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Perturbative construction of the two-dimensional O(N) nonlinear sigma model with ERG

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Abstract: We use the exact renormalization group (ERG) perturbatively to construct the Wilson action for the two-dimensional O(N) nonlinear sigma model (NLSM). The construction amounts to regularization of a nonlinear symmetry with a momentum cutoff. We find out that the model is parameterized by three functions. We show how to tune them by imposing the Ward–Takahashi (WT) identity. We construct two composite operators that generate infinitesimal change of the coupling constant and the renormalization of the scalar fields. Finally we show how the beta functions and the anomalous dimensions arise in the model up to 1-loop.

Key words: Exact renormalization group, nonlinear sigma models

1. Introduction

There has been a considerable level of interest in quantum field theories in two dimensions. It is shown that the two-dimensional nonlinear sigma model (NLSM) has many features in common with nonabelian gauge theories in four dimensions, such as geometrical significance, asymptotic freedom, mass gap in a nonperturbative spectrum, renormalizability, dynamical generation of vector bosons, existence of topologically nontrivial field configurations (solitons and instantons), and $1/N$ quantum perturbation theory [1–6]. More recently the NLSM has become one of the major ingredients of string theory quantization and related subjects. It is shown that the solutions to some basic general formulation of string theory may be regarded as given by conformally invariant two-dimensional NLSMs [7–9].

Technically, the NLSM has geometrical significance since the action is invariant under the infinitesimal field reparametrizations. In other words, two NLSMs are physically equivalent when they are related by a field redefinition alone. Moreover, its scalar fields, metric, and, hence, all of its coupling constants are dimensionless. Therefore, the action consists of an infinite number of interactions. In order to construct a compact model, some constraints may be imposed on the model. For instance, $O(N)$ NLSM is defined classically by the action in two-dimensional Euclidean space such as

$$S_{cl} = -\frac{1}{2g}\int d^2x \sum_{I=1}^{N} \partial_\mu \Phi_I \partial_\mu \Phi_I, \tag{1}$$

where the real scalar fields are restricted by the nonlinear constraints $\sum_{I=1}^{N} \Phi_I\Phi_I = 1$. Regarding the model as

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a classical spin system, g plays the role of the temperature; large g encourages fluctuations of the fields, while small g discourages them.

The asymptotic freedom of the model, first shown in [10], implies not only the validity of perturbation theory at short distances but also the generation of a mass gap due to large field fluctuations at long distances.

Dimensional regularization [11, 12] is used to show the perturbative renormalization of the model [13]. Its advantage is manifest O(N) invariance, but an external magnetic field (mass term) must be introduced to avoid IR divergences.

The purpose of this paper is to apply the method of the exact renormalization group (ERG) to renormalize the model consistently with a finite momentum cutoff Λ. This comes with a price: we must keep an increasing number of terms in the Wilson action as we go to higher orders in perturbation theory. The Wilson action does not describe the physics of low momentum $p < \Lambda$ only, it contains the physics of all momentum scales. A review article on the fundamentals of the ERG is given recently [14].

Compared with dimensional regularization, the regularization with a momentum cutoff is physically more appealing, but it is technically more complicated; the O(N) invariance is not manifest, and a naïve sharp momentum cutoff, inconsistent with shifts of loop momenta, cannot be used beyond 1-loop.

We can overcome the technical difficulties using the formulation of field theory via ERG differential equations [15, 16]. For a general perturbative construction of theories with continuous symmetry, we refer the reader to a review article [17], and in this paper we give only the minimum background necessary for our purposes. ERG was applied to the two-dimensional O(N) nonlinear σ model by various authors [18, 19]; here we aim to simplify and complete Becchi's analysis. In particular, we give a perturbative algorithm for constructing the Wilson action of the model with a finite momentum cutoff Λ. The Wilson action results from an integration of fields with momenta larger than Λ, and it is free from IR divergences without an external magnetic field. Hence, we do not need to introduce a mass term to break the symmetry explicitly to O(N-1).

2. Materials and methods

Before giving details of the inductive construction we would like to emphasize that throughout the paper we use the Euclidean metric and the following notation for momentum integrals:

$$\int_p \equiv \int \frac{d^2 p}{(2\pi)^2}. \tag{2}$$

2.1. Momentum cutoff

We regularize the model using a UV momentum cutoff Λ_0. The bare action is given by

$$S_B = -\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0)}\phi_i(-p)\phi_i(p) + S_{I,\ B}, \tag{3}$$

where the subscript i, running from 1 to $N-1$, is summed over. The interaction part is given by

$$S_{I,\ B} \quad = \quad \int d^2x \left[\Lambda_0^2\, z_0\left(\phi^2/2\right) + z_1\left(\phi^2/2\right)\, (-\partial^2)\frac{1}{2}\phi^2 + z_2\left(\phi^2/2\right)\, \phi_i(-\partial^2)\phi_i\right], \tag{4}$$

where we denote $\phi^2 = \phi_i\phi_i$. z_0, z_1, z_2 are functions of $\phi^2/2$ and depend logarithmically on the cutoff Λ_0. $S_{I,\ B}$ is the most general interaction action allowed by the manifest O(N$-$1) invariance and perturbative renormalizability in the absence of any dimensionful parameters.

The propagator, given by the free part of (3), is proportional to the smooth cutoff function $K(p/\Lambda_0)$. By choosing $K(x)$ with the following properties:

1. $K(x)$ is a positive and nonincreasing function of x^2,
2. $K(x) = 1$ for $x^2 < 1$,
3. $K(x)$ damps rapidly (faster than $1/x^2$) as $x^2 \to \infty$,

we can regularize the UV divergences of the model.

The renormalization functions z_0, z_1, and z_2 must be fine tuned, first for renormalizability and then for the O(N) invariance.

