IJPAS | INTERNATIONAL JOURNAL PURE AND APPLIED SCIENCES
ISSN: 2149-0910

Vol. 10(2); 623-631 (2024)
DOI: 10.29132/ijpas.1579207

Research Article

All Triply Telescopic Numerical Semigroups with Multiplicity 12

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Received :04.11.2024

Accepted : 26.12.2024

Abstract: Numerical semigroups form a subset of non-negative integers. Of these semigroups, symmetric ones have an important place. It is of particular importance to examine and classify telescopic numerical semigroups, which form a class of symmetric numerical semigroups. Especially finding their Frobenius numbers and spaces is a problem in itself. In this study, we will examine some telescopic numerical semigroups that will contribute to the solution to this problem. Here we will characterize all telescopic numerical semigroups produced by three elements with multiplicity 12. We will also give formulas to calculate the genus, determine number and Frobenius number in these semigroups.

Keywords: telescopic numerical semigroup; frobenius; genus

Araştırma Makalesi

Katlılığı 12 Olan Bütün Üçlü Teleskopik Sayısal Yarıgruplar

Özet: Sayısal yarıgruplar, negative olmayan tam sayıların bir alt kümesini oluştururlar. Bu sayısal yarıgruplardan simetrik olanlar önemli bir yere sahiptir. Simetrik sayısal yarıgrupların bir sınıfını oluşturan teleskopik sayısal yarıgrupları incelemek ve bunları sınıflandırmak ayrı bir önem taşımaktadır. Özellikle bu sayısal yarıgrupların Frobenius sayılarını ve boşluklarını bulmak başlı başına bir problemdir. Bu çalışmada bu probleme çözüm için bir katkı sağlayacak bazı teleskopik sayısal yarıgrupları inceleyeceğiz. Burada, katlılığı 12 olan, üç elemanla üretilen tüm teleskopik sayısal yarıgrupları karakterize edeceğiz. Ayrıca bu yarıgruplarda cins, belirteç sayısı ve Frobenius sayısını hesaplamak için formüller vereceğiz.

Anahtar Kelimeler: teleskopik sayısal yarıgrup; frobenius; cins,

1. Introduction

Numerical semigroups, which emerged towards the end of the 19th century, are an important subject within Algebra and Number Theory, which has an important place in mathematics. The issue of numerical semigroups emerged when the problem known as the "Frobenius Problem" was put forward by

Sylvester [1] . The subject of numerical semigroups has very important features in commutative algebra, algebraic geometry and coding theory (For details see [2,3,4,5]).

Numerical semigroups are especially popular in the 21st century. Because in this century, the formation of numerical semigroups and their applications in other areas of mathematics can be done faster and easier with the help of computer programs. For example GAP programing [2] .

Let $\mathbb{N} = \{u \in \mathbb{Z}: u \geq 0\}$ be nonnegative integers set , where $\mathbb{Z}$ is integers set. We called $U \subseteq \mathbb{N}$ is numerical semigroup;

if (i) $0 \in U$, (ii) $\mathbb{N} \backslash U$ is finete and (iii) $u_1 + u_2 \in U$, for all $u_1, u_2 \in U$.

The smallest $0 \neq u \in U$ is called multiplicty of U, and denoted by $m(U)$. The element of the set $\mathbb{N} \backslash U$ is called gap of U, and the set of gaps of U denoted by $g(U)$. The number $Card\big(g(U)\big)$ is genus of U, and denoted by $\lambda(U)$, i.e. $\lambda(U) = Card\big(g(U)\big)$ (here,$Card(A)$ is the numbers elements of the set A). The largest element of the set $g(U)$ is called Frobenius number of U, and denoted by $\Delta(U)$. If $U = \mathbb{N}$ then $\Delta(U) = -1$. [6, 7]. Finding the Frobenius number of a numerical semigroup is difficult. However, formulas have been developed for the Frobenius number of some special numerical semigroups [8, 9].

