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Determination of the Confidence Intervals for Multimodal Probability Density Functions

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Abstract

The shortest interval approach can be solved as an optimization problem, while the equally tailed approach is determined by using the distribution function. The equal density approach is proposed instead of the optimization problem for determining the shortest confidence interval. It is applied to multimodal probability density functions to determine the shortest confidence interval. Furthermore, the equal density and optimization approach for the shortest confidence interval and the equally tailed approach were applied to numerical examples and their results were compared. Nevertheless, the main subject of this study is the calculation of the shortest confidence intervals for any multimodal distribution.

1. INTRODUCTION

Today, inferential statistics contain the most common methods used to determine measures of development, variation and existence in almost all fields of science. Statistics obtained from a sample are used to estimate the parameters of a population. These statistics these how reflect parameters are examined statistically. It is not expected that the calculated statistic is exactly equal to the population parameter. However, the parameter within a certain interval can be estimated by considering the significance coefficient (α). This interval is called the confidence interval. The basic purpose of inferential statistics is to determine the confidence interval. Two-sided confidence limits form a confidence interval; their one-sided counterparts are referred to as lower (L) or upper (U) confidence bounds. The shortest confidence interval, which gives the same significance coefficient (α) is the best estimation for the parameter.

The problem of shortest confidence intervals has been studied by some authors. Neyman introduced the classical theory of confidence intervals [1]. Wald presented with the limit properties of the confidence intervals if the number of observations approaches infinity [2]. Blyth and Hutchinson presented table of shortest unbiased confidence intervals for the binomial parameter [3]. Sidak discussed about rectangular confidence regions for the mean values of multivariate normal distributions [4]. Levy and Narula derived the shortest confidence interval for the ratio of two variances when the populations are assumed to be normal [5]. DiCiccio and Romano considered several distinct bootstrap methods with emphasis on the mathematical correctness of bootstrap procedures for constructing confidence regions [6]. Owen constructed confidence intervals for the sample mean, for a class of M-estimates, which includes

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quantiles, and for differentiable statistical functionals, within the empirical likelihood ratio function [7]. Ferentinos described shortest confidence intervals for families of distributions involving truncation parameters [8]. Ferentios and Kourouklis constructed shortest confidence intervals for families of distributions involving two unknown truncation parameters [9]. Joula developed confidence intervals for a single unknown parameter by using a pivotal quantity [10]. He presented an elementary method for deriving the shortest such interval. Weerahandi constructed exact confidence regions for the difference in two normal means without the assumption of equal variances [11]. Newcombe compared seven methods for two-sided confidence intervals for the single proportion [12]. Willink obtained a confidence interval and test for the mean of an asymmetric distribution using a random sample of size n [13]. Zhou and Dinh evaluated several existing techniques and proposed new methods to improve coverage accuracy [14]. Kibria compiled some interval estimators for estimating the mean of the asymmetric distribution and compared the performance of these intervals [15]. Burch presented confidence intervals for the intraclass correlation coefficient comparing the unbiased confidence interval and the equal-tail probability interval [16]. Evans and Shakhatreh showed relative surprise regions to maximize both the Bayes factor in favor of the region containing the true value and the relative belief ratio with the same posterior content [17]. Baklizi and Golam proposed some confidence intervals for estimating the mean or difference of means of skewed populations and extended the median t interval to the two-sample problem [18]. Banik and Kibria considered and compared both classical and nonparametric interval estimators for estimating the mean of a positively skewed distribution [19]. And then, Banik and Kibria conducted to compare the performance of the various interval estimators for estimating the population coefficient of variation (CV) of symmetric and skewed distributions using simulation study [20]. Gulhar et al. considered several confidence intervals for estimating the population coefficient of variation based on parametric, nonparametric and modified methods and compared the performance of the existing and newly proposed interval estimators [21]. Alizadeh et al. gave a general solution to obtain an unbiased confidence interval for families of distributions involving truncation parameter using the pivotal quantity method [22]. Mammen and Polonik constructed the confidence regions for level sets [23]. The proposed construction is based on a plug in estimate of the level sets using an appropriate estimate. Fagerland et al. illustrated the performances of the confidence intervals for two independent binomial proportions and made recommendations for both small and moderate to large sample sizes [24].

The main purpose of this study is the calculation of the shortest confidence interval for multimodal distributions. Previous studies were performed to calculate confidence intervals of any parameter for the unimodal probability density function (PDF). These studies investigated the unbiased, shortest, equally tailed and uniformly most accurate (UMA) interval estimators for the parameters. Building upon these studies, this paper claims that the width of the confidence interval is shortest when densities of confidence limits are equal. The motivation of this study comes from the fact that contrary to the literature methods, the proposed method can calculate the shortest confidence interval for multimodal distributions. The equally tailed confidence interval for unimodal and univariate distributions is given in Section 2.1. The shortest confidence interval for unimodal and univariate distributions is given in Section 2.2. In Section 2.3, the proposed method is given as the different approximation for the shortest confidence interval. In Section 3, the calculation of the shortest confidence intervals for multimodal univariate distribution is given. Finally, the shortest, equally tailed and equally density confidence interval estimators for the parameters are compared experimentally to each other in Section 4.

2. CONFIDENCE INTERVAL FOR UNIMODAL PDF

If $\hat{\theta}$ is calculated via an obtained sample from a population and its distribution function is $\Phi(x)$, then the $(1-\alpha)$ two-tailed confidence interval of θ is defined as follows [25-27].

$$\begin{aligned}\Pr[L \leq \theta \leq U] &= 1-\alpha \\ &= \Phi(U) - \Phi(L), \qquad L < U\end{aligned} \tag{1}$$

L and U respectively, the lower and upper bounds of the interval for θ. According to this equation, L can have infinite values in the interval $[\Phi^{-1}(0), \Phi^{-1}(\alpha)]$, where $\Phi^{-1}(.)$ is an inverse distribution function. Likewise, U can have infinite values in the interval $[\Phi^{-1}(1-\alpha), \Phi^{-1}(1)]$ depending on L.

