International Electronic Journal of Algebra
Published Online: November 11, 2024
DOI: 10.24330/ieja.1582755

ANSWERING A QUESTION ON EQUALLY COVERED GROUPS

Andrea Lucchini

Received: 15 October 2024; Revised: 6 November 2024; Accepted: 6 November 2024
Communicated by Burcu Üngör

Abstract. Foguel, Moghaddamfar, Schmidt, and Velasquez-Berroteran asked in [Int. Electron. J. Algebra, 2024] whether there exists a positive integer n with the property that, for every finite group G, the Cartesian power G^n can be expressed as the union of a family of proper subgroups of the same order. We prove that the answer is negative.

Mathematics Subject Classification (2020): 20D05, 20D06, 20D08
Keywords: Equally covered group, Cartesian power of a group, $\mathrm{PSL}(2,p)$

1. Introduction

A finite group G is said to be covered by proper subgroups $H_1, H_2, \ldots, H_n$ if $G = H_1 \cup H_2 \cup \cdots \cup H_n$. In [2] Foguel, Moghaddamfar, Schmidt, and Velasquez-Berroteran began the study of finite groups which possess an equal covering, i.e., a covering $H_1, \ldots, H_n$ in which all the proper subgroups $H_1, \ldots, H_n$ have the same order. They called *equally covered* the finite groups with this property.

Cyclic groups do not admit any covering, and therefore obviously cannot be equal equally covered. But this is not the unique obstruction, for example the alternating group A_5 of degree 5 does not possess an equal covering. However, as it is noticed in [2], if G has order n, then the direct power G^{n+1} of $n+1$ copies of G is equally covered by the subgroups $D_{i,j}$, $1 \leq i < j \leq n+1$, where $D_{i,j}$ is the set of the elements $(g_1, \ldots, g_{n+1}) \in G^{n+1}$ with $g_i = g_j$. This led the authors to introduce the following definition: given a group G, $\xi(G)$ is defined as the smallest integer $n \geq 1$ for which the n-Cartesian power G^n has an equal covering. They noticed that $\xi(G) \leq 2$ if G is not perfect ([2, Theorem 3.4]), and this motivated the following question ([2, Question 3.6]): *is there a natural number n such that $\xi(G) \leq n$ for all finite groups G?*

In this short note we prove that the answer to the previous question is negative:

Theorem 1.1. *For every positive integer t, there exists a finite simple group G with $\xi(G) > t$.*

The following more difficult question remains open.

Question 1.2. *Let G be a finite non-abelian simple group. Does $\xi(G) \to \infty$ as $|G| \to \infty$?*

2. Proof of Theorem 1.1

Define $\mathbb{P}$ as the set of primes p with $p \equiv 13 \pmod{120}$ and set $S = \mathrm{PSL}(2, p)$. It follows from [3, Theorem 2] that for every positive integer t, there exist infinitely many $p \in \mathbb{P}$ with the property that the number $\omega(p-1)$ of distinct prime divisors of $p-1$ is at least t. Let $\mathbb{P}_t$ be the set of the primes p in $\mathbb{P}$ with $\omega(p-1) \geq t$.

Now let $t, u \in \mathbb{N}$ and consider the direct product S^u where $S = \mathrm{PSL}(2, p)$ and $p \in \mathbb{P}_t$. For $1 \leq i \leq u$, let $\pi_i : S^u \to S$ be the projection to the i-th factor and, if $J = \{i_1, \dots, i_r\} \subseteq \{1, \dots, u\}$, denote by $\pi_J : S^u \to S^r$ the homomorphism sending $g = (s_1, \dots, s_t)$ to $(s_{i_1}, \dots, s_{i_r})$. If follows from [5, Hauptsatz 8.27] that all the maximal subgroups of S are solvable and their orders belong to the set $\{\frac{p(p-1)}{2}, p+1, p-1, 12\}$. Given an odd prime q dividing $|S|$, set $\alpha_q = p(p-1)$ if p divides $p(p-1)$, $\alpha_q = p+1$ otherwise. This implies in particular that if a proper subgroup X of S contains an element of order q, then $|X|$ divides α_q.