It is known that every numerical semigroup is finitely generated. We say that U is generated by a set $Y = \{y_1, y_2, \ldots, y_s\}$ if every $x \in U$can be written as a linear combination of elements of Y such that $y_1 < y_2 < \ldots < y_s$ and. In other words,
$x = k_1 y_1 + k_2 y_2 + \ldots + k_s y_s$ where $k_1, k_2, \ldots, k_s \in \mathbb{N}$ We note that, if U is generated by set $Y = \{y_1, y_2, \ldots, y_s\}$ then we write $U =< Y >=< y_1, y_2, \ldots, y_s >$. We say that Y is the minimal system of the generators of U if no proper subset of Y generates U. In this case, we can write,
$U =< y_1, y_2, \ldots, y_s >= \{k_1 y_1 + k_1 y_1 + \ldots + k_s y_s : k_1, k_2, \ldots, k_s \in \mathbb{N}\} = \{0 = u_0, u_1, u_2, \ldots, u_{r,} \rightarrow \ldots\}$ where $u_i < u_{i+1}$ for $i = 0,1,2, \ldots, r = n(U)$, and the means arrow is $u_p \in U$ such that $p > r$ and $p \in \mathbb{N}$. If $Y = \{y_1, y_2, \ldots, y_5\}$ is the minimal system of the generator of U then we called the number of elements in Y is embedding dimension of U, denoted by $\beta(U)$. In generally, $\beta(U) \leq m(U)$ for a numerical semigroup U. If $\beta(U) = m(U)$ then U is called maximal embedding dimension (MED).

Let U be a numerical semigroup and its Frobenius number is $\Delta(U)$. Then the number $Card(\{0,1,2, \ldots, \Delta(U)\} \cap U)$ is called determine number of U, and denoted by $n(U)$ [7, 9] .

The numerical semigroup U is called symmetric if $\Delta(U) - a \in U$ for all $a \in \mathbb{Z} \backslash U$. If U is a symmetric numerical semigroup then $n(U) = \lambda(U) = \frac{\Delta(U)+1}{2}$. Moreover, it well known that every $U =< y_1, y_2 >$ numerical semigroup is symmetric and $\Delta(U) = y_1 y_2 - y_1 - y_2$ [10].

Let $U =< y_1, y_2, \ldots, y_{s-1}, y_s >$ be a numerical semigroup and $k = gcd(y_1, y_2, \ldots, y_{s-1})$. If $y_s \in < \frac{y_1}{k}, \frac{y_2}{k}, \ldots, \frac{y_{s-1}}{k} >$then U is called telescopic numerical semigroup. We note that every telescopic numerical semigroup is symmetric.$U =< y_1, y_2, y_3 >$ is called triply generated telescopic numerical semigroup if $y_3 \in < \frac{y_1}{k}, \frac{y_2}{k} >$ where $k = gcd(y_1, y_2)$ (For details see [11]).

Recently, a lot of work has been done on telescopic numerical semigroups. Kirfel and Pellikaan worked on the minimum distance of codes in a sequence coming from telescopic semigroups [12]. Garcia et al. studied Apery sets and Feng-Rao numbers on telescopic numerical semigroups [13]. İlhan showed that a numerical semigroup of the form $U =< u, u + 2,2u + 1 >$ is telescopic, where $u > 2$ is even integer [14] .

Suer and İlhan classified telescopic numerical semigroups produced with three elements and having multiples of 4,6,8,9 and 10. They derived formulas for the Frobenius number, the determine number and the genus in these classes [15, 16, 17, 18].

Wang et al. characterized all telescopic numerical semigroups with embedding dimension four and multiples of 8 and 12. Additionally, they calculated the Frobenius number and Genus in these semigroups [19].

In the numerical semigroups theory symmetric numerical semigroups ones have an important place. It is of particular importance to examine and classify telescopic numerical semigroups, which form a class of symmetric numerical semigroups. Especially finding their Frobenius numbers and spaces is a problem in itself. In this study, we will examine some telescopic numerical semigroups that will contribute to the solution to this problem. Here we will characterize all telescopic numerical semigroups produced by three elements with multiplicity 12. We will also give formulas to calculate the genus, determine number, and Frobenius number in these semigroups.

2.Triply Telescopic Numerical Semigroup with Multiplicity 12

In this section, we give characterization of all telescopic numerical semigroups with embedding dimension 3 and multiplicity 12. We obtain the number of these numerical semigroups with the following Lemma 2.2. We note that $gcd(12, x) \neq 1$ and $gcd(12, x) \neq 12$ in $U =< 12, x, y >$triply telescopic numerical semigroup.