2.1. Equally Tailed Confidence Interval

L and U (shown in Figure 1) are, respectively, the lower and upper bounds of the interval for θ, and it is called the equally tailed confidence interval, if $\Pr(\theta \in (-\infty, L)) = \Pr(\theta \in (U, \infty)) = \frac{\alpha}{2}$, where $L = \Phi^{-1}\left(\frac{\alpha}{2}\right)$ and $U = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$ [26, 28]. In addition, the width of confidence interval can be calculated as follows.

$$W = U - L \tag{2}$$

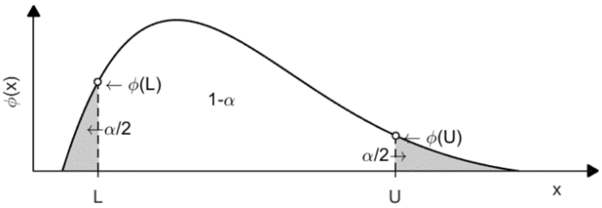


Figure 1. *The probability density function $\phi(x)$ and the equally tailed confidence bounds.*

2.1.1. Numerical Method For The Equally Tailed Confidence Interval

If the probability density function $(\phi(x))$ of the parameter (θ) is known, then the inverse distribution function of θ can be calculated and thus, the distribution function of θ is also known to find the confidence limits. Nevertheless, the inverse distribution function cannot always be calculated, and thus the following cases (Case 1 and Case 2) can occur.

Case 1: When the distribution function of θ, $\Phi(x)$, is known, but the inverse distribution function, $x_c = \Phi^{-1}(P)$, cannot be calculated analytically, the following equation to find x_c with the probability value $P = (\alpha/2 \;\; OR \; 1 - \alpha/2)$ must be solved.

$$\Phi(x) - P = 0 \tag{3}$$

If the equation cannot be solved analytically, numerical methods can be used to find the root [29].

Case 2: When the distribution function of θ, $\Phi(x)$, cannot be calculated analytically, and the inverse distribution function, $x = \Phi^{-1}(P)$, is not also calculated, and thus, $\Phi(x)$ can be found with the numerical integration method. x_c shows the critical value given as probability value P. The objective function value can be found by solving the following one-dimensional optimization problem.

$$goal\text{: } \min_{\mathrm{x}} J(x) = \left| P - \int_{-\infty}^{x} \phi(x)\, dx \right| \tag{4}$$

2.2. The Shortest Confidence Interval

In the previous section it is mentioned that there are infinite confidence limits with the same confidence level. Which confidence limit represents the estimation most precisely? While the precision of the estimation depends on the width of the interval, the shortest width typically gives the more sensitive estimation [30, 31]. Although the equally tailed confidence interval is preferred in the applications, the width of this kind of confidence interval is not the shortest [32]. However, determining confidence limits analytically in the shortest confidence interval is more difficult than doing so in the equally tailed confidence interval. Hence, numerical methods can be used to find the shortest confidence interval [29]. The shortest confidence interval of the probability density function $\phi(x)$ is shown in Figure 2.

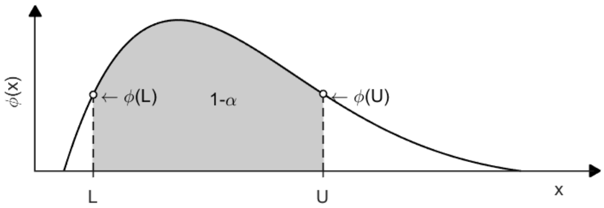


Figure 2. *The shortest confidence interval of the probability density function $\phi(x)$.*

2.2.1. Numerical Method For The Shortest Confidence Interval

In order to find the shortest confidence interval, an optimization problem with the objective function and the constraint, given respectively below, must be solved.

$$goal\text{: } \min_{L,U}(U-L) \tag{5}$$

$$constraint: \int_L^U \phi(x)\ dx = 1-\alpha \tag{6}$$

The variable U can be expressed in the variable L due to the fact that U depends on L, and thus the optimization problem becomes simplified. The lower bound L is randomly determined in the interval $[\Phi^{-1}(0), \Phi^{-1}(\alpha)]$, and the upper bound U can be calculated as follows.

$$U = \Phi^{-1}\big(1-\alpha+\Phi(L)\big) \tag{7}$$

Finally, in order to find the shortest interval, the following optimization problem must be solved.

$$goal\text{: } \min_{L} J(L) = \Phi^{-1}\big(1-\alpha+\Phi(L)\big) - L \tag{8}$$

$$constraint\text{: } \Pr(\theta \in [L,U]) = \int_L^U \phi(x)dx = 1-\alpha \tag{9}$$

If $\phi(x)$ is defined in an infinite interval, the random determination of the lower bound (L) can be difficult. Therefore, in order to make the solution simple, the initial lower bound can be determined as follows.

$$L = \Phi^{-1}(\alpha) \tag{9}$$

This is also the maximum value of the lower bound.

2.3. Different Approximation For The Shortest Confidence Interval

This paper investigates whether there are available methods other than optimization techniques to find the shortest confidence interval.

Theorem 1. The probability density values of the shortest confidence bounds (L,U) with $(1-\alpha)$ confidence coefficient are equal $\big(\phi(L) = \phi(U)\big)$.

