http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 24.12.2017 *Year*: **2018**, *Number*: **21**, *Pages*: **49-58**
Published: 27.02.2018 *Original Article*

Upper and Lower δ_{ij}-Continuous Multifunctions

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Abstaract − In this paper we introduce and study the notions of upper and lower δ_{ij}-continuous multifunctions. Several characterizations and properties concerning upper and lower δ_{ij}-continuous multifunctions and other known forms of multifunctions introduced previously are investigated.

Keywords − *Upper(lower)δ_{ij}-continuous multifunction.*

1 Introduction

A multifunction or a multivalued function is set valued function. In last thirty years the theory of multifunctions has advanced in variety of ways. Applications of this theory can be found in economic theory, viability theory, noncooperative games, decision theory, artificial intelligence, medicine and existence of solutions for differential equations. In topology there has been recently significant interest in characterizing and investigating the properties of several weak and strong forms of continuity of multifuctions. The development of such a theory is in fact very well motivated in [1, 4, 5, 6, 7, 12, 14, 15, 17]. Kucuk [10] and Cao and Reilly [3] independently defined and investigated upper(lower)δ_{ij}-continuous multifunction. The invariance of some separating properties of the bitoplogical spaces by multifunctions was studied by Smithson [18]. The notions of continuous (resp. upper semicontinuous, lower semicontinuous) multifunctions between bitopological spaces wear defined and studied by Popa [15] and Ganguly [13] introduced and studied the concept of upper (lower) almost multifunction between bitopological spaces. Several characterizations of these

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concepts were given by Kucuk and Kucuk in [11]. In this paper we introduce and study the notions of upper and lower δ_{ij}-continuous multifunctions between bitopological spaces. As a consequence, some characterizations and several proprties concerning upper (lower) δ_{ij}-continuous multifunctions are obtained. The relationship between upper (lower) δ_{ij}-continuous multifunctions and with other known forms of multifuctions introduced previously are established.

2 Preliminary

Let (X, τ_1, τ_2) be a bitopological space. The closure and interior of a subset A of X with respect to τ_i are denoted by $\tau_i.cl(A)$ and $\tau_i.int(A)$, respectively. The set $N(A, \tau_i)$ denotes the family of all τ_i-open set containing A. In particular, $N(x, \tau)$ is the family of all τ_i-open neighborhood (τ_i-nbds, for short) of x. The set of all τ_i-closed sets will be denoted by $\acute{\tau_i}$. A subset A of a bts (X, τ_1, τ_2) is called $ij-$regular closed (resp. $ij-$regular open) if $A = \tau_i.cl(\tau_j.int(A))$(resp.$A = \tau_i.int(\tau_j.cl(A))$). The set of all $ij-$regular closed (resp. $ij-$regular open) sets of (X, τ_1, τ_2) is denoted by $ijRC(X)$ (resp. $ijRO(X)$). By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \phi$ for all $x \in X$. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a set B of Y by $F^-(B)$ and $F_-(B)$ [2], respectively, that is $F^-(B) = \{x \in X : F(X) \subseteq B\}$ and $F_- = \{x \in X : F(x) \cap B \neq \phi\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$, for each $y \in Y$. For $A \subseteq X, F(A) = \cup_{x \in A} F(x)$. Then F is said to be a surjection if $F(x) = Y$, or equivalently if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Also, F is said to be injective if for any $x_1, x_2 \in X, x_1 \notin x_2$, we have $F(x_1) \cap F(x_2) = \phi$. The reader can find undefined notions of some generalizing continuities for multifunctions from the references.

Definition 2.1. Let (X, τ_1, τ_2) be a bts.[8, 13, 16]. A point x in X will be called an δ_{ij}-adherent (resp. θ_{ij}-adherent) point of a subset A of X if and only if $A \cap \tau_i.int(\tau_j.cl(U)) \neq \phi$ (resp. $A \cap \tau_j..cl(U)) \neq \phi$ for each τ_i-open nbd U of x. The set of all δ_{ij}-adherent (resp. θ_{ij}-adherent) points of A is called δ_{ij}-closure (resp. θ_{ij}-closure) of A and it is denoted by $\delta_{ij}.cl(A)$ (resp. $\theta_{ij}.cl(A)$). If $A = \delta_{ij}.cl(A)$ (resp. $A = \theta_{ij}.cl(A)$), then A is called δ_{ij}-closed (resp. θ_{ij}-closed). The complement of a δ_{ij}-closed (resp. θ_{ij}-closed) set is called a δ_{ij}-open(resp. θ_{ij}-open) set. The family of all δ_{ij}-closed (resp. δ_{ij}-open , θ_{ij}-closed, θ_{ij}-open) sets of X is denoted by $\delta_{ij}.C(X)$ (resp. $\delta_{ij}.O(X), \theta_{ij}.C(X), \theta_{ij}.O(X)$). It is clear that in any bts (X, τ_1, τ_2), we have $\theta_{ij}.O(X) \subseteq \delta_{ij}.O(X) \subseteq \tau_i$ and $ijRC(X) \subseteq \delta_{ij}.C(X)$.

