

# Chromatic indices of finite affine & projective planes and their duals

Abdurrahman DAYIOĞLU<sup>1,\*</sup>, Fatma ÖZEN ERDOĞAN<sup>1</sup>

<sup>1</sup>Bursa Uludag University Faculty of Science and Arts, Department of Mathematics, Gorukle Campus, Bursa.

Geliş Tarihi (Received Date): 11.11.2024  
Kabul Tarihi (Accepted Date): 05.05.2025

## Abstract

*In this study, rather than transitioning directly from geometric structures to graph theory, we have derived several general results and theorems concerning the coloring of points and lines within affine and projective structures. We approached this topic through the lens of vertex and edge coloring concepts, pivotal subjects within graph theory. Our investigation sheds light on the intricate relationship between geometric structures and graph theory, providing a novel perspective on coloring methodologies. Extending the principles of vertex and edge coloring to affine and projective spaces, we uncover fundamental insights into the interplay between geometry and combinatorial mathematics.*

**Keywords:** Affine plane, projective plane, chromatic index

## Sonlu Afin & Projektif düzlemlerin ve duallerinin kromatik indisleri

### Öz

*Bu çalışmada, doğrudan geometrik yapılardan graf teorisine geçiş yapmak yerine, afin ve projektif yapılardaki nokta ve doğruların renklendirilmesiyle ilgili birkaç genel sonuç ve teorem ürettik. Bu konuya, graf teorisindeki temel konular olan köşe ve kenar renklendirme kavramları merceğinden yaklaştık. Araştırmamız, geometrik yapılar ve graf teori arasındaki karmaşık ilişkiye ışık tutarak, renklendirme metodolojilerine yeni bir bakış açısı sağlıyor. Köşe ve kenar renklendirme ilkelerini afin ve projektif uzaylara genişleterek, geometri ve kombinatoriyal matematik arasındaki etkileşime dair temel içgörüler ortaya çıkarıyoruz.*

**Anahtar kelimeler:** Afin Düzlem, projektif düzlem, kromatik indis

\*Abdurrahman DAYIOĞLU, dayioglu@uludag.edu.tr, <https://orcid.org/0000-0001-8441-6406>  
Fatma Özen Erdoğan, fatmaozen@uludag.edu.tr, <https://orcid.org/0000-0002-9691-4565>

## 1. Introduction

The coloring of different geometrical structures was initially explored in various sources, particularly in the light of Erdos' article [1] titled *On The Combinatorial Problems Which I Would Most Like To See Solved*. A. Beutelspacher et al. also dealt with this issue in [2], and they calculated some boundaries for the chromatic indices of some projective spaces. This subject has continued its vitality with various studies since the 1980s.

Recent developments in this field can be found in [3], [4], [5], and [6]. On the other hand, we analyzed the concepts of graphs coloring and implemented this to finite affine and projective planes as well as their duals and revealed their relations.

Examining the historical context of geometric structures, it is evident that Hall [7] made a significant impact in his time. While he introduced the concept of partial planes, some writers, such as Batten [8], adopted his axiomatic approach and gave a similar Definition with a different name (near linear spaces), but some writers like Pickert [9] presented an alternative approach and gave a similar Definition with a different name (incidence structures).

## 2. Preliminaries

We start this section by giving some definitions and basic notions from [7] and [8], regarding affine and projective planes.

A *partial plane* is any system of points and lines such that

**PT1)** There is at most one line through any two distinct points.

**PT2)** There is at most one common point on two distinct lines.

A *space*  $S = (\mathcal{P}, \mathcal{L})$  satisfies specific conditions or axioms, where  $\mathcal{P}$  and  $\mathcal{L}$  are sets of points and lines, respectively.

If  $\mathcal{P} \cup \mathcal{L}$  is a finite set then the space  $S = (\mathcal{P}, \mathcal{L})$  is called a *finite space*. If  $\mathcal{P} \cup \mathcal{L}$  is finite then it's known that the sets  $\mathcal{P}$  and  $\mathcal{L}$  are finite and also we denote the number of points by  $|\mathcal{P}| = v$  and the number of lines by  $|\mathcal{L}| = b$ .

A *near-linear space* is a space  $S = (\mathcal{P}, \mathcal{L})$  with two conditions

**NL1)** Any line has at least two points.

**NL2)** Any two distinct points are on at most one line.

An *incidence structure* is a triple  $(\mathcal{P}, \mathcal{L}, I)$  where  $I$  is a relation called an *incidence relation*, between  $\mathcal{P}$  and  $\mathcal{L}$ . (i.e.  $I \subseteq \mathcal{P} \times \mathcal{L}$ )

**IS1)** From  $x I y$ , it follows  $x \in \mathcal{P}$ ,  $y \in \mathcal{L}$ .

**IS2)** From  $x_i I y_k$  ( $i, k = 1, 2$ ), it follows  $x_1 = x_2$  or  $y_1 = y_2$ .

The lines  $l$  and  $l'$  are called as *parallel* if  $l \cap l' = \emptyset$  or  $l = l'$  and we denote this by

$$l \parallel l'.$$

**Definition 1:** An *affine plane*  $\mathcal{A}$  is a near linear space  $(\mathcal{P}, \mathcal{L})$  satisfying following properties

- A1) Any two distinct points lie on a unique line
- A2) For each point  $p$  not on a line  $l$ , there is exactly one line  $l'$  passing through  $p$  such that  $l$  is parallel to  $l'$
- A3) There exists at least a set of three non-collinear points.

In this definition, the second condition, A2, is called *the axiom of parallelism*.

**Definition 2:** A *pencil* of parallel lines refers to a maximal set of lines in a plane that are mutually parallel. Equivalently, it can be defined as a set that includes a specific line along with all parallel lines.