Proof. Suppose L and U are, respectively, the lower bound and the upper bound of the confidence interval. When L is shifted to the left as ε_1, U must also be shifted to the left as ε_2 to maintain the current confidence coefficient. Thus, the following equation is obtained.

$$\Phi(U-\varepsilon_2) - \Phi(L-\varepsilon_1) = 1-\alpha \tag{10}$$

The following equation can be written by using $\lim_{\varepsilon_2 \to 0} \phi(U - \varepsilon_2) = \phi(U)$ and $\lim_{\varepsilon_1 \to 0} \phi(L - \varepsilon_1) = \phi(L)$.

$$\left(\Phi(U) - \varepsilon_2 \phi(U)\right) - \left(\Phi(L) - \varepsilon_1 \phi(L)\right) = 1 - \alpha \tag{11}$$

This can be converted to the following equation.

$$\left(\Phi(U) - \Phi(L)\right) - \left(\varepsilon_2 \phi(U) - \varepsilon_1 \phi(L)\right) = 1 - \alpha \tag{12}$$

Then, equation (13) can be written via equation (1),

$$(1 - \alpha) - \left(\varepsilon_2 \phi(U) - \varepsilon_1 \phi(L)\right) = 1 - \alpha \tag{13}$$

and equation (14) is obtained if it is updated.

$$\varepsilon_1 \phi(L) = \varepsilon_2 \phi(U) \tag{14}$$

Likewise, when L is shifted to the right as ε_1, U must also be shifted to the right as ε_2 to keep its current confidence coefficient. Thus, the following equation is obtained.

$$\Phi(U + \varepsilon_2) - \Phi(L + \varepsilon_1) = 1 - \alpha \tag{15}$$

The following equation can be written by using $\lim_{\varepsilon_2 \to 0} \phi(U + \varepsilon_2) = \phi(U)$ and $\lim_{\varepsilon_1 \to 0} \phi(L + \varepsilon_1) = \phi(L)$.

$$\left(\Phi(U) + \varepsilon_2 \phi(U)\right) - \left(\Phi(L) + \varepsilon_1 \phi(L)\right) = 1 - \alpha \tag{16}$$

This equation can be converted to the following equation.

$$\left(\Phi(U) - \Phi(L)\right) - \left(\varepsilon_1 \phi(U) - \varepsilon_2 \phi(L)\right) = 1 - \alpha \tag{17}$$

Then, equation (18) can be written via equation (1),

$$(1 - \alpha) - \left(\varepsilon_1 \phi(U) - \varepsilon_2 \phi(L)\right) = 1 - \alpha \tag{18}$$

and equation (19) is obtained if it is updated.

$$\varepsilon_1 \phi(L) = \varepsilon_2 \phi(U) \tag{19}$$

Thus equations (14) and (19) are the same. In these equations, the relationship between ε_1 and ε_2 depends on the relationship between $\phi(L)$ and $\phi(U)$. There are three different cases in the relationship between $\phi(L)$ and $\phi(U)$.

Case 1: If $\phi(L) < \phi(U)$, $\varepsilon_1 > \varepsilon_2$.
Case 2: If $\phi(L) > \phi(U)$, $\varepsilon_1 < \varepsilon_2$.
Case 3: If $\phi(L) = \phi(U)$, $\varepsilon_1 = \varepsilon_2$.

According to these cases, the changes of the confidence interval are examined below.

Scenario 1: When L is shifted to the left as ε_1, the width of the confidence interval is calculated as follows.

$$W' = (U - \varepsilon_2) - (L - \varepsilon_1) \tag{20}$$

$$= (U - L) + (\varepsilon_1 - \varepsilon_2)$$
$$= W + (\varepsilon_1 - \varepsilon_2)$$

Case 1. If $\varepsilon_1 > \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) > 0$. In this case, the interval width expands. If the goal is to narrow the interval, the lower bound should not be shifted to the left. For example, consider the probability density function shown in the graph below (Figure 3). When L is shifted to the left, $\phi(L)$ decreases and $\phi(U)$ increases.

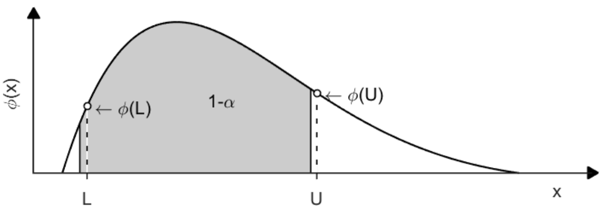


Figure 3. *Shifting the lower bound (L) to the left when* $\phi(L) < \phi(U)$.

Case 2. If $\varepsilon_1 < \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) < 0$. In this case, the interval width narrows. If the goal is to narrow the interval, the lower bound should be continually shifted to the left. For example, consider the probability density function shown in the graph below (Figure 4). When L is shifted to the left, $\phi(L)$ decreases and $\phi(U)$ increases. Thus, $\phi(L)$ closes to $\phi(U)$.

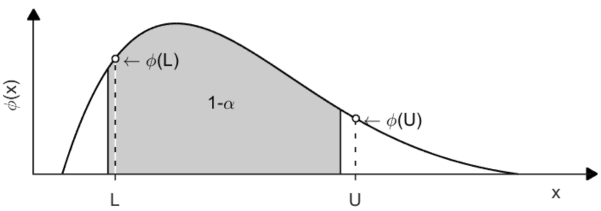


Figure 4. *Shifting the lower bound (L) to the left when* $\phi(L) > \phi(U)$.

Case 3. If $\varepsilon_1 = \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) = 0$. Thus, the interval does not change. In this case, if the lower bound is continually shifted to the left (Figure 5), $\phi(L) < \phi(U)$, and Case 1 occurs.

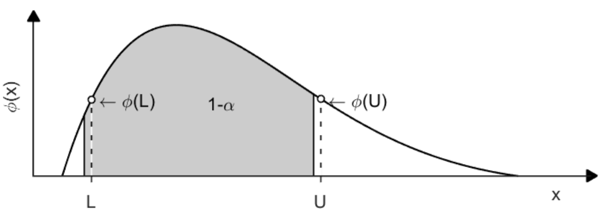


Figure 5. *Shifting the lower bound (L) to the left when* $\phi(L) = \phi(U)$.

Scenario 2: When L is shifted to the right as ε_1, the confidence interval width is found as follows.

$$\begin{aligned} W' &= (U + \varepsilon_2) - (L + \varepsilon_1) \\ &= (U - L) + (\varepsilon_2 - \varepsilon_1) \\ &= W + (\varepsilon_2 - \varepsilon_1) \end{aligned} \tag{21}$$

Case 1. If $\varepsilon_1 > \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) < 0$. In this case, the interval width narrows. If the goal is to narrow the interval, the lower bound should be continually shifted to the right. For example, consider the probability density function shown in the graph below (Figure 6). When L is shifted to the right, $\phi(L)$ increases and $\phi(U)$ decreases. Thus, $\phi(L)$ closes to $\phi(U)$.