2.2. Wilson action

The Wilson action with a finite momentum cutoff Λ has two parts:

$$S_\Lambda \equiv S_{F,\Lambda} + S_{I,\Lambda}. \tag{5}$$

The free part

$$S_{F,\Lambda} \equiv -\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda)}\phi_i(-p)\phi_i(p), \tag{6}$$

gives the propagator with a finite momentum cutoff Λ:

$$\langle \phi_i(p)\phi_j(-p)\rangle_{S_{F,\Lambda}} = \delta_{ij}\frac{K(p/\Lambda)}{p^2}. \tag{7}$$

The interaction part of the Wilson action is defined by

$$\begin{aligned} \exp\left[S_{I,\Lambda}[\phi]\right] &\equiv \int [d\phi'] \times \exp\left[-\frac{1}{2}\int_p \frac{p^2}{K(p/\Lambda_0) - K(p/\Lambda)}\phi'_i(-p)\phi'_i(p) + S_{I,\ B}[\phi+\phi']\right] \\ &= \exp\left[\frac{1}{2}\int_p \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2}\frac{\delta^2}{\delta\phi_i(p)\delta\phi_i(-p)}\right] \times \exp\left[S_{I,\ B}[\phi]\right]. \end{aligned} \tag{8}$$

Alternatively, we can define $S_{I,\Lambda}$ by the differential equation [15, 16]

$$-\Lambda\frac{\partial}{\partial\Lambda}S_{I,\Lambda} = \frac{1}{2}\int_p \frac{\Delta(p/\Lambda)}{p^2} \times \left\{\frac{\delta S_{I,\Lambda}}{\delta\phi_i(-p)}\frac{\delta S_{I,\Lambda}}{\delta\phi_i(p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta\phi_i(-p)\delta\phi_i(p)}\right\}, \tag{9}$$

and the initial condition

$$S_{I,\Lambda}\Big|_{\Lambda=\Lambda_0} = S_{I,\ B}. \tag{10}$$

For a fixed Λ, we expand $S_{I,\Lambda}$ up to two derivatives to obtain

$$S_{I,\Lambda} = \int d^2x \left[\Lambda^2\, a\left(\ln\Lambda/\mu; \phi^2/2\right) + A\left(\ln\Lambda/\mu; \phi^2/2\right)\,(-\partial^2)\frac{1}{2}\phi^2 + B\left(\ln\Lambda/\mu; \phi^2/2\right)\,\phi_i(-\partial^2)\phi_i\right] + \cdots \tag{11}$$

where the dotted part contains four or more derivatives. a, A, B are functions of $\phi^2/2$, and they can be expanded as

$$\left\{ \begin{array}{rcl} a\left(\ln\Lambda/\mu;\phi^2/2\right) & = & \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\phi^2}{2}\right)^n a_n(\ln\Lambda/\mu) \\ A\left(\ln\Lambda/\mu;\phi^2/2\right) & = & \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\phi^2}{2}\right)^n A_n(\ln\Lambda/\mu) \\ B\left(\ln\Lambda/\mu;\phi^2/2\right) & = & \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\phi^2}{2}\right)^n B_n(\ln\Lambda/\mu) \end{array} \right. \tag{12}$$

The Taylor coefficients depend logarithmically on the cutoff Λ. We have chosen the ratio of Λ to an arbitrary renormalization scale μ as the argument of the logarithm. The initial condition (10) gives

$$\left\{ \begin{array}{rcl} a(\ln\Lambda_0/\mu;\phi^2/2) & = & z_0(\phi^2/2) \\ A(\ln\Lambda_0/\mu;\phi^2/2) & = & z_1(\phi^2/2) \\ B(\ln\Lambda_0/\mu;\phi^2/2) & = & z_2(\phi^2/2) \end{array} \right. \tag{13}$$

The renormalization functions z_0, z_1, and z_2 are determined so that

$$\lim_{\Lambda_0\to\infty} S_{I,\Lambda} \tag{14}$$

exists for any finite Λ. Using the BPHZ renormalization scheme adapted to the Wilson action [20–22], we can choose $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$ as any functions. As will be explained in the next section, the O(N) invariance constrains the choice of $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$.

Alternatively, we can construct the continuum limit (14) directly without starting from a bare action. We demand that the dotted part of (11) is multiplied by the inverse powers of Λ. For given $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$, the ERG differential equation (9) uniquely determines $a\left(\ln\Lambda/\mu;\phi^2/2\right)$ and the dotted part of (11) [20–22]. This is the preferred approach we adopt in the rest of the paper. In Appendix **??**, we summarize the basic properties of the correlation functions calculated with S_Λ.

2.3. WT identity for O(N)

The Wilson action is determined uniquely in terms of $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$. For the O(N) symmetry, we must choose $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$ appropriately. In this and the following two sections, we aim to complete the analysis of Becchi given in sect. 6 of [19].

The Wilson action has manifest O(N$-$1) invariance. To insure the full O(N) invariance, we must demand the invariance of the action under the following infinitesimal transformation:

$$\delta\phi_i(p) = K\left(p/\Lambda\right)\epsilon_i[\Phi_N](p), \tag{15}$$

where ϵ_i is an infinitesimal constant and $[\Phi_N]$ is the composite operator for the N-th component of the O(N) vector, whose i-th component is proportional to ϕ_i.

The composite operators, the concept of which was first introduced in sect. 5 of [19], can be considered as infinitesimal deformations of the Wilson action, and they satisfy the same linear ERG differential equation as (16). A composite operator vanishes identically if the leading part in the derivative expansion, the part multiplied by the nonnegative powers of Λ, vanishes. For more details, see sect. 4 of [17].

More precisely, $[\Phi_N]$ is defined by the ERG differential equation

$$-\Lambda\frac{\partial}{\partial\Lambda}[\Phi_N](p) = \int_q \frac{\Delta(q/\Lambda)}{q^2}\left\{\frac{\delta S_{I,\Lambda}}{\delta\phi_i(q)}\frac{\delta}{\delta\phi_i(q)} + \frac{1}{2}\frac{\delta^2}{\delta\phi_i(q)\delta\phi_i(-q)}\right\}[\Phi_N](p), \tag{16}$$

and the derivative expansion

$$\int_p \exp^{ipx}[\Phi_N](p) = P\left(\ln\Lambda/\mu; \phi(x)^2/2\right) + \cdots, \tag{17}$$

where the dotted part, proportional to the inverse powers of Λ, contains derivatives of $\phi_i(x)$. $[\Phi_N]$ is parameterized by a function

$$P\left(0; \phi^2/2\right), \tag{18}$$

which is arbitrary as far as perturbative renormalizability of $[\Phi_N]$ is concerned.

Following Becchi [19], we now define the WT composite operator for (15) by

$$\begin{aligned}\Sigma_\Lambda &\equiv \int_p\left[\frac{\delta S_\Lambda}{\delta\phi_i(p)}\delta\phi_i(p) + \frac{\delta}{\delta\phi_i(p)}\delta\phi_i(p)\right]\\ &= \epsilon_i\int_p K(p/\Lambda)\left[\frac{\delta S_\Lambda}{\delta\phi_i(p)}[\Phi_N](p) + \frac{\delta[\Phi_N](p)}{\delta\phi_i(p)}\right],\end{aligned} \tag{19}$$

This satisfies the same ERG linear differential equation as (16). The WT identity

$$\Sigma_\Lambda = 0, \tag{20}$$

is the "quantum" invariance of the Wilson action under (15), whereby the nontrivial jacobian of (15) is taken into account. Concrete loop calculations show that the coefficient function $a(\ln\Lambda/\mu;\phi^2/2)$, corresponding to the quadratically divergent potential in the bare action, is nonvanishing. However, its noninvariance under (15) is canceled by the jacobian.

