mid |S_8(2)|$, $S_8(2) = P_i \times H$, where $i = 2, 3, 5, 7, 17$, a contradiction. If $p = 13$, we consider the maximal subgroup $O_8^+(3) : S_3$ of G . Since $13 \mid |O_8^+(3) : S_3|$, $O_8^+(3) : S_3$ is not p -decomposable, a contradiction. If $p = 11, 23$, we consider the maximal subgroup $2^{11} \cdot M_{23}$ of G . Since $p \mid |2^{11} \cdot M_{23}|$, $2^{11} \cdot M_{23} = P_i \times H$, where $i = 11, 13$. Then M_{23} is p -decomposable, it contradicts that M_{23} is a non-abelian simple group.

Suppose $G \cong J_4$, then $\pi(G) = \{2, 3, 5, 7, 11, 23, 29, 31, 37, 43\}$. By [3], if $p = 2, 3, 5, 11, 37$, we consider the maximal subgroup $U_3(11) : 2$ of G . Since $p \mid |U_3(11) : 2|$, $U_3(11) : 2$ is not p -decomposable, a contradiction. If $p = 23$, we consider the maximal subgroup $L_2(23) : 5$ of G . Since $23 \mid |L_2(23) : 5|$, $L_2(23) : 5$ is not p -decomposable, a contradiction. If $p = 7$, we consider the maximal subgroup $2_+^{1+12} \cdot M_{22} : 2$ of G . Since $7 \mid |2_+^{1+12} \cdot M_{22} : 2|$, $2_+^{1+12} \cdot M_{22} : 2$ is not p -decomposable, a contradiction. If $p = 29$, we consider the maximal subgroup $29 : 28$ of G . Since $29 \mid |29 : 28|$, $29 : 28$ is not p -decomposable, a contradiction. If $p = 43$, we consider the maximal subgroup $43 : 14$. Since $43 \mid |43 : 14|$, $43 : 14$ is not p -decomposable, a contradiction.

We can also get a contradiction when G is an other sporadic group according to the Atlas form [3].

Case 2.3. G is a simple group of Lie type.

Let G be a classical group.

Suppose $G \cong L_n(q)$, $q = r^t$, where r is a prime. If $p = r$ or $p \nmid (q^n - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGL_{n-1}(q)$, $p \mid |PGL_{n-1}(q)|$, so $PGL_{n-1}(q)$ is p -decomposable, a contradiction. If $p \mid (q^n - 1)$ but $p \nmid (q^i - 1)$, where $i < n$, G has a subgroup $N_G(P) = \frac{q^n - 1}{q - 1} : C_n$ [4]. It is not p -decomposable, so it is a contradiction.

Suppose $G \cong PSU_{2n}(q)$. If $p \mid q \prod_{i=1}^{2n-1} (q^i - (-1)^i)$ but $p \nmid (q^{2n} - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGU_{2n-1}(q)$, $p \mid |PGU_{2n-1}(q)|$, so $PGU_{2n-1}(q)$ is p -decomposable, a contradiction. If $p \mid (q^{2n} - 1)$, then by [1, Tables] and [6, Proposition

4.1.17], G has a subgroup $PSL_n(q^2).(q-1).2$. So $p \mid |PSL_n(q^2).(q-1).2|$, and thus $SL_n(q^2).(q-1).2$ is p -decomposable, a contradiction.

Suppose $G \cong PSU_{2n+1}(q)$. If $p \mid q \prod_{i=1}^{2n} (q^i - (-1)^i)$ but $p \nmid (q^{2n+1} + 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PGU_{2n}(q)$, $p \mid |PGU_{2n}(q)|$, so $PGU_{2n}(q)$ is p -decomposable, a contradiction. If $p \mid (q^{2n+1} + 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $\frac{q^{2n+1}+1}{q+1} : (2n+1)$, $p \mid |(\frac{q^{2n+1}+1}{q+1} : (2n+1))|$, so $\frac{q^{2n+1}+1}{q+1} : (2n+1)$ is p -decomposable, a contradiction.

Suppose $G \cong PSP_{2n}(q)$. If $p \mid q \prod_{i=1}^{n-1} (q^{2i} - 1)$ but $p \nmid (q^{2n} - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $E_q^{1+(2n-2)} : ((q-1) \times PSP_{2n-2}(q))$, $p \mid |E_q^{1+(2n-2)} : ((q-1) \times PSP_{2n-2}(q))|$, so $E_q^{1+(2n-2)} : ((q-1) \times PSP_{2n-2}(q))$ is p -decomposable, a contradiction. If $p \mid (q^{2n} - 1)$ but $p \nmid \prod_{i=1}^{n-1} (q^{2i} - 1)$, then by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PSO_{2n}^-(q)$, $p \mid |PSO_{2n}^-(q)|$, so $PSO_{2n}^-(q)$ is p -decomposable, a contradiction.

Suppose $G \cong P\Omega_{2n+1}(q)$. If $p \mid q(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $P\Omega_{2n}^+(q).2$, $p \mid |P\Omega_{2n}^+(q).2|$, so $P\Omega_{2n}^+(q).2$ is p -decomposable, a contradiction. If $p \mid (q^n + 1) \prod_{i=1}^{n-1} (q^{2i} - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $P\Omega_{2n}^-(q).2$, $p \mid |P\Omega_{2n}^-(q).2|$, so $P\Omega_{2n}^-(q).2$ is p -decomposable, a contradiction.