Definition 2.2. Let (X, τ_1, τ_2) be a bts.[8, 13]. A point x in X will be called an δ_{ij}-interior (resp. θ_{ij}-interior) point of a subset A of X if and only if there exists τ_i-open nbd U of x such that $\tau_i.int\tau_i.cl(U)) \subseteq A$ (resp. $\tau_i.cl(U)) \subseteq A$) equivalently, if there exists $ij-$regular open (resp. $ij-$regular closed) nbd U of x such that $U \subseteq A$. The family of all δ_{ij}-interior (resp. θ_{ij}-interior) points of A will be denoted by $\delta_{ij} - int(A)$ (resp. $\theta_{ij} - int(A)$). A subset A of a bts (X, τ_1, τ_2) is δ_{ij}-open (resp. θ_{ij}-open) if and only if $\delta_{ij} - int(A) = A$ (resp. $\theta_{ij} - int(A) = A$).

Definition 2.3. A bts (X, τ_1, τ_2) [8, 9, 15] is called:
(a) PR_2 if and only if $\forall x \in X, F \in \acute{\tau_i} s.t.x \notin F \; \exists U \in N(x, \tau_i), V \in N(F, \tau_j) s.t.U \cap$

$V = \phi$.
(b)PSR_2 if and only if $\forall x \in X, U \in N(x, \tau_i)\ \exists V \in N(x, \tau_i), \tau_i - int(\tau_i.cl(V)) \subseteq U$.
(c)PAR_2 if and only if $\forall x \in X, U \in N(x, ijRO(X))\ \exists V \in N(x, \tau_i), \tau_i.cl(V) \subseteq U$.

Theorem 2.4. Let (X, τ_1, τ_2) be a bts.[8, 15].
(a) For each $A \subseteq X$, then $\tau_i.cl(A) \subseteq \delta_{ij}.cl(A) \subseteq \theta_{ij}.cl(A)$.
(b) If $A \in \tau_j$, then $\tau_i.cl(A) = \delta_{ij}.cl(A)$.
(c) If (X, τ_1, τ_2) is PSR_2-space, then $\tau_i.cl(A) = \delta_{ij}.cl(A)$.
(d) If (X, τ_1, τ_2) is PAR_2-space, then $\delta_{ij}.cl(A) = \theta_{ij}.cl(A)$.

3 Upper and Lower δ_{ij}-Continuous Multifunctions

In this section we define and study the concept of upper and lower δ_{ij}-continuous multifunctions.some of their properties are obtained.

Definition 3.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called:
(a) Lower δ_{ij}-continuous at a point x in X if and only if for every increasing $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists increasing $\triangle_i$-open nbd U of x such that $F(x_0) \cap \triangle_i.int(\triangle_j.cl(V)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(U))$.
(b) Upper δ_{ij}-continuous at a point x in X if and only if for every decreasing $\triangle_i$-open set V in Y with $F(x) \subseteq V$, there exists decreasing τ_i-open nbd U of x such that $F(\tau_i.int(\tau_j.cl(U)) \subseteq \triangle_i.int(\triangle_j.cl(V))$.
(c) Lower (resp. upper) δ_{ij}-continuous if it has this property at each point $x \in X$.
The following theorem give us some characterizations of lower δ_{ij}-continuity of F.

Theorem 3.2. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ the following statements are equivalent:
(a) F is lower δ_{ij}-continuous,
(b) For every increasing $ij-$regular open set $V \subseteq Y$ and for each $x \in X$ with $F(x) \cap V \neq \phi$, there exists increasing $ij-$regular open nbd U of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$.
(c) For every increasing $ij-$regular open set $V \subseteq Y, F_-(V)$ is δ_{ij}-open set in X.
(d) For every increasing δ_{ij}-open set $V \subseteq Y, F_-(V)$ is δ_{ij}-open set in X.
(e) For every increasing δ_{ij}-closed set $K \subseteq Y, F^-(K)$ is δ_{ij}-closed set in X.
(f) For every increasing $ij-$regular closed set $K \subseteq Y, F^-(K)$ is δ_{ij}-closed set in X.
(g) For each $B \subseteq Y, F^-(\delta_{ij}.int(B)) \subseteq \delta_{ij}.int(F^-(B))$.
(h) For each $A \subseteq X, F(\delta_{ij}.cl(A)) \subseteq \delta_{ij}.cl(F(A))$.