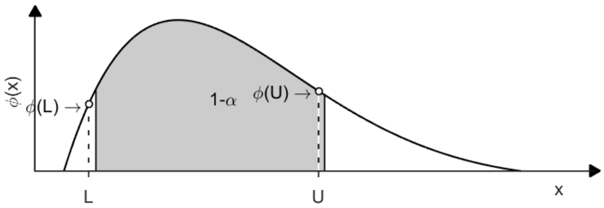


Figure 6. *Shifting the lower bound (L) to the right when $\phi(L) < \phi(U)$.*

Case 2. If $\varepsilon_1 < \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) > 0$. In this case, the interval width expands. If the goal is to achieve the shortest interval, the lower bound should not be shifted to the right. For example, consider the probability density function shown in the graph below (Figure 7). When L is shifted to the right, $\phi(L)$ increases and $\phi(U)$ decreases. Thus, $\phi(L)$ does not close to $\phi(U)$.

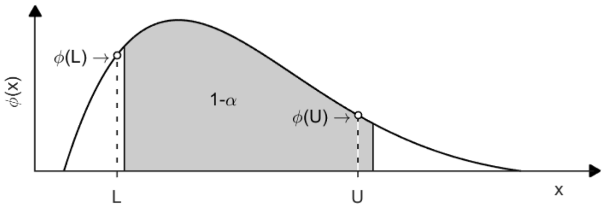


Figure 7. *Shifting the lower bound (L) to the right when $\phi(L) > \phi(U)$.*

Case 3. If $\varepsilon_1 = \varepsilon_2$, $(\varepsilon_1 - \varepsilon_2) = 0$. Thus, the interval does not change. In this case, if the lower bound is continually shifted to the right (Figure 8), $\phi(L) > \phi(U)$, and Case 2 occurs.

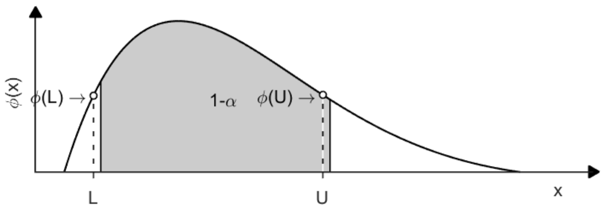


Figure 8. *Shifting the lower bound (L) to the right when $\phi(L) = \phi(U)$.*

The results of these cases can be summarized as follows.

Result 1. If $\phi(L) > \phi(U)$, the lower bound should be shifted to the left to find the shortest interval. In this case, $\phi(L)$ decreases and $\phi(U)$ increases. Thus, $\phi(L)$ closes to $\phi(U)$.

Result 2. If $\phi(L) < \phi(U)$, the lower bound should be shifted to the right to find the shortest interval. In this case, $\phi(L)$ increases and $\phi(U)$ decreases. Thus, $\phi(L)$ closes to $\phi(U)$.

Result 3. If $\phi(L) = \phi(U)$, the lower bound should not be shifted to the left or right. This is because the shortest interval has already been obtained (Figure 9).

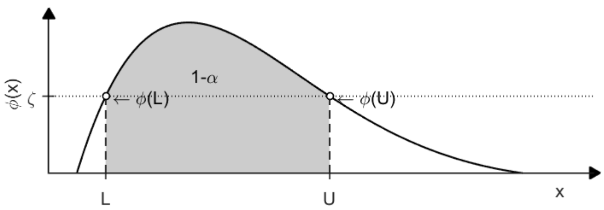


Figure 9. *Determination of the confidence bounds (L and U) when $\phi(L) = \phi(U)$.*

This approach is called the equal density approach. The confidence coefficient is a direct function of the confidence limits both in the equally tailed confidence interval and in the shortest confidence interval.

However, in the equal density approach, the confidence coefficient is calculated according to a density variable such as $\left(\zeta = \phi(L) = \phi(U)\right)$.

2.3.1. Numerical Method For The Shortest Confidence Interval By Using The Equal Density Approach

The objective function and the constraint are given respectively to determine the shortest confidence interval for equal density.

$$goal: \min_{L,U} |\Phi(U) - \Phi(L) - (1-\alpha)| \tag{22}$$

$$constraint: \zeta = \phi(L) = \phi(U), \;\; L \neq U \tag{23}$$

This can be solved as an optimization problem by using numerical methods. In this paper, the bisection method is used as a numerical method. The algorithm of the bisection method for the equal density approach can be summarized as follows.

Algorithm 1. *Bisection Method For The Equal Density Approach To Find The Shortest Confidence Interval.*

Step 1. Determine two initial values according to this method. These initial values may be $a = 0$ and $b = \zeta_{max}$. If $\zeta = 0$, the confidence coefficient can be given as follows.

$$P_a = 1, \qquad (\zeta = 0)$$

If $\zeta = \zeta_{max}$, the confidence coefficient can be given as follows.

$$P_b = 0, \qquad (\zeta = \zeta_{max})$$

Therefore, the calculation of the confidence coefficients of the initial values is not required.

Step 2. The midpoint of these initial values is calculated by using following equation.

$$\zeta = \frac{a+b}{2}$$

Step 3. Find the roots $\{L_\zeta, U_\zeta\}$ of the following equation via the midpoint.

$$\phi(x) - \zeta = 0$$

Step 4. Calculate the confidence coefficient by assuming that these roots are bounds of the confidence interval as follows.

$$P_\zeta = \Phi(U_\zeta) - \Phi(L_\zeta)$$

Step 5. If the calculated confidence coefficient approaches the desired confidence coefficient, $\left|P_\zeta - (1-\alpha)\right| < \epsilon$ assume L_ζ and U_ζ are the desired confidence limits and stop the process.