Let's think an affine plane  $\mathcal{A}$ . We already know from [10] and [11] that the number of points on each line is the same in affine planes. Denoting the number of points per line by  $k$ , it is called as the *order* of  $\mathcal{A}$ . Consequently any point is on  $k + 1$  lines [11].

We know also from [11] that each pencil of parallel lines determines an equivalence relation, the plane  $\mathcal{A} = (\mathcal{P}, \mathcal{L})$  determines  $k + 1$  equivalence classes on the set of  $k^2$  points.

If the order of  $\mathcal{A}$  is  $k$  then

- 1) There are  $v = k^2$  points in  $\mathcal{A}$ .
- 2) There are exactly  $k$  lines parallel to any line.
- 3)  $\mathcal{A}$  has  $b = k^2 + k$  lines.
- 4) For any choosen line  $l$ , precisely  $k^2$  lines intersect with  $l$  at one point.
- 5) There are  $k + 1$  pencil of parallel lines and every pencil consists of  $k$  lines.

**Definition 3:** A *projective plane*  $\Pi$  is a near linear space  $(\mathcal{P}, \mathcal{L})$  satisfying the followings:

- P1) Any two distinct points lie on a particular line
- P2) Any two distinct lines intersect at least at one point
- P3) There exists at least a set of four points no three of them are collinear.

Let's suppose  $\Pi$  is a projective plane. We know from [10] and [11] that in the projective planes each line contains the same number of points, every point lies on same number of lines. This number is referred to as  $k + 1$ , where  $k$  is an integer greater than or equal to 2. The value of  $k$  is known as the *order of the projective plane*  $\Pi$ .

In a projective plane  $\Pi$  of order  $k$ , we know that in  $\Pi$

$$v = b = k^2 + k + 1$$

In a finite affine or projective plane, a number  $k \geq 2$  exists, known as the order of the plane. However, not every number  $k$  guarantees the existence of an affine or projective plane of that order. Ongoing studies are actively exploring whether affine or projective planes can exist in various orders. Refer to sources [10], [11], [12], and [13] for more details.

*The Principal of Duality*

We know from [12] that in projective planes, a certain dual relationship between them becomes apparent upon careful examination of the propositions of incidence. In a plane, a line is determined by two points on it, while a point is determined by two lines passing through it. Therefore, if, by using the propositions of incidence, we are able to prove a theorem involving points and lines, then, by using similar reasoning, we should be able to prove a corresponding theorem establishing by interchanging points and lines. The dual theorem is obtained from the original theorem merely by the interchanging certain words such as "point" and "line", "collinear" and "concurrent", "lie on" and "intersect in", and so on. This is known as the *principle of duality* in the plane. For an axiomatic approach we present Batten's [8] dual linear space definition with a slight difference in notion.

**Definition 4:** Let  $S = (\mathcal{P}, \mathcal{L})$  be a near-linear space. The dual (near-linear) space  $D(S) = (D(\mathcal{P}), D(\mathcal{L}))$  of  $S$  is defined as follows :

$$D(\mathcal{P}) = \mathcal{L},$$

$$D(\mathcal{L}) = \{[p] \mid s([p]) \geq 2, p \in \mathcal{P}\}$$

where  $[p] = \{l \in \mathcal{L} \mid p \in l\}$ .

From [14], we know that dual of every theorem given in projective plane is also a theorem. We can also immediately deduce that this is not true in the case of an affine plane due to the propositions of incidence.

We will now present some fundamental concepts for graph theory from [15] and [16].

Let  $G = (V, E)$  be a multigraph with  $n$  vertices (*order*) and  $m$  edges (*size*). Let's take  $x$  and  $y$  be two vertices in graph  $G$ . If  $e = \{x, y\}$  is an edge, we can denote this as  $e = xy$ . The vertices  $x$  and  $y$  are referred to as *adjacent vertices* written as  $(x \sim y)$  and are considered the endpoints of the edge  $e$ . The edge  $e = \{x, y\}$  is said to be incident with  $x$  and  $y$ . In graph theory, an edge characterized by both ends being identical is termed a *loop*, while an edge distinguished by having distinct endpoints is referred to as a *link*. When multiple links share the same pair of endpoints, they are classified as *parallel edges*.  $G$  is classified as *simple* if it excludes both loops and parallel edges. The *degree of a vertex*  $v$ , denoted as  $d(v)$ , is the number of edges incident with  $v$ . A graph  $G$  is said to be *regular* if every vertex in  $G$  has the same degree. More precisely,  $G$  is said to be *k-regular* if  $d(v) = k$  for each vertex  $v$  in  $G$ , where  $k \geq 0$ .

A *graph coloring* of  $G$  assigns different colors to adjacent vertices. The *chromatic number*  $\chi(G)$  is the minimal number of colors needed in a coloring of  $G$ .

**Lemma 1:** Assume that  $G$  is a graph.  $\chi(G) \leq \Delta(G) + 1$  where  $\Delta(G)$  shows the maximum degree of a vertex in  $G$ .

When we assign a color to each edge in a graph, ensuring that any two edges sharing a common vertex have different colors, we call this process *edge coloring*. The minimum number of colors required for a proper edge coloring of a graph  $G$  is known as the *chromatic index* or *the edge chromatic number*, denoted as  $\chi'(G)$ .

### 3. Chromatic indices for affine & projective planes and their dual spaces

As a new contribution to this topic, let's present our definitions and theorems regarding point and line coloring for both affine and projective planes.

**Definition 5:** In an incidence structure  $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ , provided that the collinear points are painted in different colors, the function between the points of the plane and a set  $R = \{a_1, a_2, a_3, \dots, a_v\}$  where  $|\mathcal{P}| = v$ , which is constructed so that the image set contains the minimum number of elements, is called the *point coloring* of the structure. The number of elements of the image set of this function is the *point chromatic index* of the structure, and it is denoted by  $\chi(\mathcal{A})$ .