Proposition 2.1. **([10])** Let $U =< u_1, u_2 >$ be numerical semigroup. Then,

(1) $\Delta(U) = u_1u_2 - u_1 - u_2$

(2) $\lambda(U) = \frac{u_1u_2 - u_1 - u_2 + 1}{2}$.

Lemma 2.2. Let $U =< 12,12k + a, b >$ be triply telescopic numerical semigroup, where $a, b, k \in \mathbb{N}$ and $b > 12k + a$ is odd. Then, the number triply telescopic numerical semigroup U is 7.

Proof. Let $U =< 12,12k + a, b >$ be triply telescopic numerical semigroup, where $a, b, k \in \mathbb{N}$ and $b > 12k + a$. Let's $gcd(12,12k + a) = d$. So, we write $d|12$ and $d|a$.

In this case, we have $d = 2,3,4,6$.

i) If $d = 2$ then $2|a$. Thus, we find that $a = 2,4,6,8,10$.

ii) If $d = 3$ then $3|a$. Thus, we find that $a = 3,6,9$.

iii) If $d = 4$ then $4|a$. Thus, we find that $a = 4,8$.

iv) If $d = 6$ then $6|a$. Thus, we find that $a = 6$.

Thus, we obtain $a = 2,3,4,6,8,9,10$. That is, we have 7 triply telescopic numerical semigroups.

In Lemma 2.2 the seven triply telescopic numerical semigroups families are characterized by the following theorem.

Theorem 2.3. Let U be a numerical semigroup with multiplicity 12 and embedding dimension 3. Then U is a telescopic numerical semigroup if and only if U is member of one of the following families:

$\theta_1 = \begin{Bmatrix} < 12,12u+2, C >: C = 12u+2+(2p+1), p \neq 3m+1 \; and \\ p \neq 3r, for \;\; 0 \leq m \leq 3u-1 \; and \; 0 \leq r \leq u-1, p \in \mathbb{N} \end{Bmatrix}.$

$\theta_2 = \{< 12,12u+3, C >: C = 12u+3+p, p \neq 3q \; and \; q \in \mathbb{N}\}.$

$\theta_3 = \{< 12,12u+4, C >: C = 12u+4+(2p+1), p \in \mathbb{N}\}.$

$\theta_4 = \{< 12,12u+6, C >: C = 12u+6+(2p+1), p \in \mathbb{N}\}.$

$\theta_5 = \{< 12,12u+8, C >: C = 12u+8+(2p+1), p \in \mathbb{N}\}.$

$\theta_6 = \{< 12,12u+9, C >: C = 12u+9+p, p \neq 3t \; and \; t \in \mathbb{N}\}.$

$\theta_7 = \begin{Bmatrix} < 12,12u+10, C >: C = 12u+10+(2p+1), p \neq 3m+1 \; and \\ p \neq 3r+2, for \; 0 \leq m \leq 3u+1 \; and \; 0 \leq r \leq u-1, p \in \mathbb{N} \end{Bmatrix}.$

Proof. Let $U =< 12, X, Y >$ be a telescopic numerical semigroup with multiplicity 12 and embedding dimension 3. Let's $gcd(12, X) = d$. Thus, $Y \in < \frac{12}{d}, \frac{X}{d} >$ since $U =< 12, X, Y >$. In this case, we have $d = 1,2,3,4,6,12$.

(i) If $d = 1$ then $Y \in < 12, X >$. But, it is not true since $U =< 12, X, Y >$ has embedding dimension three.

(ii) If $d = 12$ then this contradicts that $U =< 12, X, Y >$ has embedding dimension three.

(iii) If $d = 6$ then $X = 12u + 6, u \in \mathbb{Z}^+$. Thus, we write that $12 < 12u + 6 < Y$ and $gcd(12,12u+6, Y) = 1$ since $U =< 12, X, Y >$ is a numerical semigroup.

So, $gcd(12,12u+6, Y) = gcd(6, Y) = 1$, i.e. $Y = 12u + 6 + (2p + 1)$ is odd. In this case, we obtain,

$U =< 12,12u+6,12u+6+(2p+1) > \in \theta_4$.

(iv) If $d = 4$then $X = 12u + 4$ or $X = 12u + 8, u \in \mathbb{Z}^+$.

(a) Let's $X = 12u + 4, u \in \mathbb{Z}^+$.