Taking the correlation of Σ_Λ with the elementary fields, we obtain the usual WT identity from (20):

$$\sum_{j=1}^n \epsilon_{i_j}\left\langle\phi_{i_1}(p_1)\cdots\Phi_N(p_j)\cdots\phi_{i_n}(p_n)\right\rangle^\infty = 0, \tag{21}$$

where the renormalized correlation function

$$\left\langle\phi_{i_1}(p_1)\cdots\Phi_N(p_j)\cdots\phi_{i_n}(p_n)\right\rangle^\infty \equiv \prod_{k\neq j}\frac{1}{K(p_k/\Lambda)} \times \left\langle\phi_{i_1}(p_1)\cdots[\Phi_N](p_j)\cdots\phi_{i_n}(p_n)\right\rangle_{S_\Lambda} \tag{22}$$

is independent of the cutoff Λ. (This Λ independence is a consequence of the differential equations (9, 16). See, for example, section 4.1 of [17] for more explanations.)

3. Results

The O(N-1) invariant action, S_Λ, is parameterized by two functions: $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$. In order to insure full O(N) symmetry, we find that the transformation is parameterized by another function, $P(0;\phi^2/2)$. Hence we must fine tune not only $A(0;\phi^2/2)$ and $B(0;\phi^2/2)$ but also $P(0;\phi^2/2)$. In the next two subsections, we will show the possibility of such fine tuning.

3.1. Tree level

We expand $S_\Lambda, S_{I,\Lambda}$, etc. in the number of loops. We use a superscript $^{(l)}$ to denote the l-loop level:

$$S_\Lambda = \sum_{l=0}^{\infty} S_\Lambda^{(l)}, \tag{23a}$$

$$[\Phi_N](p) = \sum_{l=0}^{\infty} [\Phi_N]^{(l)}(p), \tag{23b}$$

$$\Sigma_\Lambda = \sum_{l=0}^{\infty} \Sigma_\Lambda^{(l)}, \tag{23c}$$

$$A\left(\ln \Lambda/\mu; \phi^2/2\right) = \sum_{l=0}^{\infty} A^{(l)}\left(\ln \Lambda/\mu; \phi^2/2\right), \cdots, \tag{23d}$$

$$A_i\left(\ln \Lambda/\mu\right) = \sum_{l=0}^{\infty} A_i^{(l)}\left(\ln \Lambda/\mu\right), \cdots, \tag{23e}$$

$$A\left(0; \phi^2/2\right) = \sum_{l=0}^{\infty} A^{(l)}\left(\phi^2/2\right), \cdots. \tag{23f}$$

In this section, we show how to tune the three parameter functions

$$A^{(0)}\left(\phi^2/2\right), \quad B^{(0)}\left(\phi^2/2\right), \quad P^{(0)}\left(\phi^2/2\right) \tag{24}$$

to satisfy the WT identity at tree level, $\Sigma_\Lambda^{(0)} = 0$.

The leading part of the derivative expansion of $S_\Lambda^{(0)}$ is given by the classical action:

$$S_\Lambda^{(0)} = S_{cl} + \cdots, \tag{25}$$

S_{cl} is independent of Λ and we can write

$$S_{cl} = \int d^2x \left[-\frac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i + A^{(0)}\left(\phi^2/2\right)(-\partial^2)\frac{\phi^2}{2} + B^{(0)}\left(\phi^2/2\right)\phi_i(-\partial^2)\phi_i \right]. \tag{26}$$

Likewise, the derivative expansion of $[\Phi_N]^{(0)}$ gives

$$\int_p \exp^{ipx}[\Phi_N]^{(0)}(p) = P^{(0)}\left(\phi^2/2\right) + \cdots. \tag{27}$$

There is no quadratic divergence at tree level

$$a^{(0)}\left(\ln \Lambda/\mu; \phi^2/2\right) = 0. \tag{28}$$

As a convention, we can choose

$$A_0^{(0)} = B_0^{(0)} = 0, \quad P_0^{(0)} = 1. \tag{29}$$

At tree level, the WT identity gives

$$\Sigma_\Lambda^{(0)} \equiv \epsilon_i \int_p K\left(p/\Lambda\right) \frac{\delta S_\Lambda^{(0)}}{\delta \phi_i(p)} [\Phi_N]^{(0)}(p) = 0. \tag{30}$$

The derivative expansion gives

$$\Sigma_{cl} \equiv \epsilon_i \int d^2x\, \frac{\delta S_{cl}}{\delta \phi_i(x)} P^{(0)}\left(\phi(x)^2/2\right) = 0. \tag{31}$$

Substituting (26) into the above, we obtain

$$\begin{aligned}\Sigma_{cl} = \epsilon_i \int d^2x\, \phi_i \Big[\quad & \partial_\mu \phi_j \partial_\mu \phi_j \left\{ P^{(0)'} - (2A^{(0)'} + B^{(0)'}) P^{(0)} - 2P^{(0)'} B^{(0)} \right\} \\ & + \phi_j \partial^2 \phi_j \left\{ P^{(0)'} - 2(A^{(0)'} + B^{(0)'}) P^{(0)} - 2P^{(0)'} B^{(0)} \right\} \\ & + (\phi_j \partial_\mu \phi_j)^2 \left\{ (1 - 2B^{(0)}) P^{(0)''} - (A^{(0)''} + B^{(0)''}) P^{(0)} - 2B^{(0)'} P^{(0)'} \right\} \Big],\end{aligned} \tag{32}$$

where the prime denotes a derivative with respect to $\phi^2/2$. For Σ_{cl} to vanish, we must satisfy the following three equations:

$$(1 - 2B^{(0)}) P^{(0)'} - (2A^{(0)'} + B^{(0)'}) P^{(0)} = 0, \tag{33a}$$

$$(1 - 2B^{(0)}) P^{(0)'} - 2(A^{(0)'} + B^{(0)'}) P^{(0)} = 0, \tag{33b}$$

$$(1 - 2B^{(0)}) P^{(0)''} - 2B^{(0)'} P^{(0)'} - (A^{(0)''} + B^{(0)''}) P^{(0)} = 0. \tag{33c}$$