**Definition 6:** In an incidence structure  $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ , provided that the intersecting lines are painted in different colors, the function between the lines of the plane and a set  $R = \{a_1, a_2, a_3, \dots, a_b\}$  where  $|\mathcal{L}| = b$ , which is constructed so that the image set contains the minimum number of elements, is called the *line coloring* of the structure. The number of elements of the image set of this function is the *line chromatic index* of the structure, and it is denoted by  $\chi'(\mathcal{A})$ .

The smallest affine plane, let's call it  $\mathcal{A}_2 = (\mathcal{P}_2, \mathcal{L}_2)$  has order 2. It has 4 points and 6 lines and can be represented as below:

$$\mathcal{P}_2 = \{1,2,3,4\},$$

$$\mathcal{L}_2 = \{l_1 = \{1,2\}, l_2 = \{3,4\}, l_3 = \{1,3\}, l_4 = \{2,4\}, l_5 = \{1,4\}, l_6 = \{2,3\}\}.$$

The lines can be categorized into three sets of parallel lines:

$$l_1 \parallel l_2, \quad l_3 \parallel l_4, \quad l_5 \parallel l_6.$$

In a geometric structure, we will show the points with large dots and the lines with curves (possibly straight lines) that do not contain corners. The affine plane  $\mathcal{A}_2 = (\mathcal{P}_2, \mathcal{L}_2)$  is shown in Figure 1.

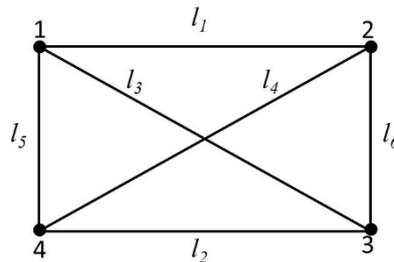


Figure 1: Affine plane of order 2

Since all the point pairs are collinear, we have to color every one of them differently. Therefore we get  $\chi(\mathcal{A}_2) = 4$ .

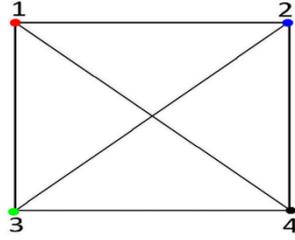


Figure 2: Point coloring for affine plane of order 2

Since there are only three pencils, we have to color each pencil with a different color. Therefore we get  $\chi'(\mathcal{A}_2) = 3$ .

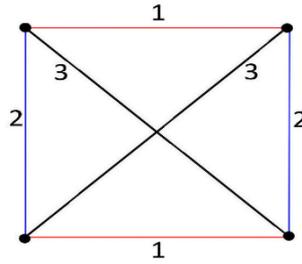


Figure 3: Line coloring for affine plane of order 2

The dual of smallest affine plane, let's call it  $D(\mathcal{A}_2) = (D(\mathcal{P}_2), D(\mathcal{L}_2))$ , has 6 points and 4 lines and can be represented as below:

$$D(\mathcal{P}_2) = \{l_1, l_2, l_3, l_4, l_5, l_6\},$$

$$D(\mathcal{L}_2) = \{L_1 = \{l_1, l_3, l_5\}, L_2 = \{l_1, l_4, l_6\}, L_3 = \{l_2, l_3, l_6\}, L_4 = \{l_2, l_4, l_5\}\}.$$

The dual of the affine plane,  $D(\mathcal{A}_2) = (D(\mathcal{P}_2), D(\mathcal{L}_2))$ , is shown in Figure 4.

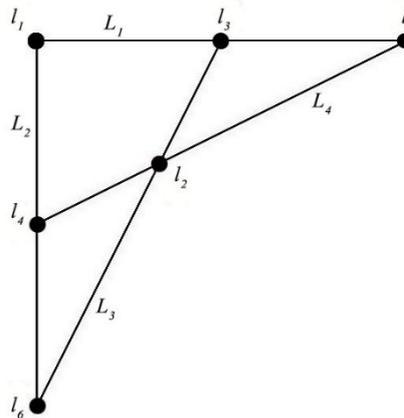


Figure 4: Dual of the affine plane of order 2

Considering the coloring of the points for incidence structures we defined earlier, it can be seen for  $D(\mathcal{A}_2)$ , the dual of the affine plane of order 2, that  $\chi(D(\mathcal{A}_2)) = 3$ .

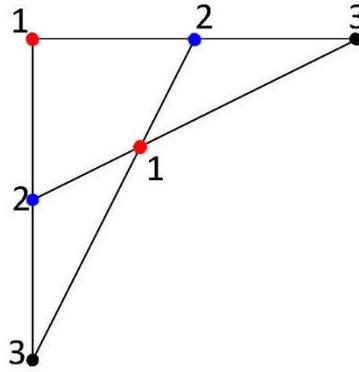


Figure 5: Point coloring for dual of the affine plane of order 2

Similarly we can see that  $\chi'(D(\mathcal{A}_2)) = 4$ .