Suppose $G \cong P\Omega_{2n}^+(q)$. If $p \mid q \prod_{i=1}^{n-1} (q^{2i} - 1)$, but $p \nmid (q^n - 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PSp_{2n-2}(q)$, $p \mid |PSp_{2n-2}(q)|$, so $PSp_{2n-2}(q)$ is p -decomposable, a contradiction. If $p \mid q^n - 1$, let the prime factor q of $|\Omega_{2n}^+(q)|$ be of power t , then by [1, Tables] and [6, Proposition 4.1.17], we can find a subgroup $E_q^{t - \frac{n(n-1)}{2}} : GL_n(q)$, $p \mid |E_q^{t - \frac{n(n-1)}{2}} : GL_n(q)|$, so $E_q^{t - \frac{n(n-1)}{2}} : GL_n(q)$ is p -decomposable, a contradiction.

Suppose $G \cong P\Omega_{2n}^-(q)$. If $p \mid q \prod_{i=1}^{n-1} (q^{2i} - 1)$, but $p \nmid (q^n + 1)$, by [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $PSp_{2n-2}(q)$. Since $p \mid |PSp_{2n-2}(q)|$, $PSp_{2n-2}(q)$ is p -decomposable, a contradiction. Suppose $p \mid (q^n + 1)$ and n is an odd number. By [1, Tables] and [6, Proposition 4.1.17], G has a subgroup $GU_n(q)$. Since $p \mid |GU_n(q)|$, $GU_n(q)$ is p -decomposable, a contradiction. Suppose $p \mid (q^n + 1)$ and n is an even number. G has a subgroup $P\Omega_n^-(q^2).2$. Since $p \mid |P\Omega_n^-(q^2).2|$, $P\Omega_n^-(q^2).2$ is p -decomposable, a contradiction.

Let G be an exceptional group.

Suppose $G \cong G_2(q)$, $|G_2(q)| = q^6(q^6 - 1)(q^2 - 1)$. If $p \mid q^6(q^3 - 1)(q^2 - 1)$, but $p \nmid (q^3 + 1)$, by [7, Table 4.1], G has a maximal subgroup $SL_3(q) : 2$. Since $p \mid |SL_3(q) : 2|$, $SL_3(q) : 2$ is p -decomposable, a contradiction. If $p \mid q^3 + 1$, G has a maximal subgroup $SU_3(q) : 2$. Since $p \mid |SU_3(q) : 2|$, $SU_3(q) : 2$ is p -decomposable, a contradiction.

Suppose $G \cong {}^3D_4(q)$, $|{}^3D_4(q)| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. If $p \mid q^{12}(q^6 - 1)(q^2 - 1)$, by [7, Theorem 4.3.], G has a maximal subgroup $G_2(q)$. Since $p \mid |G_2(q)|$, then $G_2(q)$ is p -decomposable, a contradiction. If $p \mid (q^2 + q + 1)$, by [7, Theorem 4.3.], G has a maximal subgroup $(C_{q^2+q+1} \times C_{q^2+q+1}) : SL_2(3)$. Since $p \mid |(C_{q^2+q+1} \times C_{q^2+q+1}) : SL_2(3)|$, $(C_{q^2+q+1} \times C_{q^2+q+1}) : SL_2(3)$ is p -decomposable, a contradiction. If $p \mid (q^2 - q + 1)$, by [7, Theorem 4.3.], G has a maximal subgroup $(C_{q^2-q+1} \times C_{q^2-q+1}) : SL_2(3)$. Since $p \mid |(C_{q^2-q+1} \times C_{q^2-q+1}) : SL_2(3)|$, $(C_{q^2-q+1} \times C_{q^2-q+1}) : SL_2(3)$ is p -decomposable, a contradiction. If $p \mid (q^4 - q^2 + 1)$, by [7, Theorem 4.3.], G has a maximal subgroup $C_{q^4-q^2+1} : 4$. Since $p \mid |C_{q^4-q^2+1} : 4|$, $C_{q^4-q^2+1} : 4$ is p -decomposable, a contradiction.

We can also get a contradiction when G is an other exceptional group.

From (b) of [2], we have $G = Q : P$, where G is an inner nilpotent group, P is a Sylow p -subgroup of G and where Q is a Sylow q -subgroup of G . (2.2) is proved. □

4. Proof of Corollary 1.2

Proof. It is obviously that (b) and (c) are true from (2.1) and (2.2) of Theorem 1.1. From 1 of Theorem 1.1, if G is p -decomposable, then $G = P \times K$, where P is the Sylow p -subgroup of G , and K is the Hall p' -subgroup of G . Suppose K is nilpotent, then G is nilpotent, (a) is proved. Suppose K is not nilpotent. Let $N < K$. Then $P \times N < G$, so $p \mid |P \times N|$. So, N is nilpotent, and we can get K is inner-nilpotent. If P is not a group of order p . Let $H < P$ and $|H| = p$. Then $p \mid |H \times K|$. So $H \times K$ is nilpotent, and we get K is nilpotent, a contradiction. Hence, P is a group of order p , (d) is proved. \square

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References

- [1] J. Bray, D. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, London Math. Soc. Lecture Note Ser., 407, Cambridge University Press, Cambridge, 2013.
- [2] Z. M. Chen, *Inner- p -closed group*, Mathematical progress, 04, 385-388, 1986.
- [3] J. H. Conway, R. T. Curtis, S. P. Norton et al., *Atlas of Finite Groups*, London-New York, Oxford Univ Press, 1985.
- [4] B. Huppert, *Endliche Gruppen I*, Berlin, Springer-Verlag, 1967.
- [5] J. T. Shi and Y. F. Tian, *On finite groups in which every maximal subgroup of order divisible by p is nilpotent (or abelian)*, Rend. Sem. Mat. Univ. Padova, published online first, 2024.
- [6] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, vol. 129. London Math. Soc. Lecture Note Ser., Cambridge University Press, Cambridge, 1990.
- [7] R. A. Wilson, *The Finite Simple Groups*, London, Springer-Verlag, 2009.