Step 6. If the calculated confidence coefficient is smaller than the desired value $(1-\alpha)$; $a = \zeta$, $\left(P_\zeta < 1-\alpha\right)$ else $b = \zeta$, $\left(P_\zeta > 1-\alpha\right)$ go to Step 3.

3. CONFIDENCE INTERVAL FOR THE MULTIMODAL PROBABILITY DENSITY FUNCTIONS

The afore-mentioned assumptions are used for the unimodal probability density functions. Therefore, in this section, the validity of the assumptions is investigated for the multimodal functions. The confidence interval of the multimodal probability density function is given as follows [33].

$$\Pr\{\theta \in ([L_1, U_1] \cup [L_2, U_2] \cup \dots [L_n, U_n]\} = 1 - \alpha \tag{24}$$

The multiple intervals are defined and these intervals provide the following condition, meaning that they are independent from each other.

$$L_1 < U_1 < L_2 < U_2 < \cdots < L_i < U_i < \cdots < L_n < U_n \tag{25}$$

If these intervals do not intersect with each other, they can be written as follows.

$$Pr\{\theta \in [L_1, U_1]\} + Pr\{\theta \in [L_2, U_2]\} + \cdots + Pr\{\theta \in [L_n, U_n]\} = 1 - \alpha \tag{26}$$

Each interval $[L_i, U_i]$ gives a probability value as Equation (27).

$$P_i = Pr\{\theta \in [L_i, U_i]\} = \Phi(U_i) - \Phi(L_i) \tag{27}$$

In the present case, the total width of the confidence intervals is calculated by using Equation (28).

$$W = \sum_i W_i = \sum_i U_i - L_i \tag{28}$$

The probability values of the bimodal probability density functions are shown as follows.

$$P_1 = \Phi(U_1) - \Phi(L_1) \tag{29}$$

$$P_2 = \Phi(U_2) - \Phi(L_2) \tag{30}$$

The desired confidence level is obtained by adding these probability values as Equation (31).

$$P_1 + P_2 = \left(\Phi(U_1) - \Phi(L_1)\right) + \left(\Phi(U_2) - \Phi(L_2)\right) = 1 - \alpha \tag{31}$$

Theorem 2. The confidence limits that have the minimum total width of the confidence intervals, which has the (1-α) confidence coefficient for a multimodal probability density function, are the confidence limits that have the equal density value.

Proof. To check whether the selected confidence intervals are the shortest intervals the lower bound of the first region must be shifted to the left as ε_1 and the upper bound of the first region must be shifted to the right as ε_1 according to Theorem 1. In this case, the probability value (P_1) of the first region increases. Thus, the second region must be narrowed to be stable confidence coefficient. The probability density values of the upper and lower bounds of each region should be equal according to Theorem 1. Therefore, the lower bound must be shifted to the right as ε_2. This is parallel with the upper bound, which must be shifted to the left as ε_2 for the contraction of the second region. This process can be summarized as follows.

$$P_1 + P_2 = \left(\Phi(U_1 + \varepsilon_1) - \Phi(L_1 - \varepsilon_1)\right) + \left(\Phi(U_2 - \varepsilon_2) - \Phi(L_2 + \varepsilon_2)\right) \tag{32}$$

Here, the probability density values of the upper and lower bounds of the first region are equal to each other and these are shown as ζ_1 in Equation (33).

$$\zeta_1 = \phi(L_1) = \phi(U_1) \tag{33}$$

Similarly, the probability density values of the upper and lower bounds of the second region are equal to each other and these are shown as ζ_2 in Equation (34).

$$\zeta_2 = \phi(L_2) = \phi(U_2) \tag{34}$$

Equation (35) is obtained by using Equation (32), (33) and (34).

$$P_1 + P_2 = (\Phi(U_1\ ) + \varepsilon_1\zeta_1 - \Phi(L_1\ ) + \varepsilon_1\zeta_1) + (\Phi(U_2) - \varepsilon_2\zeta_2 - \Phi(L_2) - \varepsilon_2\zeta_2) \tag{35}$$

Equation (36) is obtained by using Equation (29) and (30).

$$P_1 + P_2 = (P_1 + 2\varepsilon_1\zeta_1) + (P_2 - 2\varepsilon_2\zeta_2) \tag{36}$$

Equation (37) is obtained by updating previous equations. If the bounds of the first region are narrowed to ε_1 from the right and left, the same result is obtained as follows.

$$\varepsilon_1\zeta_1 = \varepsilon_2\zeta_2 \tag{37}$$

Scenario 1: If the first region is extended from the left and right bounds as ε_1, the second region will be narrowed from the left and right bounds as ε_2. The value of ζ_1 decreases and the value of ζ_2 increases, provided that the first region is extended. In this case, the total width of the confidence intervals is found as follows.

$$W' = W_1' + W_2' = [U_1 + \varepsilon_1 - L_1 + \varepsilon_1] + [U_2 - \varepsilon_2 - \ L_2 - \varepsilon_2] \tag{38}$$

Equation (39) is obtained via Equation (28).

$$W' = W + 2(\varepsilon_1 - \varepsilon_2) \tag{39}$$

Case 1. If $\varepsilon_1 > \varepsilon_2$, the total width of the confidence intervals will extend (Figure 10), because the second part of Equation (39) is positive $\big((\varepsilon_1 - \varepsilon_2) > 0\big)$. Then, if the intention is to narrow the total width, the first region must be narrowed.

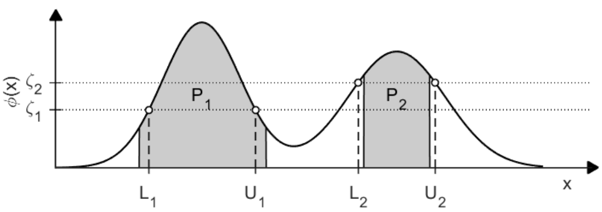


Figure 10. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 > \varepsilon_2$).*

Case 2. If $\varepsilon_1 < \varepsilon_2$, the total width of the confidence intervals will narrow (Figure 11), because the second part of Equation (39) is negative $\big((\varepsilon_1 - \varepsilon_2) < 0\big)$. Then, if the intention is to narrow the total width, the first region must be extended.