From (33a) and (33b), we get

$$B^{(0)'}(x) P^{(0)}(x) = 0, \tag{34}$$

where we write $x \equiv \phi^2/2$ for short. Since $P^{(0)}(x) \neq 0$, we obtain $B^{(0)'}(x) = 0$; hence using (29) we obtain

$$B^{(0)}(x) = 0. \tag{35}$$

Thus, (33a) gives

$$P^{(0)'}(x) = 2A^{(0)'}(x) P^{(0)}(x). \tag{36}$$

Using (29), we obtain

$$P^{(0)}(x) = \exp\left[2A^{(0)}(x)\right]. \tag{37}$$

Finally, (33c) gives

$$P^{(0)''}(x) = A^{(0)''}(x) P^{(0)}(x). \tag{38}$$

This is solved by

$$A^{(0)}(x) = \frac{1}{4} \ln\left(1 - 2c\, x\right), \tag{39}$$

where c is an arbitrary constant. Hence, we obtain

$$P^{(0)}(x) = \sqrt{1 - 2cx}. \tag{40}$$

The constant c may be chosen either positive or negative. If we choose a positive $c = g > 0$, then we obtain

$$\Phi_N = \sqrt{1 - g\phi^2}, \tag{41}$$

appropriate for the classical O(N) nonlinear σ model. If we choose a negative $c = -g < 0$ instead, we obtain

$$\Phi_N = \sqrt{1 + g\phi^2}, \tag{42}$$

appropriate for the classical O(N−1, 1) nonlinear σ model. We make the first choice.

To summarize, we have obtained

$$\begin{cases} A^{(0)}(x) &= \frac{1}{4}\ln(1 - 2gx) \\ B^{(0)}(x) &= 0 \\ P^{(0)}(x) &= \sqrt{1 - 2gx}, \end{cases} \tag{43}$$

where g is an arbitrary **positive** coupling constant. Although we did not predict a coupling constant to the model in the Wilson action, it is automatically produced by the algorithm. The corresponding classical action is given by the familiar expression

$$S_{cl} = -\frac{1}{2g}\int d^2x \left[g\partial_\mu\phi_i\partial_\mu\phi_i + \partial_\mu\sqrt{1 - g\phi^2}\cdot\partial_\mu\sqrt{1 - g\phi^2}\right]. \tag{44}$$

3.2. Loop levels

Let us now assume that we have determined S_Λ and $[\Phi_N]$ up to l-loop level ($l \geq 0$) such that

$$\Sigma_\Lambda^{(0)} = \cdots = \Sigma_\Lambda^{(l)} = 0. \tag{45}$$

Under this induction hypothesis, we wish to determine $S_\Lambda^{(l+1)}$ and $[\Phi_N]^{(l+1)}$ (or equivalently $A^{(l+1)}(x)$, $B^{(l+1)}(x)$ and $P^{(l+1)}(x)$) such that

$$\Sigma_\Lambda^{(l+1)} = 0. \tag{46}$$

Note that Σ_Λ is a composite operator, satisfying the same ERG differential equation as (16). Applying the loop expansion and using the induction hypothesis, we find

$$-\Lambda\frac{\partial}{\partial\Lambda}\Sigma_\Lambda^{(l+1)} = \int_p \frac{\Delta(p/\Lambda)}{p^2}\frac{\delta S_{I,\Lambda}^{(0)}}{\delta\phi_i(-p)}\frac{\delta\Sigma_\Lambda^{(l+1)}}{\delta\phi_i(p)}. \tag{47}$$

Denoting the leading part of the derivative expansion of $\Sigma_\Lambda^{(l+1)}$ as $\tilde{\Sigma}^{(l+1)}$, we obtain

$$-\Lambda\frac{\partial}{\partial\Lambda}\tilde{\Sigma}^{(l+1)} = 0, \tag{48}$$

since $\Delta(p/\Lambda) = 0$ for $p^2 < \Lambda^2$. Hence, $\tilde{\Sigma}^{(l+1)}$ is independent of Λ. Thus, we obtain

$$\tilde{\Sigma}^{(l+1)}[\phi] = \epsilon_i \int d^2x\, \phi_i \left[\partial_\mu \phi_j \partial_\mu \phi_j \cdot s_1\left(\phi^2/2\right) + \phi_j \partial^2 \phi_j \cdot s_2\left(\phi^2/2\right) + (\phi_j \partial_\mu \phi_j)^2 \cdot s_3\left(\phi^2/2\right)\right], \tag{49}$$

where $s_i(\phi^2/2)\,(i = 1, 2, 3)$ are functions of $\phi^2/2$, independent of $\ln \Lambda/\mu$.

The definition (20) of Σ_Λ gives the decomposition

$$\Sigma_\Lambda^{(l+1)} = \Sigma_\Lambda^{(l+1),t} + \Sigma_\Lambda^{(l+1),u}, \tag{50}$$

where

$$\Sigma_\Lambda^{(l+1),t} = \epsilon_i \int_p K(p/\Lambda) \left[\frac{\delta S_\Lambda^{(l+1)}}{\delta\phi_i(p)}[\Phi_N]^{(0)}(p) + \frac{\delta S_\Lambda^{(0)}}{\delta\phi_i(p)}[\Phi_N]^{(l+1)}(p)\right], \tag{51}$$

$$\Sigma_\Lambda^{(l+1),u} = \epsilon_i \int_p K(p/\Lambda) \left[\sum_{k=1}^{l} \frac{\delta S_\Lambda^{(k)}}{\delta\phi_i(p)}[\Phi_N]^{(l+1-k)}(p) + \frac{\delta[\Phi_N]^{(l)}(p)}{\delta\phi_i(p)}\right]. \tag{52}$$