Proof. (a)$\to$ (b): Let x in X and let V by an $ij-$regular open set in Y with $F(x) \cap V \neq \phi$. Then V is $\triangle_i-$open set in Y. By (a), there exists $W \in N(x, \tau_i)$ such that $F(x_0) \cap \triangle_i.int(\triangle_i.cl(V)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(W))$. But V is ij-regular open set, so $F(x_0) \cap V \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(W))$. Put $U = \tau_i.int(\tau_j.cl(W))$. Then U is ij-regular open set in X. So $F(x_0) \cap V \neq \phi$ for $x_0 \in U$.
(b)$\to$ (c): Let $V \subseteq Y$ be an ij-regular open set and let x in X with $x \in F^-(V)$. Then $F(x) \cap V \neq \phi$. By (b), there exists ij-regular open nbd U of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Which implies that $U \subseteq F_-(V)$. Consequently $F_-(V)$ is δ_{ij}-open set in X.
(c)$\to$ (d): Let $V \subseteq Y$ be a δ_{ij}-open set and let x in X with $x \in F_-(V)$. So,

$F(x) \cap V \neq \phi$ and so there exists $y \in Y$ such that $y \in F(x) \cap V$. Hence, $y \in F(x)$ and $y \in V$. Since V is δ_{ij}-open set, then there exist ij-regular open set $W \subseteq Y$ such that $y \in W \subseteq V$. Thus $F(x) \cap W \neq \phi$ and so $x \in F_-(W)$. Since W is ij-regular open set, by (c), $F_-(W)$ is a δ_{ij}-open set of X and from $x \in F_-(W)$, there exists an ij-regular open set $U \subseteq X$ such that $x \in U \subseteq F_-(W) \subseteq F_-(V)$. Thus $F_-(V)$ is a δ_{ij}-open set in X.
(d)→ (e): Let $K \subseteq Y$ be any δ_{ij}-closed set. Then $T \backslash K$ is a δ_{ij}-open set. By (d), $F_-(Y \backslash K)$ is a δ_{ij}-open set. As we can write $F^-(K) = X \backslash F_-(Y \backslash K)$ so $F^-(K)$ is a δ_{ij}-closed set in X.
(e)→ (f): Let $K \subseteq Y$ be any δ_{ij}-regular closed set. Then K is a δ_{ij}-closed set. By (e), $F^-(K)$ is a δ_{ij}-closed set in X.
(f)→ (c): Let $V \subseteq Y$ be an ij-regular open set. Then $Y \backslash V$ is an ij-regular closed set of Y. By (f), $F^-(Y \backslash V)$ is δ_{ij}-closed set in X. Thus $F_-(V)$ is δ_{ij}-open set in X.
(c)→ (a): Let x in X and let $V \subseteq Y$ be any $\triangle_i$-open set with $F(x) \cap V \neq \phi$. Since $V \subseteq \triangle_i.int(\triangle_j.cl(V))$, then $F(x) \cap \triangle_i.int(\triangle_j.cl(V)) \neq \phi$. So, x is $F^-(\triangle_i.int(\triangle_j.cl(V)))$. By (c), there exists $ij-$regular open nbd U of x such that $U \subseteq F^-(\triangle_i.int(\triangle_j.cl(V)))$. Thus $F(x_0) \cap \triangle_i.int(\triangle_j.cl(V)) \neq \phi$ for each x_0 in U. Thus F is lower δ_{ij}-continuous.
(d)→ (g): Let $B \subseteq Y$. Since $\delta_{ij}.int(B) \subseteq B$, then $F_-(\delta_{ij}.int(B)) \subseteq F_-(B)$. Since $\delta_{ij}.int(B)$ is δ_{ij}-open set of Y, then by (d), $F_-(\delta_{ij}.int(B)) = \delta_{ij}int(F_-(\delta_{ij}.int(B))) \subseteq \delta_{ij}.int(F_-(B))$. Thus $F_-(\delta_{ij}.int(B))) \subseteq \delta_{ij}.int(F_-(B))$.
(g)→ (d): Let V be δ_{ij}-open set of Y. By (g), we have $F_-(V) = F_-(\delta_{ij}.int(V)) \subseteq \delta_{ij}.int(F_-(V))$. Thus $F_-(V)$ is δ_{ij}-open set of X.
(d)→ (h): Under the assumption (e) suppose that (h) is not true, i.e. for some $A \subseteq X$, we have $F(\delta_{ij}.cl(A)) \nsubseteq \delta_{ij}.cl(F(A))$. Then there exists y in Y such that $y \in F(\delta_{ij}.cl(A))$, but $y \notin \delta_{ij}.cl(F(A))$. So, $Y \backslash (\delta_{ij}.cl(F(A)))$ is δ_{ij}-open set containing y. By (d), we have $F_-(Y \backslash (\delta_{ij}.cl(F(A))))$ is δ_{ij}-open set in X and $F_-(Y) \subseteq F_-(Y \backslash (\delta_{ij}.cl(F(A))))$. Since $Y \backslash (\delta_{ij}.cl(F(A))) \cap F(A) = \phi$ and $A \subseteq F^-(F(A))$ we have $F_-(Y \backslash (\delta_{ij}.cl(F(A)))) \cap F^-(F(A)) = \phi$ and $F_-(Y \backslash (\delta_{ij}.cl(F(A)))) \cap A = \phi$. Since $F_-(Y \backslash (\delta_{ij}.cl(F(A))))$ is δ_{ij}-open set in X, then $F_-(Y \backslash (\delta_{ij}.cl(F(A)))) \cap \delta_{ij}.cl(A) = \phi$. On the other hand, because of $y \in F(\delta_{ij}.cl(A))$, we have $F_-(Y) \cap \delta_{ij}.cl(A) \neq \phi$, which is contradiction with $F_-(Y \backslash (\delta_{ij}.cl(F(A)))) \cap \delta_{ij}.cl(A) = \phi$. Thus $y \in F(\delta_{ij}.cl(A))$ implies $y \in \delta_{ij}.cl(F(A))$. Consequently, $F(\delta_{ij}.cl(A)) \subseteq \delta_{ij}.cl(F(A))$.
(h)→ (e): Let $K \subseteq Y$ be any δ_{ij}-closed set. Since we have always $FF^-(K) \subseteq K$, then we obtain $\delta_{ij}.cl(FF^-(K)) \subseteq \delta_{ij}.cl(K) = K$. By (h), $F(\delta_{ij}.cl(F^-(K))) \subseteq \delta_{ij}.cl(FF^-(K))$. Thus $F(\delta_{ij}.cl(F^-(K))) \subseteq K$ and so