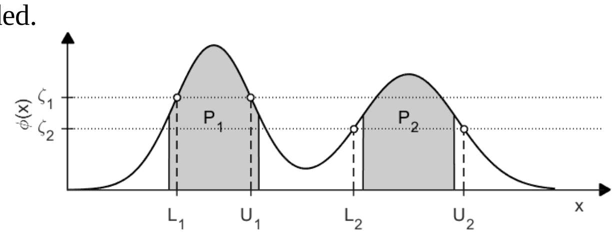


Figure 11. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 < \varepsilon_2$).*

Case 3. If $\varepsilon_1 = \varepsilon_2$, the total width will not change (Figure 12), because the second part of Equation (39) is zero $\left((\varepsilon_1 - \varepsilon_2) = 0\right)$. In this case, if the first region continues to extend, Case 1 will occur, because $\zeta_1 < \zeta_2$.

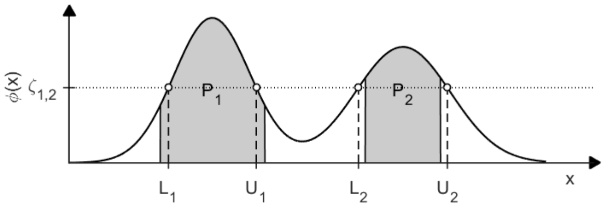


Figure 12. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 = \varepsilon_2$).*

Scenario 2: If the first region is narrowed from the left and right bounds as ε_1, the second region will be extended from the left and right bounds as ε_2. The value of ζ_1 increases and the value of ζ_2 decreases, provided that the first region is narrowed. In this case, the total width of the confidence intervals is calculated as follows.

$$W' = W_1' + W_2' = [U_1 - \varepsilon_1 - L_1 - \varepsilon_1] + [U_2 + \varepsilon_2 - L_2 + \varepsilon_2] \tag{40}$$

Equation (41) is obtained via Equation (28).

$$W' = W + 2(\varepsilon_2 - \varepsilon_1) \tag{41}$$

Case 1. If $\varepsilon_1 < \varepsilon_2$, the total width of the confidence intervals will extend (Figure 13), because the second part of Equation (41) is positive $\left((\varepsilon_2 - \varepsilon_1) > 0\right)$. Then, if the intention is to narrow the total width, the first region must be extended.

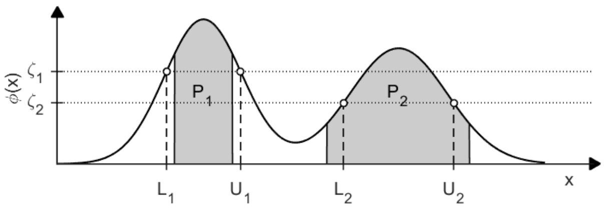


Figure 13. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 < \varepsilon_2$).*

Case 2. If $\varepsilon_1 > \varepsilon_2$, the total width of the confidence intervals will narrow (Figure 14), because the second part of Equation (41) is negative $\left((\varepsilon_2 - \varepsilon_1) < 0\right)$. In this case, if the intention is to narrow the total width, the first region must be narrowed.

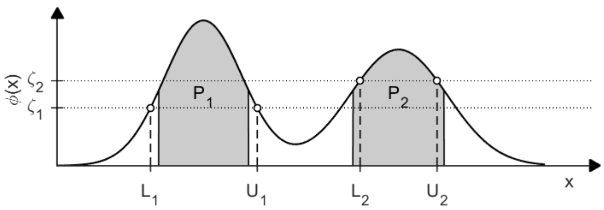


Figure 14. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 > \varepsilon_2$).*

Case 3. If $\varepsilon_1 = \varepsilon_2$, the total width of the confidence intervals will not change (Figure 15), because the second part of Equation (41) is zero $\left((\varepsilon_2 - \varepsilon_1) = 0\right)$. In this case, if the first region continues to narrow, Case 1 will occur, because $\zeta_1 < \zeta_2$.

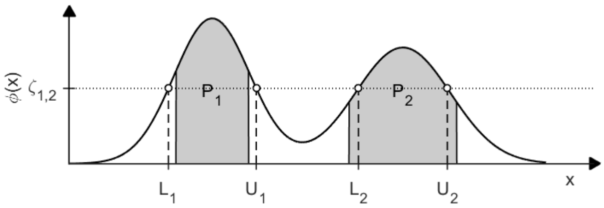


Figure 15. *Variation of the boundaries in the bimodal probability density function ($\varepsilon_1 = \varepsilon_2$).*

Consequently, the shortest interval is performed as three cases. Firstly, the first region must be expanded in the case of $\zeta_1 > \zeta_2$. Secondly, the first region must be narrowed in the case of $\zeta_1 < \zeta_2$. Lastly, in the case of $\zeta_1 = \zeta_2$, there must be no changes made.

The cases of the bimodal probability density function can be easily applied to the multimodal probability density function. Likewise, when any interval narrows, the other interval or intervals can be expanded. However, an answer to the question of which interval is expanded and at what rate this interval is changed can be elusive. In the case of multi-confidence intervals, when one interval is narrowed, the other interval is expanded and the others can be fixed. If this process is performed to any two intervals, the method will not change because of Theorem 2. Thus, Theorem 2 can also be applied to the multimodal probability density functions.

3.1. Numerical Method For The Multi-Confidence Intervals By Using The Equal Density Approach

The determination of confidence intervals according to the equal density approach can be performed by using the bisection method as the unimodal case. Algorithm 2 has been developed to perform this method.

Algorithm 2. *Bisection Method For The Equal Density Approach To Find The Multi-Confidence Intervals*

Step 1. Determine two initial values according to this method. These initial values may be $a = 0$ and $b = \zeta_{max}$. If $= 0$, the confidence coefficient can be given as follows.

$$P_a = 1, \qquad (\zeta = 0)$$

If $\zeta = \zeta_{max}$, the confidence coefficient can be given as follows.

$$P_b = 0, \qquad (\zeta = \zeta_{max})$$

Therefore, the calculation of the confidence coefficients of the initial values is not required.