Only $\Sigma_\Lambda^{(l+1),t}$ depends on $A^{(l+1)}(x), B^{(l+1)}(x), and P^{(l+1)}(x)$, and $\Sigma_\Lambda^{(l+1),u}$ are determined by S_Λ and $[\Phi_N]$ up to l-loop. Therefore, the functions $s_i(x)$ are given as the sum

$$s_i(x) = t_i(x) + u_i(x) \quad (i = 1, 2, 3), \tag{53}$$

where $t_i(x)$ are linear in $A^{(l+1)}(x), B^{(l+1)}(x), and P^{(l+1)}(x)$, and $u_i(x)$ are determined by the lower loop functions. We obtain explicitly

$$t_1(x) = P^{(l+1)'} - (2A^{(l+1)'} + B^{(l+1)'})P^{(0)} - 2A^{(0)'}P^{(l+1)} - 2P^{(0)'}B^{(l+1)}, \tag{54}$$

$$t_2(x) = P^{(l+1)'} - 2(A^{(l+1)'} + B^{(l+1)'})P^{(0)} - 2A^{(0)'}P^{(l+1)} - 2P^{(0)'}B^{(l+1)}, \tag{55}$$

$$t_3(x) = P^{(l+1)''} - (A^{(l+1)''} + B^{(l+1)''})P^{(0)} - A^{(0)''}P^{(l+1)} - 2P^{(0)''}B^{(l+1)} - 2B^{(l+1)'}P^{(0)'}. \tag{56}$$

There is no relation among the $t(x)$'s. Thus, whatever $u(x)$'s are, we can solve the equations

$$s_i(x) = t_i(x) + u_i(x) = 0 \quad (i = 1, 2, 3). \tag{57}$$

Using (43), the solution is obtained explicitly as follows:

$$B^{(l+1)}(x) = B^{(l+1)}(0) + \int_0^x dy \frac{-u_1(y) + u_2(y)}{\sqrt{1 - 2gy}}, \tag{58}$$

$$\frac{d}{dx}A^{(l+1)}(x) = \frac{1}{(1-2gx)^2}\Bigg[A^{(l+1)'}(0) + \int_0^x dy \Big\{-2g^2 B^{(l+1)}(y) + (1-2gy)B^{(l+1)''}(y) + g\sqrt{1-2gy}\,(-2u_1(y) + u_2(y)) + (1-2gy)^{\frac{3}{2}}\left(2u_1'(y) - u_2'(y) - u_3(y)\right)\Big\}\Bigg], \tag{59}$$

$$P^{(l+1)}(x) = \sqrt{1 - 2gx}\left[P^{(l+1)}(0) + \int_0^x dy \left\{2A^{(l+1)'}(y) - \frac{2g}{1-2gy}B^{(l+1)}(y) + \frac{-2u_1(y) + u_2(y)}{\sqrt{1-2gy}}\right\}\right]. \tag{60}$$

Note that

$$A^{(l+1)'}(0)\,, \quad B^{(l+1)}(0)\,, \quad P^{(l+1)}(0), \tag{61}$$

are left undetermined as constants of integration. This is expected, since $A^{(l+1)'}(0)$ normalizes the coupling g, $B^{(l+1)}(0)$ normalizes the field ϕ^i, and $P^{(l+1)}(0)$ normalizes the composite operator $[\Phi_N]$. For example, we can adopt the convention [20–22]

$$A_1\left(\ln \Lambda/\mu\right)\Big|_{\Lambda=\mu} = \frac{\partial}{\partial x} A(0;x)\Big|_{x=0} = -\frac{g}{2}, \tag{62}$$

$$B_0\left(\ln \Lambda/\mu\right)\Big|_{\Lambda=\mu} = B(0;0) = 0, \tag{63}$$

$$P_0\left(\ln \Lambda/\mu\right)\Big|_{\Lambda=\mu} = P(0;0) = 1, \tag{64}$$

analogous to the minimal subtraction for dimensional regularization [23]. This concludes our inductive construction of the O(N) nonlinear σ model.

3.3. 1-Loop results

Let us give explicitly the 1-loop corrections to 2- and 4-point vertices in the Wilson action. (More details are given in Appendix **??**.) For the 2-point vertex we find

$$a_1^{(1)} = \frac{g}{2}\int_q \Delta(q) = g\int_q K(q), \tag{65}$$

$$B_0^{(1)} = \frac{g}{4\pi}\ln \Lambda/\mu, \tag{66}$$

and for the 4-point vertex we find

$$\begin{aligned} a_2^{(1)} &= 2g^2 \int_q \Delta(q) K(q) \\ &= 2g^2 \int_q K(q)^2, \end{aligned} \tag{67}$$

$$A_1^{(1)} = N\frac{g^2}{4\pi}\ln \Lambda/\mu, \tag{68}$$

$$B_1^{(1)} = \text{const.} \tag{69}$$

We also find the 2- and 4-point vertices for the composite operator $[\Phi_N](p)$:

$$P_0^{(1)} = \frac{N-1}{4\pi} g \ln \Lambda/\mu, \tag{70}$$

$$P_1^{(1)} = (N-1)\frac{g^2}{4\pi}\ln \Lambda/\mu + \text{const.} \tag{71}$$

We have fixed the Λ independent part of $A_1^{(1)}$, $B_0^{(1)}$, and $P_0^{(1)}$ using the convention (62, 63, 64).

The two constants in $B_1^{(1)}$ and $P_1^{(1)}$ are left undetermined by the ERG differential equations. They are determined by the WT identity. Calculating

$$\epsilon_i \int_p K(p/\Lambda) \frac{\delta[\Phi_N]^{(0)}(p)}{\delta\phi_i(p)} \tag{72}$$

only up to cubic in fields and up to two derivatives, we obtain

$$u_1^{(1)}(0) = -g^2 \left(\int_q \frac{K(q)(1-K(q))}{q^2} + \frac{1}{4\pi} \right), \tag{73}$$

$$u_2^{(1)}(0) = g^2 \int_q \frac{K(q)}{q^2} \left(\frac{1}{4}\tilde{\Delta}(q) - 2(1-K(q)) \right), \tag{74}$$

where

$$\tilde{\Delta}(q) \equiv -2q^2 \frac{d}{dq^2}\Delta(q). \tag{75}$$

Hence, we obtain

$$B_1^{(1)} = g^2 \left(\frac{1}{4\pi} - \int_q \frac{K(1-K)}{q^2} + \frac{1}{4}\int_q \frac{K\tilde{\Delta}}{q^2} \right), \tag{76}$$

$$P_1^{(1)}\Big|_{\Lambda=\mu} = g^2 \left(\frac{1}{2\pi} + \frac{1}{4}\int_q \frac{K\tilde{\Delta}}{q^2} \right). \tag{77}$$

In **??** we explain how to obtain the beta function of g and anomalous dimension of ϕ_i in the ERG approach. The above 1-loop results reproduce the well-known results first obtained in [10]:

$$\beta(g) \simeq (N-2)\frac{g^2}{2\pi}, \quad \gamma(g) \simeq \frac{g}{4\pi} \tag{78}$$

4. Discussion

In this paper we have applied the ERG formulation of quantum field theory for the perturbative construction of the two-dimensional nonlinear σ model. We start with the Wilson action, which has manifest O(N-1) invariance. We get the full O(N) symmetry by the invariance of the action under the infinitesimal transformation of the scalar fields given in (15). We define the WT identity as the quantum invariance of the Wilson action under (15). Here the nontrivial jacobian of (15) is taken into account. We see that a quadratically divergent potential is generated by the momentum cutoff, but its noninvariance is compensated by the jacobian of the nonlinear symmetry transformation. Then we find that the model is parameterized by three renormalization functions, $A(0;x), B(0;x), P(0;x)$, and we show how to tune these functions by imposing the WT identity (20).