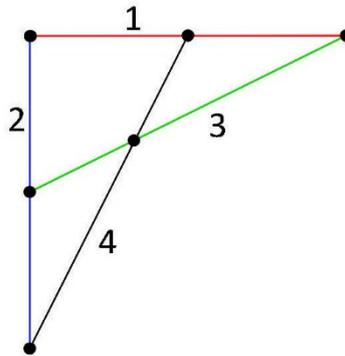


Figure 6: Line coloring for dual of the affine plane of order 2

Let us take  $\mathcal{A}_3$  as the affine plane of order 3 with points and lines given, respectively.  $\mathcal{A}_3 = (\mathcal{P}_3, \mathcal{L}_3)$  illustrated by Figure 7.

$$\mathcal{P}_3 = \{1,2,3,4,5,6,7,8,9\},$$

$$\mathcal{L}_3 = \{l_1 = \{1,2,3\}, l_2 = \{4,5,6\}, l_3 = \{7,8,9\}, l_4 = \{1,4,7\}, l_5 = \{2,5,8\}, l_6 = \{3,6,9\}, \\ l_7 = \{1,5,9\}, l_8 = \{2,6,7\}, l_9 = \{3,4,8\}, l_{10} = \{3,5,7\}, l_{11} = \{1,6,8\}, l_{12} = \{2,4,9\}\}$$

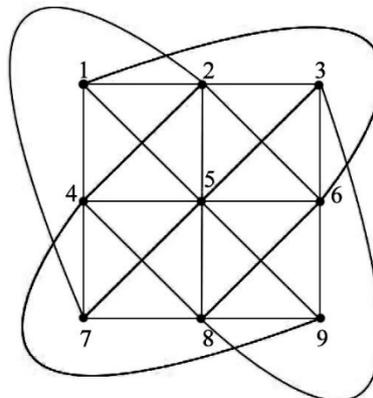


Figure 7: Affine plane of order 3

Since all the point pairs are collinear, we have to color every one of them differently. Therefore we get  $\chi(\mathcal{A}_3) = 9$ .

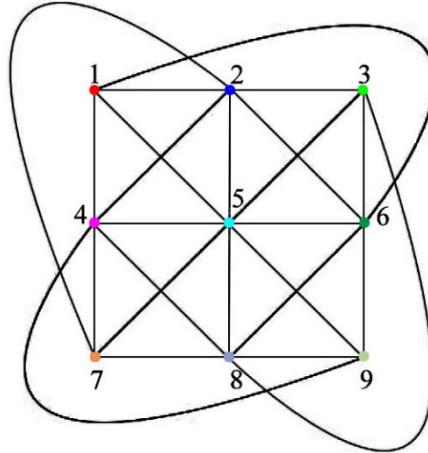


Figure 8: Point coloring for affine plane of order 3

Since there are only four pencils, we have to color each pencil with a different color. Therefore we get  $\chi'(\mathcal{A}_3) = 4$ .

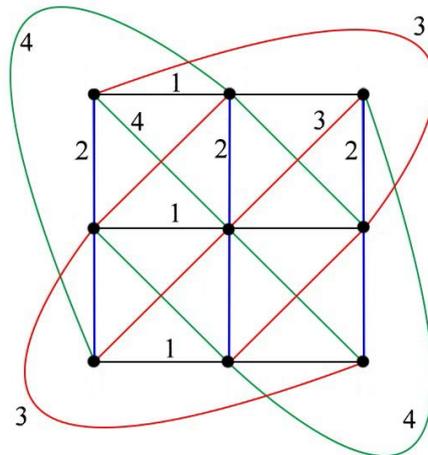


Figure 9: Line coloring for affine plane of order 3

The dual of the affine plane of order 3, let's call it  $D(\mathcal{A}_3) = (D(\mathcal{P}_3), D(\mathcal{L}_3))$ , has 12 points and 9 lines and can be represented as below:

$$\begin{aligned}
 D(\mathcal{P}_3) &= \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10}, l_{11}, l_{12}\}, \\
 D(\mathcal{L}_3) &= \{L_1 = \{l_1, l_4, l_9, l_{10}\}, L_2 = \{l_1, l_5, l_8, l_{11}\}, L_3 = \{l_1, l_6, l_7, l_{12}\}, \\
 &L_4 = \{l_2, l_4, l_8, l_{12}\}, L_5 = \{l_2, l_5, l_7, l_{10}\}, L_6 = \{l_2, l_6, l_9, l_{11}\}, \\
 &L_7 = \{l_3, l_4, l_7, l_{11}\}, L_8 = \{l_3, l_5, l_9, l_{12}\}, L_9 = \{l_3, l_6, l_8, l_{10}\}\}.
 \end{aligned}$$

The dual of the affine plane of order 3,  $D(\mathcal{A}_3) = (D(\mathcal{P}_3), D(\mathcal{L}_3))$ , is shown in Figure 10.

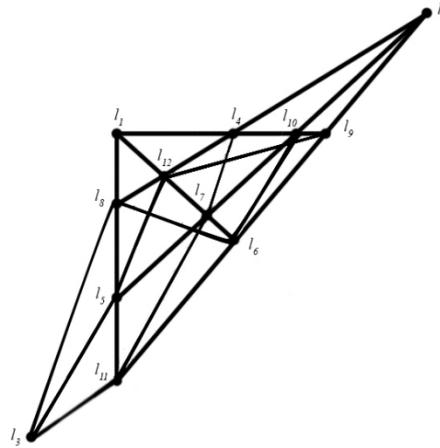


Figure 10: Dual of the affine plane of order 3

Considering the coloring of the points for incidence structures we defined earlier, it can be seen for  $D(\mathcal{A}_3)$ , the dual of the affine plane of order 3, that  $\chi(D(\mathcal{A}_3)) = 4$ .

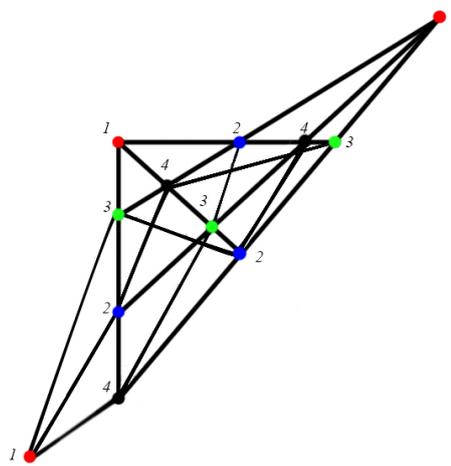


Figure 11: Point coloring for dual of the affine plane of order 3

Similarly we can see that  $\chi'(D(\mathcal{A}_3)) = 9$ .