$\delta_{ij}.cl(F^-(K)) \subseteq F^-F(\delta_{ij}.cl(F^-(K))) \subseteq F^-(K)$. Hence $F^-(K)$ is δ_{ij}-closed set in X. □

Theorem 3.3. For multifunction $F : (X, \tau_1, \tau_2) \rightarrow (Y, \triangle_1, \triangle_2)$ the following statements are equivalent:
(a) F is upper δ_{ij}-continuous,
(b) For every ij-regular open set $V \subseteq Y$ for each $x \in X$ with $F(x) \subseteq V$, there exists ij-regular open nbd U of x such that $F(U) \subseteq V$.
(c) For each ij-regular open set $V \subseteq Y, F^-(V)$ is δ_{ij}-open set in X.
(d) For each ij-open set $V \subseteq Y, F^-(\triangle_i.int(\triangle_j.cl(V)))$ is δ_{ij}-closed set in X.
(e) For each δ_{ij}-closed set $K \subseteq Y, F_-(\triangle_j.cl(\triangle_i.int(K)))$ is δ_{ij}-closed set in X.
(f) For each δ_{ij}-regular closed set $K \subseteq Y, F_-(K)$ is δ_{ij}-open set in X.

Proof. It is quite similar to that of Theorem 3.2 and so it is omitted. □

Definition 3.4. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called pairwise point compact if the induced multifunctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are point compact.

Theorem 3.5. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be a pairwise point compact multifunction and $(Y, \triangle_1, \triangle_2)$ be PAR_2-space. Then the following statements are equivalent:
(a) F is upper δ_{ij}-continuous,
(b) For each δ_{ij}-open set $V \subseteq Y, F^-(V)$ is δ_{ij}-open set in X.
(c) For each δ_{ij}-closed set $K \subseteq Y, F_-(K)$ is δ_{ij}-closed set in X.
(d) For each $B \subseteq Y, \delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B))$.