From the infinitesimal change of the Wilson action a composite operator is defined that has two degrees of freedom, corresponding to the infinitesimal variation of g and that of the normalization of ϕ_i. We obtain the beta functions and anomalous dimensions of the model. These are shown explicitly in the Appendices.

Before ending we would like to remark that only short-distance physics can be explored perturbatively, and long-distance physics needs nontrivial approximations, such as $1/N$. For the $1/N$ expansions it is common to linearize the O(N) symmetry using an auxiliary field; it would be interesting to extend the ERG formulation to accommodate the auxiliary field.

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References

[1] Novikov, V. A.; Shifman, M. A.; Vainshtein, A. I.; Zakharov, V. I. *Phys. Rep.* **1984**, *116*, 103-171.

[2] D'Adda, A.; Luscher, M.; Vecchia, P. Di. *Nucl. Phys. B* **1978**, *146*, 63-76.

[3] Witten, E. *Nucl. Phys. B* **1979**, *160*, 57-115.

[4] Gross, D.; Neveu, A. *Phys. Rev. D* **1974**, *10*, 3235-3253.

[5] Bardeen, W.; Lee, B. W.; Shrock, R. E. *Phys. Rev. D* **1976**, *14*, 985-1005.

[6] Coleman, S. *Aspects of Symmetry*; Cambridge University Press: New York, NY, USA, 1985.

[7] Callan, C. G.; Friedan, D.; Martinec, E. J.; Perry, M. J. *Nucl. Phys. B* **1985**, *262*, 593-609.

[8] Osborn, H. *Nucl. Phys. B* **1988**, *308*, 629-661.

[9] Curci, G.; Paffuti, G. *Nucl. Phys. B* **1989**, *312*, 227-252.

[10] Polyakov, A. M. *Phys. Lett. B* **1975**, *59*, 79-81.

[11] Bollini, C. G.; Giambiagi, J. J. *Nuovo Cim. B* **1972**, *12*, 20-26.

[12] Hooft, G.; Veltman, M. J. G. *Nucl. Phys. B* **1972**, *44*, 189-213.

[13] Brezin, E.; Zinn-Justin, J.; Le Guillou, J. C. *Phys. Rev. D* **1976**, *14*, 2615-2661.

[14] Rosten, O. J. *Phys. Rep.* **2012**, *511*, 177-272.

[15] Wilson, K. G.; Kogut, J. B. *Phys. Rep.* **1974**, *12*, 75-199.

[16] Polchinski, J. *Nucl. Phys. B* **1984**, *231*, 269-295.

[17] Igarashi, I.; Itoh, K.; Sonoda, H. *Prog. Theor. Phys. Suppl.* **2010**, *181*, 1-166.

[18] Mitter, P. K.; Ramadas, T. D. *Commun. Math. Phys.* **1989**, *122*, 575-596.

[19] C. Becchi, "On the construction of renormalized gauge theories using renormalization group techniques," arXiv:hep-th/9607188.

[20] Pernici, M.; Raciti, M. *Nucl. Phys. B* **1998**, *531*, 560-592.

[21] Sonoda, H. *Phys. Rev. D* **2003**, *67*, 065011.

[22] Sonoda, H. *J. Phys. A* **2007**, *40*, 5733-5749.

[23] Hooft, G. *Nucl. Phys. B* **1973**, *61*, 455-468.

A. Basic properties of the correlation functions

The correlation functions of a Wilson action S_Λ are dependent on the cutoff Λ. Using the inverse of the cutoff function, however, we can easily construct Λ independent correlation functions:

$$\langle \phi_i(p)\phi_j(-p)\rangle^\infty \equiv \frac{1}{K(p/\Lambda)^2}\langle \phi_i(p)\phi_j(-p)\rangle_{S_\Lambda} + \delta_{ij}\frac{1-1/K(p/\Lambda)}{p^2}, \tag{A.1}$$

$$\langle \phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle^\infty \equiv \prod_{j=1}^{n}\frac{1}{K(p_j/\Lambda)}\langle \phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle_{S_\Lambda}, \tag{A.2}$$

for $n \geq 4$. The Λ independence of these correlation functions is a consequence of the ERG differential equation (9). See sect. 2 of [17] for more details.

Similarly, given a composite operator $\mathcal{O}_\Lambda$ that satisfies the same linear ERG differential equation as (16), we can construct Λ independent correlation functions by

$$\langle \mathcal{O}\,\phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle^\infty \prod_{j=1}^{n}\frac{1}{K(p_j/\Lambda)}\cdot\langle \mathcal{O}_\Lambda\,\phi_{i_1}(p_1)\cdots\phi_{i_n}(p_n)\rangle_{S_\Lambda}. \tag{A.3}$$

See sect. 4 of [17] for more details.

B. 1-Loop calculations

The interaction part of the classical action is given by

$$\begin{aligned} S_{I,cl} &\equiv \int d^2x(-\partial^2)\frac{\phi^2}{2}\cdot\frac{1}{4}\ln\left(1-g\phi^2\right) \\ &= \int d^2x(-\partial^2)\frac{\phi^2}{2}\cdot\frac{-1}{4}\sum_{n=1}^{\infty}(2g)^n(n-1)!\cdot\frac{1}{n!}\left(\frac{\phi^2}{2}\right)^n. \end{aligned} \tag{B.4}$$

Hence,

$$A_n^{(0)} = -\frac{1}{4}(n-1)!(2g)^n. \tag{B.5}$$

Thus, for the graph in Figure 1, we obtain the Feynman rule

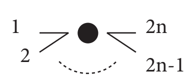

Figure 1. Tree level vertex ($n \geq 2$).

$$\delta_{i_1 i_2}\cdots\delta_{i_{2n-1}i_{2n}}\left\{(p_1+p_2)^2+\cdots+(p_{2n-1}+p_{2n})^2\right\}A_{n-1}^{(0)}. \tag{B.6}$$

As the simplest example, we consider the 1-loop contribution to the two-point vertex given by the Feynman graph in Figure 2.