Then, we have $12 < 12u + 4 < Y$ and $gcd(12,12u+4, Y) = 1$ since $U =< 12, X, Y >$ is a numerical semigroup. So, $gcd(12,12u+4, Y) = gcd(4, Y) = 1$, i.e. $Y = 12u + 4 + (2p + 1)$ is odd. In this case, we obtain

$U =< 12,12u+4,12u+4+(2p+1) > \in \theta_3$.

(b) Let's $X = 12u + 8, u \in \mathbb{Z}^+$. Then, we have $12 < 12u + 8 < Y$ and $gcd(12,12u+8, Y) = 1$ since $U =< 12, X, Y >$ is a numerical semigroup. So, $gcd(12,12u+8, Y) = gcd(8, Y) = 1$, i.e. $Y = 12u + 8 + (2p + 1)$ is odd. In this case, we obtain

$U =< 12,12u+8,12u+8+(2p+1) > \in \theta_5$.

(v) If $d = 3$ then $X = 12u + 3$ or $X = 12u + 9, u \in \mathbb{Z}^+$.

(a) Let's $X = 12u + 3, u \in \mathbb{Z}^+$. Then, we have $12 < 12u + 3 < Y$ and $gcd(12,12u+3, Y) = 1$ and $Y \neq 3k, k \in \mathbb{N}$ since $U =< 12, X, Y >$ is a numerical semigroup. So, $gcd(12,12u+3, Y) = gcd(3, Y) = 1$. In this case, we obtain

$U =< 12,12u+3,12u+3+p > \in \theta_2$, for $p \neq 3q, q \in \mathbb{N}$.

(b) Let's $X = 12u + 9, u \in \mathbb{Z}^+$. Then, we have $12 < 12u + 9 < Y$ and $gcd(12,12u+9, Y) = 1$ and $Y \neq 3n, n \in \mathbb{N}$ since $U =< 12, X, Y >$ is a numerical semigroup. So, $gcd(12,12u+9, Y) = gcd(9, Y) = 1$. In this case, we obtain

$U =< 12,12u+9,12u+9+p > \in \theta_6$ for $p \neq 3t, t \in \mathbb{N}$.

(vi) $d = 2$ then $X = 12u + 2$ or $X = 12u + 10, u \in \mathbb{Z}^+$.

(a) Let's $X = 12u + 2, u \in \mathbb{Z}^+$. Then, we have $12 < 12u + 2 < Y$ and $gcd(12,12u + 2, Y) = 1$ since $U =< 12, X, Y >$ is a numerical semigroup. So, $gcd(12,12u + 2, Y) = gcd(2, Y) = 1$, i.e. $Y > X$ is odd. In this case, we obtain

$Y = 12u + 2 + (2p + 1)$

Here, it is not $p = 3m + 1 or\, p = 3r$, where $0 \leq m \leq 3u - 1 and 0 \leq r \leq u - 1, p \in \mathbb{N}$. If $p = 1$ (for $m = 0$) then we have

$Y = 12u + 2 + (2 + 1) = (12u + 2) + 3 \in < 6,6u + 1 >$

namely, there exist $t, n \in \mathbb{N}$ such that $Y = (12u + 2) + 3 = 6t + (6u + 1)n$.

This is cannot. Because, it not exist $t \in \mathbb{N}$ such that $3 = 6t$.

If $p = 0$ (for $r = 0, i.e. for\ u = 1$) then we have,

$Y = 12u + 2 + 1 = (12u + 2) + 1 = 15 \in < 6,6u + 1 > = < 6,7 >$.

But, it is not true. Because, not there exist $q, v \in \mathbb{N}$ such that $Y = 15 = 6q + 7v$.

Thus, it must $p \neq 3m + 1\ or\ p \neq 3r$, where $0 \leq m \leq 3u - 1\ and\ 0 \leq r \leq u - 1, p \in \mathbb{N}$. Finally, we find that, $U =< 12,12u + 2,12u + 2 + (2p + 1) > \in \theta_1$.

(b) Let's $X = 12u + 10, u \in \mathbb{Z}^+$. Then, we have $12 < 12u + 10 < Y$ and gcd $(12,12u + 10, Y) = 1$ since gcd $(12,12u + 10, Y) = 1$ is a numerical semigroup So, $gcd(12,12u + 10, Y) = gcd(10, Y) = 1$, i.e. $Y > X$ is odd.