Proof. (a)→ (b): Let V be a δ_{ij}-open set in Y and let x in X with $x \in F^-(V)$. Then $F(x) \subseteq V$. Since V is δ_{ij}-open, then for each $y \in F(x)$, there exists ij-regular open set W_y such that $y \in W_y \subseteq V$. Since $(Y, \triangle_1, \triangle_2)$ is PAR_2-space. Then there exists an $\triangle_i$-open set τ_y such that $y \in \tau_y \subseteq \triangle_j.cl(\tau_y) \subseteq \triangle_i.int(\triangle_j cl(W_y)) = W_y$. Hence we have $F(x) \subseteq \cup\{T_y : y \in F(x)\} \subseteq \cup\{\triangle_j.cl(\tau_y) : y \in F(x)\} \subseteq \cup\{W_y : y \in F(x)\} \subseteq V$. Since $F(x)$ is a $\triangle_i$-compact set, there exists points $y_1, y_2, ..., y_n \in F(x)$ such that $F(x) \subseteq \cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq U\{\triangle_j.cl(\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\} \subseteq \cup\{W_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq V$. Therefore, we obtain $F(x) \subseteq \triangle_i.int(\cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} = \cup\{\tau_{y_s} : y_s \in F(x), s = 1, 2, ..., n\} \subseteq \triangle_i.int(\triangle_j.cl(\cup\{\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\}) \subseteq V$. Put $H = \triangle_i.int((\cup\{\triangle_j.cl\{\tau_{y_s}) : y_s \in F(x), s = 1, 2, ..., n\}))$. Then H is ij-regular open set of Y with $F(x) \subseteq H$. By (a), there exists ij-regular open nbd U of x such that $U \subseteq F^-(H) \subseteq F^-(V)$. Therefore, $x \subseteq U \subseteq F^-(V)$ and this mean that $F^-(V)$ is δ_{ij}-open set in X.
(b)→ (c): Let $K \subseteq Y$ be δ_{ij}-closed set. Then $Y \backslash K$ is δ_{ij}-open set in Y. By (b) we conclude that $F^-(Y \backslash K)$ is a δ_{ij}-open set in X, so $F^-(K)$ is δ_{ij}-closed set in X.
(c)→ (a): Let x in X and let $V \subseteq Y$ be ij-regular open set of Y such that $F(x) \subseteq V$. So, $Y \backslash V$ is a δ_{ij}-closed set in Y. By (c) $F^-(Y \backslash V)$ is a δ_{ij}-closed set in X. Thus $F^-(V) = X \backslash F_-(Y \backslash V)$ is δ_{ij}-open set in X. Since $x \in F^-(V)$, there exists ij-regular open nbd U of x such that $x \in U \in F^-(V)$. Thus F is upper δ_{ij}-continuous.
(c)→ (d): Let $B \subseteq Y$. Since $B \subseteq \delta_{ij}.cl(B)$, then $F_-(B) \subseteq F_-(\delta_{ij}.cl(B))$. Since $\delta_{ij}.cl(B)$ is a δ_{ij}-closed set of Y, then by (c), $F_-(\delta_{ij}.cl(B))$ is δ_{ij}-closed set of X. Hence, we have $\delta_{ij}.cl(F_-(B)) \subseteq \delta_{ij}.cl(F_-(\delta_{ij}.cl(B))) = F_-(\delta_{ij}.cl(B))$ and so $\delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B))$.
(d)→ (c): Let B a δ_{ij}-closed set in Y. Then $F_-(B) = F_-(\delta_{ij}.cl(B))$. By (d), we have $\delta_{ij}.cl(F_-(B)) \subseteq F_-(\delta_{ij}.cl(B)) = F_-(B)$ and $F_-(B)$ is δ_{ij}-closed set in X. □

Theorem 3.6. Let $F_1 : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ and $F_2 : (Y, \triangle_1, \triangle_2) \to (Z, \Gamma_1, \Gamma_2)$ are lower δ_{ij}-continuous function then $F_2 \circ F_1 : (X, \tau_1, \tau_2) \to (Z, \Gamma_1, \Gamma_2)$ is lower δ_{ij}-continuous function.

Proof. Let K be δ_{ij}-closed set in Z. From lower δ_{ij}-continuity of F_2, we have $F_2^-(K)$ is δ_{ij}-closed set in Y. Since F_1 is lower δ_{ij}-continuous, then $F_1^-(F_2^-(K))$ is δ_{ij}-closed set in Y. But $(F_2 \circ F_1)^-(K) = F_1^-(F_2^-(K))$. Therefore $F_2 \circ F_1$ is lower δ_{ij}-continuous function.