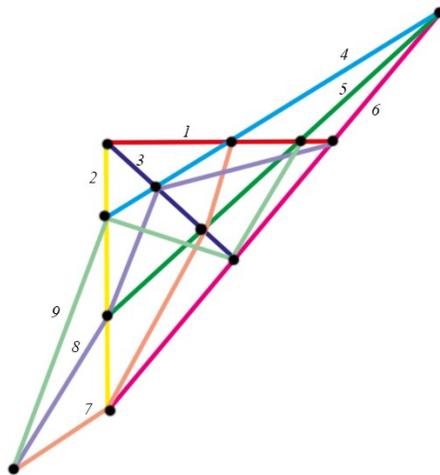


Figure 12: Line coloring for dual of the affine plane of order 3

To be able to see the pattern more clearly we can continue with the 4th order affine plane, but with a slight difference, we will not illustrate it this time but we give a table for it. Let us take  $\mathcal{A}_4 = (\mathcal{P}_4, \mathcal{L}_4)$  as the affine plane of order 4 with points and lines given, respectively.

$$\begin{aligned} \mathcal{P}_4 &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}, \\ \mathcal{L}_4 &= \{l_1 = \{1,2,3,4\}, l_2 = \{5,6,7,8\}, l_3 = \{9,10,11,12\}, l_4 = \{13,14,15,16\}, \\ & l_5 = \{1,5,9,13\}, l_6 = \{2,6,10,14\}, l_7 = \{3,7,11,15\}, l_8 = \{4,8,12,16\}, \\ & l_9 = \{1,6,11,16\}, l_{10} = \{2,5,12,15\}, l_{11} = \{3,8,9,14\}, l_{12} = \{4,7,10,13\} \\ & l_{13} = \{1,7,12,14\}, l_{14} = \{2,8,11,13\}, l_{15} = \{3,5,10,16\}, l_{16} = \{4,6,9,15\}, \\ & l_{17} = \{1,8,10,15\}, l_{18} = \{2,7,9,16\}, l_{19} = \{3,6,12,13\}, l_{20} = \{4,5,11,14\}\} \end{aligned}$$

Since all the point pairs are collinear, we have to color every one of them differently. Therefore we get  $\chi(\mathcal{A}_4) = 16$ .

Since there are only five pencils of parallel lines, we have to color each pencil with a different color. Therefore we get  $\chi'(\mathcal{A}_4) = 5$ .

The dual of affine plane of order 4, let's call it  $D(\mathcal{A}_4) = (D(\mathcal{P}_4), D(\mathcal{L}_4))$ , has 20 points and 16 lines and can be represented as below.

$$\begin{aligned} D(\mathcal{P}_4) &= \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10}, l_{11}, l_{12}, l_{13}, l_{14}, l_{15}, l_{16}, l_{17}, l_{18}, l_{19}, l_{20}\}, \\ D(\mathcal{L}_4) &= \{L_1 = \{l_1, l_5, l_9, l_{13}, l_{17}\}, L_2 = \{l_1, l_6, l_{10}, l_{14}, l_{18}\}, L_3 = \{l_1, l_7, l_{11}, l_{15}, l_{19}\}, \\ & L_4 = \{l_1, l_8, l_{12}, l_{16}, l_{20}\}, L_5 = \{l_2, l_5, l_{10}, l_{15}, l_{20}\}, L_6 = \{l_2, l_6, l_9, l_{16}, l_{19}\}, \\ & L_7 = \{l_2, l_7, l_{12}, l_{13}, l_{18}\}, L_8 = \{l_2, l_8, l_{11}, l_{14}, l_{17}\}, L_9 = \{l_3, l_5, l_{11}, l_{16}, l_{18}\}, \\ & L_{10} = \{l_3, l_6, l_{12}, l_{15}, l_{17}\}, L_{11} = \{l_3, l_7, l_9, l_{14}, l_{20}\}, L_{12} = \{l_3, l_8, l_{10}, l_{13}, l_{19}\}, \\ & L_{13} = \{l_4, l_5, l_{12}, l_{14}, l_{19}\}, L_{14} = \{l_4, l_6, l_{11}, l_{13}, l_{20}\}, L_{15} = \{l_4, l_7, l_{10}, l_{16}, l_{17}\}, \\ & L_{16} = \{l_4, l_8, l_9, l_{15}, l_{18}\}\}. \end{aligned}$$

We investigate the collinearity of points in the dual of the affine plane of order 4 in the Appendix section. Considering the coloring of the points for incidence structures we defined earlier and the Appendix, it can be seen for  $D(\mathcal{A}_4)$ , which is the dual affine plane of order 4, that  $\chi(D(\mathcal{A}_4)) = 5$ .

Similarly we can see that  $\chi'(D(\mathcal{A}_4)) = 16$ .

Now we can generalize our results as below.

**Theorem 1:** Let  $\mathcal{A}_n = (\mathcal{P}_n, \mathcal{L}_n)$  be a an affine plane of order  $n$ . The point chromatic index of  $\mathcal{A}_n$  is

$$\chi(\mathcal{A}_n) = n^2.$$

*Proof.* Since A1 must hold in every affine plane, we know that any two points must contain within a unique line. So, we don't have any non-collinear point pairs. Therefore,

in an affine plane of order  $n$ , there must be  $n^2$  points that we have to color differently. Thus, the point chromatic index of  $\mathcal{A}_n$  must be equal to  $n^2$ , which is the number of points.