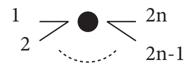

Figure 2. $\mathcal{V}_2^{(1)}(p)$: 1-loop correction to the two-point vertex with momentum p.

The ERG differential equation gives

$$\begin{aligned}
-\Lambda\frac{\partial}{\partial\Lambda}\mathcal{V}_2^{(1)}(p) &= \int_q \frac{\Delta(q/\Lambda)}{q^2}2(p+q)^2 A_1^{(0)} \\
&= -\frac{g}{2}\int_q \frac{\Delta(q/\Lambda)}{q^2}2(p+q)^2 \\
&= -g\int_q \frac{\Delta(q/\Lambda)}{q^2}(p^2+q^2) \\
&= -g\left[\Lambda^2\int_q \Delta(q) + \frac{1}{2\pi}p^2\right], \qquad \text{(B.7)}
\end{aligned}$$

where we have used

$$\int_q \frac{\Delta(q)}{q^2} = \frac{1}{2\pi}. \qquad \text{(B.8)}$$

Hence, integrating this over Λ, we obtain

$$\mathcal{V}_2^{(1)}(p) = \Lambda^2\frac{g}{2}\int_q \Delta(q) + 2p^2\frac{g}{4\pi}\ln\Lambda/\mu. \qquad \text{(B.9)}$$

This gives

$$a_1^{(1)} = \frac{g}{2}\int_q \Delta(q), \quad B_0^{(1)}\left(\ln\Lambda/\mu\right) = \frac{g}{4\pi}\ln\Lambda/\mu, \qquad \text{(B.10)}$$

where we have used the normalization condition $B_0^{(1)}(0) = 0$.

As another example, let us consider the 1-loop contribution to the 1-point vertex of the jacobian:

$$\int_q K\left(q/\Lambda\right)\frac{\delta[\Phi_N]^{(0)}(q)}{\delta\phi_i(q)}. \qquad \text{(B.11)}$$

Now the leading part of the derivative expansion of $[\Phi_N]^{(0)}$ is given by

$$\int_p \exp^{ipx}[\Phi_N]^{(0)}(p) = P^{(0)}\left(\phi(x)^2/2\right) + \cdots, \qquad \text{(B.12)}$$

where

$$\begin{aligned}
P^{(0)}(x) &= \sqrt{1-2gx} \\
&= 1 - \sum_{n=1}^{\infty}\frac{(2n-2)!}{2^{n-1}(n-1)!}g^n\cdot\frac{x^n}{n!}. \qquad \text{(B.13)}
\end{aligned}$$

Hence, we obtain

$$P_0^{(0)} = 1, \tag{B.14}$$

$$P_{n\geq 1}^{(0)} = -\frac{(2n-2)!}{2^{n-1}(n-1)!}\, g^n, \tag{B.15}$$

Let us denote the $2n$-point vertex $P_n^{(0)}$ for $[\Phi_N]$ by Figure 3. Then the one-point vertex for the 1-loop jacobian is given by Figure 4 and calculated as

$$\begin{aligned}\int_q K\left(q/\Lambda\right) P_1^{(0)} &= -\Lambda^2\, g \int_q K(q)\\ &= -\Lambda^2\, \frac{g}{2} \int_q \Delta(q).\end{aligned} \tag{B.16}$$

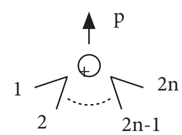

Figure 3. $2n$-point vertex for $[\Phi_N](p)$.

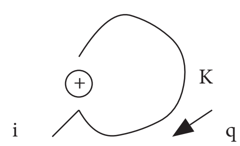

Figure 4. 1-loop contribution to the 1-point vertex of the jacobian (B.11).

This cancels the contribution of the $a_1^{(1)}$ term to $\Sigma_\Lambda^{(1)}$.

C. Beta function and anomalous dimension

The derivation of the mass independent beta functions and anomalous dimensions in the ERG formalism has been discussed in [20] and [22].

C.1. μ dependence of the Wilson action

The Wilson action S_Λ for a different choice of μ satisfies the same ERG differential equation (9). Hence,

$$\Psi_\Lambda \equiv -\mu\partial_\mu S_\Lambda, \tag{C.17}$$

is a composite operator satisfying the ERG differential equation

$$-\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda = \int_p \frac{\Delta(p/\Lambda)}{p^2}\left\{\frac{\delta S_{I,\Lambda}}{\delta\phi_i(p)}\frac{\delta\Psi_\Lambda}{\delta\phi_i(-p)} + \frac{1}{2}\frac{\delta^2\Psi_\Lambda}{\delta\phi_i(p)\delta\phi_i(-p)}\right\}. \tag{C.18}$$

Ψ_Λ has the correlation functions

$$\langle \Psi\, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = -\mu\partial_\mu \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty \tag{C.19}$$

Expanding Ψ_Λ up to two derivatives, we obtain

$$\Psi_\Lambda = \int d^2x \left[\Lambda^2\, \dot{a}(\ln \Lambda/\mu; \phi^2/2) + \dot{A}(\ln \Lambda/\mu; \phi^2/2)\, (-\partial^2) \frac{\phi^2}{2} + \dot{B}(\ln \Lambda/\mu; \phi^2/2)\, \phi_i(-\partial^2)\phi_i \right] + \cdots , \tag{C.20}$$

where

$$\begin{cases} \dot{a}(\ln \Lambda/\mu; x) & \equiv \frac{\partial}{\partial \ln \Lambda/\mu} a(\ln \Lambda/\mu; x) \\ \dot{A}(\ln \Lambda/\mu; x) & \equiv \frac{\partial}{\partial \ln \Lambda/\mu} A(\ln \Lambda/\mu; x) \\ \dot{B}(\ln \Lambda/\mu; x) & \equiv \frac{\partial}{\partial \ln \Lambda/\mu} B(\ln \Lambda/\mu; x) \end{cases} \tag{C.21}$$

Especially at $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$\partial_x \dot{A}(0; x)\Big|_{x=0}, \tag{C.22}$$

and that of $\phi_i(-\partial^2)\phi_i$ is

$$\dot{B}(0; 0). \tag{C.23}$$

Since Ψ_Λ is an infinitesimal change of the Wilson action, it has two degrees of freedom, corresponding to the infinitesimal variation of g and that of the normalization of ϕ_i. Thus, we can construct two composite operators:

1. $\mathcal{O}_g$ that generates an infinitesimal change of g:

$$\mathcal{O}_g \equiv -\partial_g S_\Lambda. \tag{C.24}$$

The correlation functions of $\mathcal{O}_g$ are given by

$$\langle \mathcal{O}_g\, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = -\partial_g \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty . \tag{C.25}$$

2. $\mathcal{N}$ that generates an infinitesimal renormalization of ϕ_i:

$$\mathcal{N} \equiv -\int_p \phi_i(p) \frac{\delta S_\Lambda}{\delta \phi_i(p)} - \int_p \frac{K(p/\Lambda)\,(1 - K(p/\Lambda))}{p^2} \left\{ \frac{\delta S_\Lambda}{\delta \phi_i(p)} \frac{\delta S_\Lambda}{\delta \phi_i(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi_i(p) \delta \phi_i(-p)} \right\}. \tag{C.26}$$

$\mathcal{N}$ counts the number of fields:

$$\langle \mathcal{N}\, \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = n \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty . \tag{C.27}$$

Ψ_Λ must be a linear combination of $\mathcal{O}_g$ and $\mathcal{N}$; hence

$$\Psi_\Lambda = \beta(g)\, \mathcal{O}_g + \gamma(g)\, \mathcal{N}, \tag{C.28}$$

where neither $\beta(g)$ nor $\gamma(g)$ depends on Λ. This gives the differential equation

$$(-\mu\partial_\mu + \beta\partial_g) \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty = n\gamma \langle \phi_{i_1}(p_1) \cdots \phi_{i_n}(p_n) \rangle^\infty . \tag{C.29}$$

Hence, $\beta(g)$ is the beta function of g and $\gamma(g)$ is the anomalous dimension of ϕ_i.

C.2. $\mathcal{O}_g$

The derivative expansion gives

$$\mathcal{O}_g = \int d^2x \Big[\Lambda^2(-\partial_g)\, a(\ln\Lambda/\mu; \phi^2/2) + (-\partial_g)A(\ln\Lambda/\mu; \phi^2/2)\,(-\partial^2)\frac{\phi^2}{2} + (-\partial_g)B(\ln\Lambda/\mu; \phi^2/2)\,\phi_i(-\partial^2)\phi_i\Big] + \cdots . \tag{C.30}$$

At $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$-\partial_g\partial_x A(0;x)\Big|_{x=0} = \frac{1}{2}, \tag{C.31}$$

and the coefficient of $(1/2)\phi_i(-\partial^2)\phi_i$ is

$$-\partial_g B(0;0) = 0. \tag{C.32}$$

These are consequences of the conventions (62, 63).

C.3. $\mathcal{N}$

Using the interaction part of the action, we can rewrite

$$\begin{aligned}\mathcal{N} &= \int_p p^2\phi_i(p)\phi_i(-p)\int_p\Big[-1+2\Big(1-K(p/\Lambda)\Big)\Big]\phi_i(p)\frac{\delta S_{I,\Lambda}}{\delta\phi_i(p)} \\ &\quad -\int_p \frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2}\left\{\frac{\delta S_{I,\Lambda}}{\delta\phi_i(p)}\frac{\delta S_{I,\Lambda}}{\delta\phi_i(-p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta\phi_i(p)\delta\phi_i(-p)}\right\}.\end{aligned} \tag{C.33}$$

Hence, the derivative expansion gives

$$\begin{aligned}\mathcal{N}(\Lambda) = \int d^2x \Big[& \Lambda^2 a_{\mathcal{N}}\left(\ln\Lambda/\mu; \phi^2/2\right) + A_{\mathcal{N}}\left(\ln\Lambda/\mu; \phi^2/2\right)\left(-\partial^2\right)\frac{\phi^2}{2} \\ + & B_{\mathcal{N}}\left(\ln\Lambda/\mu; \phi^2/2\right)\phi_i\left(-\partial^2\right)\phi_i\Big] + \cdots ,\end{aligned} \tag{C.34}$$

where

$$A_{\mathcal{N}}(\ln\Lambda/\mu; x) = -2A(\ln\Lambda/\mu; x) - 2x\frac{\partial}{\partial x}A(\ln\Lambda/\mu; x) + \cdots , \tag{C.35}$$

$$B_{\mathcal{N}}(\ln\Lambda/\mu; x) = 1 - 2B(\ln\Lambda/\mu; x) + \cdots , \tag{C.36}$$

up to loop corrections. At $\Lambda = \mu$, the coefficient of $(\phi^2/2)(-\partial^2)(\phi^2/2)$ is

$$\partial_x A_{\mathcal{N}}(0;x)\Big|_{x=0} = 2g + \cdots , \tag{C.37}$$

and the coefficient of $\frac{1}{2}\phi_i(-\partial^2)\phi_i$ is

$$B_{\mathcal{N}}(0;0) = 1 + \cdots . \tag{C.38}$$

C.4. β and γ

Comparing the derivative expansion of Ψ_Λ with those of $\mathcal{O}_g$ and $\mathcal{N}$, we obtain $\beta(g)$ and $\gamma(g)$ as follows:

$$\partial_x \dot{A}(0;x)\Big|_{x=0} = \frac{1}{2}\beta(g) + \gamma(g)\partial_x A_{\mathcal{N}}(0;x)\Big|_{x=0}, \tag{C.39}$$

$$\dot{B}(0;0) = \gamma(g) B_{\mathcal{N}}(0;0), \tag{C.40}$$

where we have used (C.31, C.32).

Using (68, 66), we obtain

$$\partial_x \dot{A}^{(1)}(0;x)\Big|_{x=0} = N\frac{g^2}{4\pi}, \tag{C.41}$$

$$\dot{B}^{(1)}(0;0) = \frac{g}{4\pi}. \tag{C.42}$$

Using (C.37, C.38), we also obtain

$$\partial_x A_{\mathcal{N}}^{(0)}(0;x)\Big|_{x=0} = 2g, \tag{C.43}$$

$$B_{\mathcal{N}}^{(0)}(0;0) = 1. \tag{C.44}$$

Hence, at 1-loop (C.39, C.40) give

$$N\frac{g^2}{4\pi} = \frac{1}{2}\beta^{(1)} + \gamma^{(1)}\cdot 2g, \tag{C.45}$$

$$\frac{g}{4\pi} = \gamma^{(1)}\cdot 1. \tag{C.46}$$

Thus, we obtain (78).