□

Proposition 3.7. Let (X, τ_1, τ_2) be a bts, $A \subseteq X$ be τ_i-open set and $U \subseteq X$ be ij-regular open set. Then $W = A \cap U$ is ij-regular open set in $(A, \tau_{1A}, \tau_{2A})$.

Proof. It is very similar to that of Proposition 2.6 in[10].

□

Theorem 3.8. For a multifunction $F_1 : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$, the following statement are true:
(a) If F is lower(resp. upper) δ_{ij}-continuous and A is an τ_i-open set in X, then $F\mid_A$: $(A, \tau_{1|A}, \tau_{2|A}) \to (Y, \triangle_1, \triangle_2)$ is lower (resp. upper) δ_{ij}-continuous.
(b) Let $U = \{U_\alpha : \alpha \in \Omega\}$ be ij-regular open cover of X. Then a $p-$multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is lower (resp. upper) δ_{ij}-continuous if and only if the restrictions $F_\alpha = F \mid U_\alpha : (U_\alpha, \tau_{1|U_\alpha}, \tau_{2|U_\alpha}) \to (Y, \triangle_1, \triangle_2)$ are lower (resp. upper) δ_{ij}-continuous, for each $\alpha \in \Omega$.

Proof. (a): Let $x \in A$ and V be any ij-regular open set in Y with $F\mid_A (x) \cap V \neq \phi$. Hence $F(x) \cap V \neq \phi$. Since F is lower δ_{ij}-continuous, then there exists $U \in N(x, ijRO(x))$ such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Then $U \subseteq F_-$. Put $W = U \cap A$. Then W is ij-regular open set in A with $W \subseteq A \cap F_- = F\mid_A (V)$. Hence $F\mid_A (x_0) \cap V \neq \phi$, for each $x_0 \in W$. Thus $F\mid_A$ is lower δ_{ij}-continuous. The proof is the upper δ_{ij}-continuous of F is similar.
(b): Let F be lower δ_{ij}-continuous and $\alpha \in \Omega$ be such that $x \in U_\alpha$ and let V be any ij-regular open set in Y such that $F_\alpha(x) \cap V \neq \phi$. Since $F(x) = F_\alpha(x)$ and F is lower δ_{ij}-continuous, then there exists an ij-regular open nbd U_0 of x such that $F(x_0) \cap V \neq \phi$, for each $x_0 \in U_0$. Hence $U_0 \in V_0$. Put $U = U_\alpha \cap U_0$, thus U is ij-regular open subset of U_α and $x \in U$. Therefore $U = U_\alpha \cap U_0 \subseteq U_\alpha \cap F_-(V) = F_{-\alpha}(V)$. Thus F_α is lower δ_{ij}-continuous at x. Conversely, suppose that F_α is lower δ_{ij}-continuous, for each $\alpha \in \Omega$. Let $x \in X$ and V be an ij-regular open set in Y such that $F(x) \cap V \neq \phi$. Then there exists $\alpha \in \Omega$ such that $x \in U_\alpha$. Hence $F(x) = F_\alpha(x)$ and so $F_\alpha(x) \cap V \neq \phi$. Since F_α is lower δ_{ij}-continuous, there exists ij-regular open set U in U_α with $x \in U$ such that $F_\alpha(x_0) \cap V \neq \phi$, for each $x_0 \in U$. Then $U \subseteq F_\alpha(V) = F_-(V) \cap U_\alpha \subseteq F_-(V)$. Thus $F_\alpha(U) \cap V \neq \phi$ implies $U \subseteq F_{-\alpha}$, but $F_{-\alpha}(V) = F_-(V) \cap U_\alpha$. Take ij-regular open set W in X such that $U = U_\alpha \cap W$. Thus U is ij-regular open set W in X. Hence F is lower δ_{ij}-continuous.
The proof of the upper δ_{ij}-continuous of F is similar. □

4 Mutual Relationships

This section explain some of types of multifunction with some examples.

Definition 4.1. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called [15]:
(a) pairwise lower semicontinuous (p. l. s. c, for short) at a point $x \in X$ if the induced multifuctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are lower semicontinuous at a point $x \in X$.
(b) pairwise upper semicontinuous (p. u. s. c, for short) at a point $x \in X$ if the induced multifuctions $F : (X, \tau_i) \to (Y, \triangle_i), i = 1, 2$ are upper semicontinuous at a point $x \in X$.
(c) pairwise lower (resp. pairwise upper) semicontinuous if it has this property at

each point $x \in X$.
Now we give two examples in order to show that the concepts of upper (resp. lower) δ_{ij}-continuity and pairwise upper (resp. pairwise lower) semicontinuous are independent.