In this case, we write $Y = 12u + 10 + (2p + 1)$.

Here, it is not $p = 3m + 1\ or\ p = 3r + 2$, where $0 \leq m \leq 3u + 1\ and\ 0 \leq r \leq u - 1, p \in \mathbb{N}$.

If $p = 1$ (for $m = 0$) then we have $Y = 12u + 10 + (2 + 1) = (12u + 10) + 3 \in < 6,6u + 1 >$ namely, there exist $t, n \in \mathbb{N}$ such that $Y = (12u + 10) + 3 = (12u + 12) + 1 = 6t + (6u + 1)n$. So, we find that $t = 2, n = 2$ from this equailty. But, this is not true. Because, the left side of this equation is odd but the right side is even.

ıf $p = 3r + 2$ then we have, $Y = 12u + 10 + (2(3r + 2) + 1) = (12u + 6r) + 15 = 6(2u + r + 2) + 3 \in < 6,6u + 1 >$. But, it is impossible. Because, there is not exist $q, v \in \mathbb{N}$ such that, $Y = 6(2u + r + 2) + 3 = 6q + (6u + 1)v$. Thus, it must $p \neq 3m + 1\ or\ p \neq 3r + 2$, where $0 \leq m \leq 3u + 1\ and\ 0 \leq r \leq u - 1, p \in \mathbb{N}$.

Finally, we find that we obtain

$U =< 12,12u + 10,12u + 10 + (2p + 1) > \in \theta_7$.

On the contrary, let $U \in \theta_i (for 1 \leq i \leq 7)$ be the numerical semigroup in the Theorem.

(i) If $U =< 12,12u + 2, C >$ then $gcd(12,12u + 2) = 2$.

So,we write,

$C = 12u + 2 + (2p + 1) = 12u + 2 + 2(3m - 1) + 1 = 12u + 6m + 1 \in < \frac{12}{2}, \frac{12u+2}{2} > = < 6,6u + 1 >$.

That is,

$U =< 12,12u + 2, C > \in \theta_1$ is telescopic numerical semigroup,

where

$C = 12u + 2 + (2p + 1), p \neq 3m + 1\ and\ p \neq 3r, for\, 0 \leq m \leq 3u - 1\ and\ 0 \leq r \leq u - 1, p \in \mathbb{N}$.

(ii) If $U =< 12,12u + 3, C >$ then $gcd(12,12u + 3) = 3$.

So, we write, $C = 12u + 3 + p, p \neq 3q$ *and* $q \in \mathbb{N}$. In this case

$p = 3q + 1$ *or* $p = 3q + 2, q \in \mathbb{N}$.

(a) If $q = 0$

then we write,

$C = 12u + 3 + 1 = 4(3u + 1) \in < 4, 4u + 1 >$ or

$C = 12u + 3 + 2 = 1.(4u + 1) + 4.(2u + 1) \in < 4, 4u + 1 >$

That is,

$U = < 12, 12u + 3, C > \in \theta_2$ is telescopic numerical semigroup.

(b) If $q = 1$ then,

$C = 12u + 3 + 4 = 3(4u + 1) + 1.4 \in < 4, 4u + 1 >$ or

$C = 12u + 3 + 5 = 4(3u + 2) \in < 4, 4u + 1 >$

That is, $U = < 12, 12u + 3, C > \in \theta_2$ is telescopic numerical semigroup.

If we continue like this, we see that the numerical semigroup

$U = < 12, 12u + 3, C > \in \theta_2$ is telescopic.

(iii) If $U = < 12, 12u + 4, C >$ where $C = 12u + 4 + (2p + 1), p \in \mathbb{N}$.

Then, we find that $C = 12u + 4 + (2p + 1) \in < 3, 3u + 1 >$.

Because,

(a) If $p = 0$ then $C = 12u + 4 + 1 = 2(3u + 1) + 3(2u + 1) \in < 3, 3u + 1 >$;

(b) If $p = 1$ then $C = 12u + 4 + 3 = 4(3u + 1) + 1.3 \in < 3, 3u + 1 >$;

(c) If $p = 2$ then $C = 12u + 4 + 5 = 3(4u + 3) \in < 3, 3u + 1 >$;

…

If we continue like this, we see that $U = < 12, 12u + 4, C > \in \theta_3$ is a telescopic numerical semigroup. The remaining classes of numerical semigroups can be seen in a similar way to be telescopic.