Lemma 2.1. *Let $S = \mathrm{PSL}(2, p)$ with $p \in \mathbb{P}_t$. Fix a positive integer u and let $g = (x_1, \dots, x_u) \in S^u$, with $|x| = q_i$, where $q_1, \dots, q_u$ are odd primes and $a_{q_1} = \cdots = a_{q_u}$. Suppose that H is a subgroup of S^u containing g. Then $H \cong X \times Y$, where X is solvable, $|X|$ divides $(\alpha_{q_1})^u$ and $Y \cong S^v$ for some non-negative integer v. Moreover, if $X = 1$, then $\pi_i(H) = S$ for every $1 \leq i \leq u$.*

Proof. For $1 \leq i \leq u$, $\pi_i(H)$ is a subgroup of S containing x_i, so either $\pi_i(H) = S$ or $\pi_i(H)$ is a solvable subgroup of S whose order divides α_{q_i}. Let

$$J = \{i \mid \pi_i(H) = S\}.$$

We may assume $J \neq \emptyset$, otherwise H is a solvable group whose order divides $\alpha_{q_1} \cdots \alpha_{q_u} = (\alpha_{q_1})^u$. By [1, Proposition 1.1.39], $\pi_J(H) \cong S^v$ for a suitable positive integer v. Let $I = \{1, \dots, u\} \setminus J$. We have that H is a subdirect product of $\pi_I(H) \times \pi_J(H)$. Since $\pi_I(H)$ is solvable and $\pi_H(H) \cong S^u$, it follows from the Goursat's Lemma (see for example [4, Theorem 5.5.1]) that $H \cong \pi_I(H) \times \pi_J(H)$. Finally notice that, since q_i divides $|\pi_i(H)|$ for every $1 \leq i \leq u$, $\pi_I(H) = 1$ if and only if $I = \emptyset$. □

Theorem 2.2. *If $p \in \mathbb{P}_t$ and the direct power $(\mathrm{PSL}(2, p))^u$ admits an equal covering, then $u > t$.*

Proof. Let $S = \mathrm{PSL}(2,p)$. Assume that $G = S^u$ has an equal covering

$$G = H_1 \cup \cdots \cup H_n$$

and suppose by contradiction that $u \leq t$. Let $q_1, \ldots, q_t$ be distinct odd prime divisors of $p-1$. Let $x = (x_1, \ldots, x_u) \in G$ with the property that, for $1 \leq i \leq u$, $|x_i| = q_i$ and let $z = (y, \ldots, y)$ where $|y|$ is an odd prime divisor of $p+1$ (notice that the existence of such a prime is ensured by the fact that $p \in P_t$ implies that 4 does not divide $p+1$). It is not restrictive to assume $z \in H_1$. Moreover there exists $r \in \{1, \ldots, n\}$ such that $x \in H_r$. Notice that $a_{q_1} = \cdots = a_{q_u} = p(p-1)$ and $a_{|y|} = p+1$, hence, by Lemma 2.1, there exist two nonnegative integers u_1, u_2, a divisor a_1 of $(p+1)^u$ and a divisor a_2 of $(p(p-1))^u$ such that

$$|H_1| = a_1|S|^{u_1} = a_2|S|^{u_2} = |H_r|. \tag{2.1}$$

Since p does not divides a_1 but divides $|S|$, if follows from (2.1) that $u_1 \geq u_2$. In particular

$$a_2 = |S|^{u_1-u_2}a_1. \tag{2.2}$$

If $a_1 \neq 1$, then if follows again from Lemma 2.1 and its proof, that $|y|$ divides a_1. However $|y|$ does not divide a_2, so we must have $a_1 = 1$. This implies $a_2 = |S|^{u_1-u_2}$, and since again $|y|$ divides $|S|$ but not a_2, we conclude $a_2 = 1$. This implies that $\pi_k(H_r) = S$ for every $1 \leq k \leq u$ and therefore, by [1, Proposition 1.1.39], there exist $1 \leq i < j \leq u$ and $\alpha \in \mathrm{Aut}(S)$, such that H_r is contained in the subgroup of S^u consisting of the elements $(s_1, \ldots, s_u) \in S^u$ such that $s_j = \alpha(s_i)$. Since $(x_1, \ldots, x_u) \in H_r$, this would imply $q_i = |x_i| = |x_j| = q_j$, against our original choice. □

Disclosure statement. The author reports that there are no competing interests to declare.

References

[1] A. Ballester-Bolinches and L. M. Ezquerro, Classes of Finite Groups, Mathematics and Its Applications (Springer), 584, Springer, Dordrecht, 2006.

[2] T. Foguel, A. R. Moghaddamfar, J. Schmidt and A. Velasquez-Berroteran, *Concerning equally covered groups*, Int. Electron. J. Algebra, (2024) DOI: 10.24330/ieja.1567377.

[3] H. Halberstam, *On the distribution of additive number-theoretic functions, III*, J. London Math. Soc., 31 (1956), 14-27.

[4] M. Hall, Jr., The Theory of Groups, The Macmillan Company, New York, 1959.

[5] B. Huppert, Endliche Gruppen, I, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

Andrea Lucchini
Università di Padova
Dipartimento di Matematica
"Tullio Levi-Civita" Via Trieste
63 - 35121 Padova, Italy
e-mail: lucchini@math.unipd.it