Example 4.2. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{2\}\}$ and $\triangle_2 = \{Y, \phi, \{3\}\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1, 2\}, F(b) = \{2, 3\}$ and $F(c) = \{1, 3\}$. Then F is pairwise lower semicontinuous multifunction but it is not lower δ_{ij}-continuous multifunction, since $\{2\} \in 12RO(Y)$ and $\{3\} \in 21RO(Y)$, but $F_-(\{2\}) = \{a, b\} \notin \delta_{12}O(X)$ and $F_-(\{3\}) = \{a, b\} \notin \delta_{21}O(X)$.

Example 4.3. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a, b\}\}, \tau_2 = \{X, \phi, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{2\}\}$ and $\triangle_2 = \{Y, \phi, \{3\}\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{2\}, F(b) = \{3\}$ and $F(c) = \{1, 2\}$. Then F is pairwise upper semicontinuous multifunction but it is not upper δ_{ij}-continuous multifunction. Indeed, $\{2\} \in 12RO(Y)$ and $\{3\} \in 21RO(Y)$, but $F^-(\{2\}) = \{a\} \notin \delta_{12}O(X)$ and $F^-(\{3\}) = \{b\} \notin \delta_{21}O(X)$.

Theorem 4.4. Ever upper (resp. lower) δ_{ij}-continuous multifunction from any bts to a PSR_2-space is p-upper (resp. p-lower) semicontinuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) δ_{ij}-continuous multifunction and $(Y, \triangle_1, \triangle_2)$ is PSR_2-space. Let $V \subseteq Y$ be $\triangle_i$-open set. Since $(Y, \triangle_1, \triangle_2)$ is PSR_2-space, then V is ij-regular open. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$ is δ_{ij}-open set in X, then $F^-(V)$ (resp. $F_-(V)$) is τ_i-open set in X. So F is p-upper (resp. p-lower) semicontinuous.

□

Theorem 4.5. Ever p-upper (resp. p-lower) semicontinuous multifunction from a PSR_2-space to any *bts*-space is upper (resp. lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be p-upper (resp. p-lower) continuous multifunction and (X, τ_1, τ_2) is PSR_2-space. Let $V \subseteq Y$ be ij-regular open, then V is $\triangle_i$-open set. By p-upper (resp. p-lower) continuity of F, $F^-(V)$ (resp. $F_-(V)$ is τ_i-open set in X. Since (X, τ_1, τ_2) is PSR_2-space, then $F^-(V)$ (resp. $F_-(V)$) is ij-regular open set in X. So F is upper (resp. lower) δ_{ij}-continuous. □

Definition 4.6. A p-multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called:
(a) lower strongly θ_{ij}-continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists $\tau_i -$open nbd U of x such that $F(x_0) \cap V \neq \phi$ for each $x_0 \in \tau_i.cl(U)$.
(b) upper strongly θ_{ij}-continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \subseteq Y$, there exists τ_i-open nbd U of x such that $F(\tau_i.cl(U)) \subseteq V$.
(c) lower (resp. upper) strongly θ_{ij}-continuous if it has this property at each point $x \in X$.

Theorem 4.7. Every upper (resp. lower) strongly θ_{ij}-continuous multifunction is upper (resp. lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) strongly θ_{ij}-continuous multifunction and $V \subseteq Y$ be ij-regular open set, then V is $\triangle_i$-open. By upper (resp. lower) strongly θ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is θ_{ij}-open set in X. Hence $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. So F is upper (resp. lower) δ_{ij}-continuous. The following example shows the converse of Theorem 4.7 is not true in general. □

Example 4.8. Let $X = \{a, b, c\}$ with $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b, c\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{1\}\}$ and $\triangle_2 = \{Y, \phi\}$. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1\}, F(b) = \{2\}$ and $F(c) = \{2, 3\}$. Then F is upper (resp. lower) δ_{ij}-continuous multifunction but it is not upper (resp. lower) strongly θ_{ij}-continuous multifunction. Indeed, $\{1\} \in \triangle_1$ but $F_-(\{1\}) = \{a\} \notin \theta_{12}O(X)$ and $F^-(\{1\}) = \{a\} \notin \theta_{12}O(X)$.
The following theorem give us the condition for converse.