Step 2. The midpoint of these initial values is calculated by using the following equation.

$$\zeta = \frac{a+b}{2}$$

Step 3. Find the roots $\{L_{\zeta 1}, U_{\zeta 1}, L_{\zeta 2}, U_{\zeta 2}, \dots, L_{\zeta n}, U_{\zeta \mathrm{n}},\}$ of the following equation via the midpoint.

$$\phi(x) - \zeta = 0$$

Step 4. Calculate the confidence coefficient by assuming these roots as bounds of the confidence intervals as follows.

$$P_\zeta = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\Phi(U_{\zeta i}) - \Phi(L_{\zeta i})\right)$$

Step 5. If the calculated confidence coefficient approaches the desired confidence coefficient, $\left|P_\zeta - (1-\alpha)\right| < \epsilon$, approve $\{L_{\zeta 1}, U_{\zeta 1}, L_{\zeta 2}, U_{\zeta 2}, \dots, L_{\zeta n}, U_{\zeta n},\}$ as the desired confidence bounds and stop the process.

Step 6. If the calculated confidence coefficient is smaller than the desired value $(1-\alpha)$; $a = \zeta$, $\left(P_\zeta < 1-\alpha\right)$ else $b = \zeta$, $\left(P_\zeta > 1-\alpha\right)$ go to Step 3.

4. EXPERIMENTAL RESULTS

This section covers four different examples. The limits and width of confidence intervals which are calculated according to the equally tailed, the shortest interval and the equal density approaches in the following examples.

Example 1. The confidence bounds (L, U) and the width of confidence intervals (W) are calculated for the significance coefficient $\alpha = \{0.01, 0.025, 0.05, 0.10\}$ according to the equally tailed (et), the shortest interval (sh) and the equal density (ed) approaches in case the desired statistics $(\hat{\theta})$ from a standard normal distribution $\left(\hat{\theta} \sim \mathcal{N}(0,1)\right)$. These bounds are given in Table 1. The tolerance value is determined as
$\epsilon = 1 \times 10^{-5}$ for all approaches.

***Table 1**. The confidence limits and the width of the confidence interval in a symmetrical probability density function.*

$\boldsymbol{\alpha}$	$\boldsymbol{L_{et}}$	$\boldsymbol{U_{et}}$	$\boldsymbol{W_{et}}$	$\boldsymbol{L_{sh} = L_{ed}}$	$\boldsymbol{U_{sh} = U_{ed}}$	$\boldsymbol{W_{sh}}$	$\boldsymbol{W_{et} - W_{sh}}$
0.0100	-2.5758	2.5758	5.1517	-2.5758	2.5758	5.1517	0.0000
0.0250	-2.2414	2.2414	4.4828	-2.2414	2.2414	4.4828	0.0000
0.0500	-1.9600	1.9600	3.9199	-1.9600	1.9600	3.9199	0.0000
0.1000	-1.6449	1.6449	3.2897	-1.6449	1.6448	3.2897	0.0000

If the unimodal probability density function is symmetric for the two-tailed confidence interval, as in this example, all approaches have the same confidence limits. Therefore, the differences $(W_{et} - W_{sh})$ in the width of the confidence intervals are zero, as seen in Table 1.

Example 2. The confidence bounds (L, U) and the width of the confidence interval (W) are calculated for the significance coefficient $\alpha = \{0.01, 0.025, 0.05, 0.10\}$ according to the equally tailed (et), the shortest interval (sh) and the equal density (ed) approaches in case the desired statistics $(\hat{\theta})$ from an asymmetrical distribution as log-normal distribution $\left(\hat{\theta} \sim Log\mathcal{N}(0,1)\right)$. These bounds are given in Table 2. The tolerance value is determined as $\epsilon = 1 \times 10^{-5}$ for all approaches.

***Table 2**. The confidence limits and the width of the confidence interval in an asymmetrical probability density function.*

$\boldsymbol{\alpha}$	$\boldsymbol{L_{et}}$	$\boldsymbol{U_{et}}$	$\boldsymbol{W_{et}}$	$\boldsymbol{L_{sh} = L_{ed}}$	$\boldsymbol{U_{sh} = U_{ed}}$	$\boldsymbol{W_{sh}}$	$\boldsymbol{W_{et} - W_{sh}}$
0.0100	0.0761	13.1422	13.0661	0.0132	10.2434	10.2302	2.8359
0.0250	0.1063	9.4065	9.3002	0.0190	7.1036	7.0846	2.2156
0.0500	0.1409	7.0991	6.9582	0.0261	5.1869	5.1609	1.7974
0.1000	0.1930	5.1803	4.9872	0.0375	3.6127	3.5753	1.4119

In this example, we discussed two-tailed confidence intervals for asymmetrical probability density function. While the shortest interval and the equal density approaches have the same confidence limits, the equally tailed approach has a wider confidence interval than the others. Furthermore, the differences in the width of the confidence intervals are shown in Table 2.

Example 3. The confidence limits and the width of the confidence intervals are calculated for the significance coefficient $\alpha = \{0.01, 0.025, 0.05, 0.10\}$ in case the desired statistics $(\hat{\theta})$ has a multimodal mixed distribution as log-normal distribution and normal distribution $\left(\hat{\theta} \sim [Log\mathcal{N}(1,0.4)/2 + \mathcal{N}(8,0.9)/2]\right)$. However, the equally tailed confidence interval is not logically valid, because the investigated distribution has a multimodal distribution. In addition to this, the shortest interval (sh) approach may not find the best solution in multiple infinite solutions. In such case, this approach will not be valid for multimodal distributions. Furthermore, the researcher will need to select how many confidence regions are required for these distributions. Therefore, the equal density approach is simulated for the shortest multi-confidence intervals. The obtained confidence limits are shown in Table 3.