**Theorem 2:** Let  $D(\mathcal{A}_n) = (D(\mathcal{P}_n), D(\mathcal{L}_n))$  be the dual of an affine plane of order  $n$ . The point chromatic index of  $D(\mathcal{A}_n)$  is

$$\chi(D(\mathcal{A}_n)) = n + 1.$$

*Proof.* In an affine plane of order  $n$ , there are  $n + 1$  pencils of parallel lines, and each pencil consists of  $n$  lines. Moreover, lines within the same pencil are pairwise parallel. Let  $l_i, l_j$  be in the same pencil. This means that if two lines  $l_i$  and  $l_j$  belong to the same pencil, they do not intersect in the affine plane. Mathematically, this means that there does not exist a point  $a_k$  (where  $1 \leq k \leq n^2$ ) such that  $a_k$  belongs to both  $l_i$  and  $l_j$ . In other words,

$$(1 \leq k \leq n^2) \wedge (a_k \in l_i \wedge a_k \in l_j)$$

never holds when  $l_i \parallel l_j$ .

Applying the duality principle, points in the dual plane correspond to lines in the original affine plane, and lines in the dual plane correspond to points in the original affine plane. Since no two lines in the same pencil intersect in the affine plane, their corresponding points in the dual plane do not lie on the same line.

Therefore, in the dual plane, the points corresponding to a single pencil in the affine plane form an independent set in terms of graph coloring, meaning they can be assigned the same color. Since there are  $n^2 + n$  lines in the affine plane and every pencil consists of  $n$  lines, it follows that the minimum number of required colors is,

$$\frac{n^2 + n}{n} = n + 1$$

**Theorem 3:** Let  $\mathcal{A}_n = (\mathcal{P}_n, \mathcal{L}_n)$  be a an affine plane of order  $n$ . The line chromatic index of  $\mathcal{A}_n$  is

$$\chi'(\mathcal{A}_n) = n + 1.$$

*Proof.* Due to the Definition 2, the line chromatic index of a structure, we are trying to label the non-intersecting lines in an affine plane. This can very well be perceived as the parallel lines or pencils of that affine plane. Since there are only  $n + 1$  pencils in an affine plane of order  $n$ , the line chromatic index must be also  $n + 1$ .

**Theorem 4:** Let  $D(\mathcal{A}_n) = (D(\mathcal{P}_n), D(\mathcal{L}_n))$  be the dual of an affine plane of order  $n$ . The line chromatic index of  $D(\mathcal{A}_n)$  is

$$\chi'(D(\mathcal{A}_n)) = n^2.$$

*Proof.* Since A1 applies to every affine plane, we know that any two points must be contained in a unique line. By duality, this implies that any two distinct lines in the dual

plane must intersect at a unique point. Hence, there are no parallel (non-intersecting) lines in the dual plane.

Consequently, in the dual of the affine plane of order  $n$ , the  $n^2$  lines form an incidence structure in which every pair of distinct lines meets at a unique point. Therefore, the line chromatic index of  $D(\mathcal{A}_n)$  is equal to the total number of lines in  $D(\mathcal{A}_n)$ .

**Theorem 5:** Let  $\Pi_n = (\mathcal{P}_n, \mathcal{L}_n)$  be a projective plane of order  $n$  and  $D(\Pi_n) = (D(\mathcal{P}_n), D(\mathcal{L}_n))$  be a dual projective plane of order  $n$ . The point chromatic index and the line chromatic index of  $\Pi_n$  and  $D(\Pi_n)$  are:

$$\chi(\Pi_n) = \chi'(\Pi_n) = \chi(D(\Pi_n)) = \chi'(D(\Pi_n)) = n^2 + n + 1.$$

*Proof.* Since P1 and P2 must hold in every projective plane, we know that any pair of distinct points determine a single line and that any two lines intersect at a single point. Therefore we must color  $n^2 + n + 1$  points and  $n^2 + n + 1$  lines with different colors. Thus, the point chromatic index and the line chromatic index of  $\Pi_n$  and  $D(\Pi_n)$  must be equal to the number of points and number of lines in the projective plane of order  $n$ .

#### 4. Conclusion

We defined point and line chromatic indices for affine and projective planes as well as their duals. We also reach some interesting results and comprehensive theorems for such geometric structures.

Our findings contribute to the broader discourse on graph coloring techniques, offering new avenues for exploration in both geometric and combinatorial domains.

Furthermore, our results have practical implications for various fields, including computer science, where graph coloring algorithms play a crucial role in solving optimization problems. By bridging the gap between geometric structures and graph theory, our study paves the way for future research endeavors to leverage the rich interconnections between these disciplines.

In order to observe what we have done in this study in a more general framework, we are doing a similar study for linear spaces and their duals. In this context, we think we can find more unique results when the scope of geometric structures is expanded.

There are also various geometric structures that do not have the condition for the points that any two of them must lie on a unique line. For example, near-linear spaces are one of them, and examining the chromatic indices in such spaces will lead to many different results. Those results holds immense promise for applications in fields like computer networks, image processing, and neural networks, where mathematical irregularities can spark innovation and drive technological advancement.