Theorem 4.9. Every upper (resp.lower) δ_{ij}-continuous multifunction from a PAR_2-space is upper (resp. lower) strongly δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) θ_{ij}-continuous multifunction, (X, τ_1, τ_2) be a PAR_2-space and $(Y, \triangle_1, \triangle_2)$ be a PR_2-space. Let $V \subseteq Y$ be $\triangle_i$-open set. Since $(Y, \triangle_1, \triangle_2)$ is PR_2-space, then V is ij-regular open set. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. Since (X, τ_1, τ_2) is a PAR_2-space. Then $F^-(V)$ (resp. $F_-(V)$)is θ_{ij}-open set in X. Thus F is upper (resp. lower) strongly θ_{ij}-continuous. □

Definition 4.10. A multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ is called:
(a) pairwise lower almost continuous at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \cap V \neq \phi$, there exists τ_i-open nbd U of x such that $F(x_0) \cap \triangle_i.int(\triangle_j.cl(v)) \neq \phi$, for each $x_0 \in \tau_i.int(\tau_j.cl(U))$.
(b) Pairwise upper almost at a point x in X if and only if for every $\triangle_i$-open set V in Y with $F(x) \subseteq V$, there exists $\triangle_i$-open nbd U of x such that $F(\tau_i.int(\tau_j.cl(U)) \subseteq \triangle_i.int(\triangle_i.cl(V)))$.
(c) pairwise lower(resp. pairwise upper) continuous if it has this property at each point $x \in X$.

Theorem 4.11. Every upper (resp.lower) δ_{ij}-continuous multifunction is P- upper (resp. P-lower) almost continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be upper (resp. lower) δ_{ij}-continuous multifunction and let $V \subseteq Y$ be ij-regular open set. By upper (resp. lower) δ_{ij}-continuity of F, $F^-(V)$ (resp. $F_-(V)$) is δ_{ij}-open set in X. Thus $F^-(V)$ (resp. $F_-(V)$)is τ_i-open set in X. So F is $P-$upper (resp. $P-$lower) almost continuous.
The following examples show the converse of Theorem 4.11 is not true in general. □

Example 4.12. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{a, b\}\}, Y = \{1, 2, 3\}$, $\triangle_1 = \{Y, \phi, \{1\}, \{2, 3\}\}$ and $\triangle_2 = 2^Y$,. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = \{1, 2\}, F(b) = \{1, 3\}$ and $F(c) = \{2, 3\}$. Then F is P-lower almost continuous multifunction but it is not lower δ_{ij}-continuous multifunction. Indeed, $\{1\} \in ijRO(Y)$ but $F_-(\{1\}) = \{a, b\} \notin \delta_{ij}O(X)$.

Example 4.13. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as in Example 4.12. Define a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ as follows: $F(a) = F(b) = \{1\}$ and $F(c) = Y$. Then F is P-upper almost continuous multifunction but it is not upper δ_{ij}-continuous multifunction. Indeed, $\{1\} \in ijRO(Y)$, but $F_{-}(\{1\}) = \{a, b\} \notin \delta_{ij}O(X)$. The following theorem gives us the condition for converse.

Theorem 4.14. Every $P-$upper (resp.$P-$lower) almost continuous multifunction from a PSR_2- space to any *bts*-space is $P-$upper (resp. P-lower) δ_{ij}-continuous.

Proof. Let $F : (X, \tau_1, \tau_2) \to (Y, \triangle_1, \triangle_2)$ be $P-$upper (resp. $P-$lower) almost continuous multifunction and (X, τ_1, τ_2) is PSR_2- space . Let $V \subseteq Y$ be ij-regular open set. By $P-$upper (resp. $P-$lower) almost continuity of F, $F^{-}(V)$ (resp. $F_{-}(V)$) is τ_i-open set in X. Since (X, τ_1, τ_2) is PSR_2- space, then $F^{-}(V)$ (resp. $F_{-}(V)$)is ij-regular open set in X. So F is upper (resp. lower) δ_{ij}-continuous.
The applications of multifunctions with closed graphs, cluster (inverse cluster) set of functions, separation axioms and weak and strong forms of compactness in bitopological spaces are now under consideration and will e the subject of the next paper. □

5 Conclusion

The filed of mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Therefore, generalization of continuity is one of the most important subject in topology. On the other hand, topology plays a significant role in quantum physics, high energy physics and supersting theory [5, 6]. Thus we studies upper and lower δ_{ij}-continuous multifunctions which are some generalized continuity may have possible applications in quantum physics, high energy physics and supersting theory.

Acknowledgement

The authors are grateful for reviewers for this paper.

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