We will obtain the Frobenius number, genus and determine number of each of the telescopic numerical semigroup classes in the theorem given above with the help of the following Lemma 2.4.

Lemma 2.4. ([1]) Let $U = < y_1, y_2, \ldots, y_s >$ be a numerical semigroup and $k = gcd(y_1, y_2, \ldots, y_{s-1})$.

If $W = < \frac{y_1}{k}, \frac{y_2}{k}, \ldots, \frac{y_{s-1}}{k}, y_s >$ then

(**1**) $\Delta(U) = k\Delta(W) + (k - 1)y_s$.

(**2**) $\lambda(U) = k\lambda(W) + \frac{(k-1)(y_s-1)}{2}$.

Proposition 2.5. Let the numerical semigroup U belong to one of the classes given in Theorem 2.3. In this case,

(1) If $U \in \theta_1$ then $\Delta(U) = 60u + C - 2$ and $\lambda(U) = 30u + \frac{C-1}{2}$.

(2) If $U \in \theta_2$ then $\Delta(U) = 36u + 2C - 3$ and $\lambda(U) = 18u + C - 1$.

(3) If $U \in \theta_3$ then $\Delta(U) = 24u + 3C - 4$ and $\lambda(U) = 12u + \frac{3(C-1)}{2}$.

(4) If $U \in \theta_4$ then $\Delta(U) = 12u + 5C - 6$ and $\lambda(U) = 6u + \frac{5(C-1)}{2}$.

(5) If $U \in \theta_5$ then $\Delta(U) = 24u + 3C + 4$ and $\lambda(U) = 12u + \frac{3C+5}{2}$.

(6) If $U \in \theta_6$ then $\Delta(U) = 36u + 2C + 15$ and $\lambda(U) = 18u + C + 8$.

(7) If $U \in \theta_7$ then $\Delta(U) = 60u + C + 38$ and $\lambda(U) = 30u + \frac{C+39}{2}$.

Proof. Let $U \in \theta_1$**.** Then $U =< 12,12u + 2, C >$ and $gcd(12,12u + 2) = 2$ where, $C = 12u + 2 + (2p + 1), p \neq 3m + 1 \; and \; p \neq 3r, for 0 \leq m \leq 3u - 1 \; and$ $0 \leq r \leq u - 1, p \in \mathbb{N}$.

So, $W =< \frac{12}{2}, \frac{12u+2}{2} >=< 6,6u + 1 >$

and we find that,

$\Delta(W) = 6(6u + 1) - 6 - 6u - 1 = 30u - 1$ from Proposition 2.1. Thus, we obtain
$\Delta(U) = 2\Delta(W) + C = 2(30u - 1) + C = 60u + C - 2$ and

$$\lambda(U) = 2\lambda(W) + \frac{(2-1)(C-1)}{2} = 2(15u) + \frac{C-1}{2} = 30u + \frac{C-1}{2}$$

from Lemma 2.4. Other cases in the Proposition can be proven in a similar way.

Example 2.6. We put $u = 1$ and $C = 20$ in $U =< 12,12u + 3, C >\in \theta_2$.

Then,

$$U =< 12,15,20 >= \begin{Bmatrix} 0,12,15,20,24,27,30,32,35,36,39,40,42,44,45,47,48,50,51,52,54,55,56,57, \\ 59,60,62,63,64,65,66,67,68,69,70,71,72,74, \rightarrow \ldots \end{Bmatrix}$$

is a telescopic numerical semigroup since $gcd(12,15) = 3$ and $20 \in< \frac{12}{3}, \frac{15}{3} >=< 4,5 >$.

This, we find the Frobenius number and genus of U is $\Delta(U) = 36.1 + 2.20 - 3 = 73$ and $\lambda(U) = 18.1 + 20 - 1 = 37$ from Proposition 2.5/ (2).

We write, $\beta(U) = 3, m(U) = 12, n(U) = \lambda(U) = \frac{\Delta(U)+1}{2} = 37$,

and we find that U not MED since $\beta(U) \neq m(U)$.

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