## References

- [1] Erdős P., On The Combinatorial Problems Which I Would Most Like To See Solved, **Combinatorica** pages 25–42, <https://doi.org/10.1007/BF02579174>, (1981).
- [2] Beutelspacher, A., Jungnickel, D.; Vanstone, S.A., On the chromatic index of a finite projective space, **Geom Dedicata** 32, 313–318, <https://doi.org/10.1007/BF00147923>, (1989).
- [3] Araujo-Pardo, G., Kiss, G., Rubio-Montiel, C., Vázquez-Avila, A., On chromatic indices of finite affine spaces, **arXiv preprint**, arXiv:1711.09031, (2017).
- [4] Xu, L., Feng, T., The chromatic index of finite projective spaces, **J. Combin. Des.**, 31, 432–446, <https://doi.org/10.1002/jcd.21904>, (2023).
- [5] Meszka M., The Chromatic Index of Projective Triple Systems, **J. Combin. Designs**, 21: 531-540, <https://doi.org/10.1002/jcd.21368>, 2013.
- [6] Ozeki, K., Kempe Equivalence Classes of Cubic Graphs Embedded on the Projective Plane, **Combinatorica**, 42 (Suppl 2), 1451–1480, <https://doi.org/10.1007/s00493-021-4330-2>, (2022).
- [7] Hall, M., Projective Planes, **Trans. Am. Math. Soc.** , 54, 229-77, (1943) and correction, 65, 473-4, (1949).
- [8] Batten, L. M., **Combinatorics of Finite Geometries, 2nd edition**, Cambridge University Press: New York, (1997).
- [9] Pickert G., **Projektive Ebenen**, Springer-Verlag: Berlin, Göttingen, Heidelberg, (1955).
- [10] Hughes, D. R., Piper, F. C., **Projective Planes**, Springer: New York, (1973).
- [11] Bennett, M. K., **Affine and Projective Geometry**, John Wiley-Interscience: New York, (1995).
- [12] Faulkner T. E., **Projective Geometry**, Dover: New York, (1949).
- [13] Lam, C. W. H., Thiel, L., Swiercz, S., The Non-Existence of Finite Projective Planes of Order 10. **Canadian Journal of Mathematics**, 41-6: 1117–1123, <https://doi.org/10.4153/CJM-1989-049-4>, (1989).
- [14] Kaya, R., **Projektif Geometri**, Osmangazi Üni. Yayınları: Eskişehir, (2005).
- [15] Jungnickel, D., **Graphs, Networks and Algorithms**, Springer: Augsburg, (2013).
- [16] Bondy A., Murty, U.S.R., **Graph Theory**, Springer: London, (2008).

Appendix

The point chromatic index of  $D(\mathcal{A}_4)$  can easily be calculated using the information given in the table below:

Table for  $D(\mathcal{A}_4)$

$l_1 \vee l_2 = \Phi$	$l_2 \vee l_3 = \Phi$	$l_3 \vee l_4 = \Phi$	$l_4 \vee l_5 = L_{13}$	$l_5 \vee l_6 = \Phi$	$l_6 \vee l_7 = \Phi$	$l_7 \vee l_8 = \Phi$	$l_8 \vee l_9 = L_{16}$	$l_9 \vee l_{10} = \Phi$	$l_{10} \vee l_{11} = \Phi$
$l_1 \vee l_3 = \Phi$	$l_2 \vee l_4 = \Phi$	$l_3 \vee l_5 = L_9$	$l_4 \vee l_6 = L_{14}$	$l_5 \vee l_7 = \Phi$	$l_6 \vee l_8 = \Phi$	$l_7 \vee l_9 = L_{11}$	$l_8 \vee l_{10} = L_{12}$	$l_9 \vee l_{11} = \Phi$	$l_{10} \vee l_{12} = \Phi$
$l_1 \vee l_4 = \Phi$	$l_2 \vee l_5 = L_8$	$l_3 \vee l_6 = L_{10}$	$l_4 \vee l_7 = L_{15}$	$l_5 \vee l_8 = \Phi$	$l_6 \vee l_9 = L_6$	$l_7 \vee l_{10} = L_{15}$	$l_8 \vee l_{11} = L_8$	$l_9 \vee l_{12} = \Phi$	$l_{10} \vee l_{13} = L_{14}$
$l_1 \vee l_5 = L_1$	$l_2 \vee l_6 = L_6$	$l_3 \vee l_7 = L_{11}$	$l_4 \vee l_8 = L_{16}$	$l_5 \vee l_9 = L_1$	$l_6 \vee l_{10} = L_2$	$l_7 \vee l_{11} = L_3$	$l_8 \vee l_{12} = L_4$	$l_9 \vee l_{13} = L_1$	$l_{10} \vee l_{14} = L_2$
$l_1 \vee l_6 = L_2$	$l_2 \vee l_7 = L_7$	$l_3 \vee l_8 = L_{12}$	$l_4 \vee l_9 = L_{16}$	$l_5 \vee l_{10} = L_5$	$l_6 \vee l_{11} = L_{14}$	$l_7 \vee l_{12} = L_7$	$l_8 \vee l_{13} = L_{12}$	$l_9 \vee l_{14} = L_{11}$	$l_{10} \vee l_{15} = L_{10}$
$l_1 \vee l_7 = L_3$	$l_2 \vee l_8 = L_8$	$l_3 \vee l_9 = L_{11}$	$l_4 \vee l_{10} = L_{15}$	$l_5 \vee l_{11} = L_9$	$l_6 \vee l_{12} = L_{10}$	$l_7 \vee l_{13} = L_7$	$l_8 \vee l_{14} = L_8$	$l_9 \vee l_{15} = L_{16}$	$l_{10} \vee l_{16} = L_6$
$l_1 \vee l_8 = L_4$	$l_2 \vee l_9 = L_6$	$l_3 \vee l_{10} = L_{12}$	$l_4 \vee l_{11} = L_{14}$	$l_5 \vee l_{12} = L_{13}$	$l_6 \vee l_{13} = L_{14}$	$l_7 \vee l_{14} = L_{11}$	$l_8 \vee l_{15} = L_{16}$	$l_9 \vee l_{16} = L_6$	$l_{10} \vee l_{17} = L_{10}$
$l_1 \vee l_9 = L_1$	$l_2 \vee l_{10} = L_8$	$l_3 \vee l_{11} = L_9$	$l_4 \vee l_{12} = L_{13}$	$l_5 \vee l_{13} = L_1$	$l_6 \vee l_{14} = L_2$	$l_7 \vee l_{15} = L_8$	$l_8 \vee l_{16} = L_4$	$l_9 \vee l_{17} = L_1$	$l_{10} \vee l_{18} = L_2$
$l_1 \vee l_{10} = L_2$	$l_2 \vee l_{11} = L_8$	$l_3 \vee l_{12} = L_{10}$	$l_4 \vee l_{13} = L_{14}$	$l_5 \vee l_{14} = L_{13}$	$l_6 \vee l_{15} = L_{10}$	$l_7 \vee l_{16} = L_{15}$	$l_8 \vee l_{17} = L_8$	$l_9 \vee l_{18} = L_{16}$	$l_{10} \vee l_{19} = L_6$
$l_1 \vee l_{11} = L_3$	$l_2 \vee l_{12} = L_7$	$l_3 \vee l_{13} = L_{12}$	$l_4 \vee l_{14} = L_{13}$	$l_5 \vee l_{15} = L_8$	$l_6 \vee l_{16} = L_6$	$l_7 \vee l_{17} = L_{15}$	$l_8 \vee l_{18} = L_{16}$	$l_9 \vee l_{19} = L_6$	$l_{10} \vee l_{20} = L_{14}$
$l_1 \vee l_{12} = L_4$	$l_2 \vee l_{13} = L_7$	$l_3 \vee l_{14} = L_{11}$	$l_4 \vee l_{15} = L_{16}$	$l_5 \vee l_{16} = L_9$	$l_6 \vee l_{17} = L_{10}$	$l_7 \vee l_{18} = L_7$	$l_8 \vee l_{19} = L_{12}$	$l_9 \vee l_{20} = L_{11}$	
$l_1 \vee l_{13} = L_1$	$l_2 \vee l_{14} = L_8$	$l_3 \vee l_{15} = L_{10}$	$l_4 \vee l_{16} = L_{15}$	$l_5 \vee l_{17} = L_1$	$l_6 \vee l_{18} = L_2$	$l_7 \vee l_{19} = L_3$	$l_8 \vee l_{20} = L_4$		
$l_1 \vee l_{14} = L_2$	$l_2 \vee l_{15} = L_5$	$l_3 \vee l_{16} = L_9$	$l_4 \vee l_{17} = L_{15}$	$l_5 \vee l_{18} = L_9$	$l_6 \vee l_{19} = L_6$	$l_7 \vee l_{20} = L_{14}$			
$l_1 \vee l_{15} = L_3$	$l_2 \vee l_{16} = L_6$	$l_3 \vee l_{17} = L_{10}$	$l_4 \vee l_{18} = L_{16}$	$l_5 \vee l_{19} = L_{13}$	$l_6 \vee l_{20} = L_{14}$				
$l_1 \vee l_{16} = L_4$	$l_2 \vee l_{17} = L_8$	$l_3 \vee l_{18} = L_9$	$l_4 \vee l_{19} = L_{13}$	$l_5 \vee l_{20} = L_5$					
$l_1 \vee l_{17} = L_1$	$l_2 \vee l_{18} = L_7$	$l_3 \vee l_{19} = L_{12}$	$l_4 \vee l_{20} = L_{14}$						
$l_1 \vee l_{18} = L_2$	$l_2 \vee l_{19} = L_6$	$l_3 \vee l_{20} = L_{11}$							
$l_1 \vee l_{19} = L_3$	$l_2 \vee l_{20} = L_8$								
$l_1 \vee l_{20} = L_4$									
$l_{11} \vee l_{12} = \Phi$	$l_{12} \vee l_{13} = L_{12}$	$l_{13} \vee l_{14} = \Phi$	$l_{14} \vee l_{15} = \Phi$	$l_{15} \vee l_{16} = \Phi$	$l_{16} \vee l_{17} = L_{15}$	$l_{17} \vee l_{18} = L_{15}$	$l_{18} \vee l_{19} = \Phi$	$l_{19} \vee l_{20} = \Phi$	
$l_{11} \vee l_{13} = L_7$	$l_{12} \vee l_{14} = L_8$	$l_{13} \vee l_{15} = \Phi$	$l_{14} \vee l_{16} = \Phi$	$l_{15} \vee l_{17} = L_{10}$	$l_{16} \vee l_{18} = L_9$	$l_{17} \vee l_{19} = \Phi$	$l_{18} \vee l_{20} = \Phi$		
$l_{11} \vee l_{14} = L_{11}$	$l_{12} \vee l_{15} = L_{16}$	$l_{13} \vee l_{16} = \Phi$	$l_{14} \vee l_{17} = L_8$	$l_{15} \vee l_{18} = L_{16}$	$l_{16} \vee l_{19} = L_6$	$l_{17} \vee l_{20} = \Phi$			
$l_{11} \vee l_{15} = L_3$	$l_{12} \vee l_{16} = L_4$	$l_{13} \vee l_{17} = L_1$	$l_{14} \vee l_{18} = L_2$	$l_{15} \vee l_{19} = L_3$	$l_{16} \vee l_{20} = L_4$				
$l_{11} \vee l_{16} = L_{15}$	$l_{12} \vee l_{17} = L_8$	$l_{13} \vee l_{18} = L_7$	$l_{14} \vee l_{19} = L_{13}$	$l_{15} \vee l_{20} = L_5$					
$l_{11} \vee l_{17} = L_{15}$	$l_{12} \vee l_{18} = L_{16}$	$l_{13} \vee l_{19} = L_{12}$	$l_{14} \vee l_{20} = L_{11}$						
$l_{11} \vee l_{18} = L_7$	$l_{12} \vee l_{19} = L_{12}$	$l_{13} \vee l_{20} = L_{13}$							
$l_{11} \vee l_{19} = L_3$	$l_{12} \vee l_{20} = L_4$								
$l_{11} \vee l_{20} = L_{11}$									