Table 3. *The confidence limits and the width of the confidence interval in a bimodal probability density function.*

α	L_1	U_1	L_2	U_2	W_1	W_2	$W_1 + W_2$
0.0100	0.9829	9.9793	-	-	8.9964	-	8.9964
0.0250	1.0297	5.3240	5.8017	9.8766	4.2943	4.0749	8.3692
0.0500	1.0911	4.9389	6.0850	9.7496	3.8478	3.6646	7.5124
0.1000	1.2110	4.4324	6.3984	9.5224	3.2214	3.1240	6.3454

Although a bimodal distribution function is obtained in this example, only a confidence interval is calculated for $\alpha = \{0.01\}$ (Figure 16(a)). This result stems from the density value (ζ) which is smaller than the minimum density value between the two peaks. Two confidence intervals are obtained for other α values (Figure 16(b)). Furthermore, the proposed equal density approach provides a more reliable solution, as seen in Table 3.

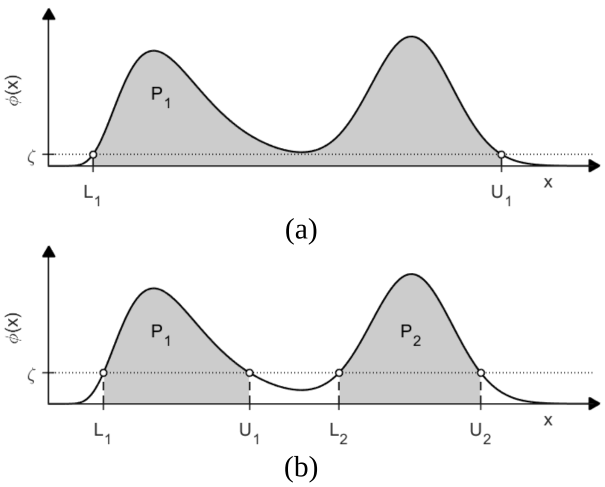


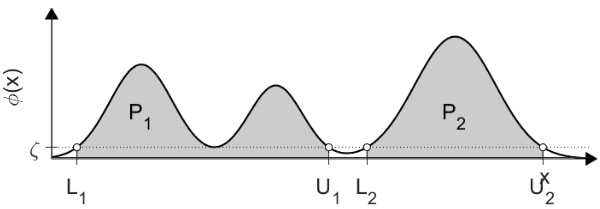
Figure 16. *The confidence intervals in a bimodal probability density function, (a) The confidence level of* 99% $(\alpha = 0.01)$*; (b) The confidence level of* 90% $(\alpha = 0.1)$.

Example 4. The confidence limits and the width of the confidence intervals are calculated for the significance coefficient $\alpha = \{0.01, 0.025, 0.05, 0.10\}$ in case the desired statistics $(\hat{\theta})$ has a multimodal mixed normal distribution $\left(\hat{\theta} \sim [\mathcal{N}(2,0.7) \times 0.3 + \mathcal{N}(5,0.6) \times 0.2 + \mathcal{N}(9,0.9) \times 0.5]\right)$. However, the equally tailed confidence interval is not logically valid as in Example 3, because the investigated distribution has a multimodal distribution. In addition to this, the shortest interval (sh) approach may not find the best solution in multiple infinite solutions. In such case, this approach will not be valid for multimodal distributions. Furthermore, the researcher will need to select how many confidence regions are required for these distributions. Therefore, the equal density approach is simulated for the shortest multi-confidence intervals. The obtained confidence limits are shown in Table 4.

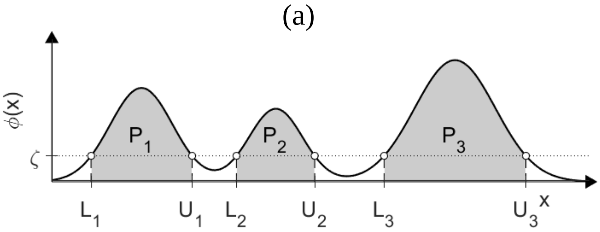
Table 4. *The confidence limits and the width of the confidence interval in a multimodal probability density function.*

α	L_1	U_1	L_2	U_2	L_3	U_3	W_1	W_2	W_3	$W_1+W_2+W_3$
0.010	0.373	6.433	6.741	11.190	-	-	6.060	4.450	0.000	10.510
0.025	0.563	6.182	7.033	10.958	-	-	5.619	3.925	0.000	9.544
0.050	0.677	3.368	3.893	6.065	7.177	10.821	2.691	2.172	3.644	8.507
0.100	0.878	3.132	4.122	5.868	7.417	10.582	2.255	1.746	3.165	7.165

Although a multimodal distribution function is obtained in this example, only two confidence intervals are calculated for $\alpha = \{0.01, 0.025\}$ (Figure 17 (a)). This result stems from the density value (ζ) which is smaller than the minimum density value between the two peaks which is in the range of $[L_1, U_1]$. Three confidence intervals are obtained for other α values (Figure 17 (b)). Furthermore, the proposed equal density approach provides a more reliable solution, as seen in Table 4.



(a)



(b)

Figure 17. *The confidence intervals in a multimodal probability density function, (a) The confidence level of* 97.5% ($\alpha = 0.025$)*; (b) The confidence level of* 90% ($\alpha = 0.1$).

5. CONCLUSION

In this paper, the commonly used equally tailed approach is compared with the rarely used shortest interval approach. Proofs and algorithms of the equal density approach, which is an alternative to the shortest interval approach, are studied. The equal density and the shortest interval approaches have the same confidence limits, as in the proofs. This was demonstrated in the simulations.

All approaches have the same confidence limits for the two-tailed confidence level in the unimodal symmetric distribution. When considering an asymmetric probability density function for the two-tailed confidence level, the shortest interval and the equal density approaches have the same width of confidence interval. However, the equally tailed approach has a wider width of the confidence interval. Furthermore, the equally tailed and the shortest interval approaches in multimodal distributions are not logically valid for the confidence intervals. Nevertheless, the equal density approach gave consistent results for all distributions in the examples.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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