ON THE PARANORMED TAYLOR SEQUENCE SPACES

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Abstract. In this paper, the sequence spaces $t_0^r(p)$, $t_1^r(p)$ and $t^r(p)$ of non-absolute type which are the generalization of the Maddox sequence spaces have been introduced and it is proved that the spaces $t_0^r(p)$, $t_1^r(p)$ and $t^r(p)$ are linearly isomorphic to spaces $c_0(p)$, $c(p)$ and $l(p)$, respectively. Furthermore, the $\alpha$- and $\beta$- and $\gamma$-duals of the spaces $t_0^r(p)$, $t_1^r(p)$ and $t^r(p)$ have been computed and their bases have been constructed and some topological properties of these spaces have been investigated. Besides this, the class of matrices $(t_0^r(p) : \mu)$ has been characterized, where $\mu$ is one of the sequence spaces $l_\infty$, $c$ and $c_0$ and derives the other characterizations for the special cases of $\mu$.

1. Introduction

By $w$, we shall denote the space of all real-valued sequences. Any vector subspace of $w$ is called a sequence space. We shall write $l_\infty$, $c$ and $c_0$ for the spaces of all bounded, convergent and null sequences, respectively. Also by $bs$, $cs$, $l_1$ and $l_p$, we denote the spaces of all bounded, convergent, absolutely and $p$-absolutely convergent series, respectively, where $1 < p < \infty$.

A linear topological space $X$ over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha$’s in $\mathbb{R}$ and all $x$’s in $X$, where $\theta$ is the zero vector in the linear space $X$.

Assume here and after that $(p_k)$ be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $l(p)$ were defined by Maddox [12] (see also Simons [14] and
Nakano \[13\]) as follows:
\[
\ell_\infty(p) = \{x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty\},
\]
\[
c(p) = \{x = (x_k) \in w : \lim_{k \to \infty} |x_k - l_k|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\},
\]
\[
c_0(p) = \{x = (x_k) \in w : \lim_{k \to \infty} |x_k|^{p_k} = 0\},
\]
\[
\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\},
\]
which are the complete spaces paranormed by
\[
g_1(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/L} \iff \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/L},
\]
respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \(\infty\). By \(F\) and \(N_k\), we shall denote the collection of all finite subsets of \(\mathbb{N}\) and the set of all \(n \in \mathbb{N}\) such that \(n \geq k\), respectively. We shall assume throughout that \(p_k^{-1} + (p'_k)^{-1} = 1\) provided \(1 < \inf p_k \leq H < \infty\).

Let \(\lambda, \mu\) be any two sequence spaces and \(A = (a_{nk})\) be an infinite matrix of real numbers \(a_{nk}\), where \(n, k \in \mathbb{N}\). Then, we say that \(A\) defines a matrix mapping from \(\lambda\) into \(\mu\), and we denote it by \(A : \lambda \to \mu\), if for every sequence \(x = (x_k) \in \lambda\), the sequence \(Ax = \{(Ax)_n\}\), the \(A\)–transform of \(x\), is in \(\mu\), where
\[
(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}).
\]

By \((\lambda : \mu)\), we denote the class of all matrices \(A\) such that \(A : \lambda \to \mu\). Thus, \(A \in (\lambda : \mu)\) if and only if the series on the right-hand side of (1.1) converges for each \(n \in \mathbb{N}\) and every \(x \in \lambda\), and we have \(Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu\) for all \(x \in \mu\). A sequence \(x\) is said to be \(A\)–summable to \(\alpha\) if \(Ax\) converges to \(\alpha\) which is called the \(A\)–limit of \(x\).

2. **The Sequence Spaces \(t_0^r(p)\), \(t_c^r(p)\) and \(t_f(p)\) of Non-Absolute Type**

In this section, we define the sequence spaces \(t_0^r(p)\), \(t_c^r(p)\) and \(t_f(p)\), and prove that \(t_0^r(p)\), \(t_c^r(p)\) and \(t_f(p)\) are the complete paranormed linear spaces.

For a sequence space \(\lambda\), the matrix domain \(\lambda_A\) of an infinite matrix \(A\) is defined by
\[
X_A = \{x = (x_k) \in w : Ax \in X\}.
\]

In [5], Choudhary and Mishra have defined the sequence space \(\overline{\ell}(p)\) which consists of all sequences such that \(S–\)transforms are in \(\ell(p)\), where \(S = (s_{nk})\) is defined by
\[
s_{nk} = \begin{cases} 
1, & 0 \leq k \leq n, \\
0, & k > n.
\end{cases}
\]
Başar and Altay [3] have studied the space \(bs(p)\) which is formerly defined by Başar in [4] as the set of all series whose sequences of partial sums are in \(\ell_\infty(p)\).

More recently, Altay and Başar have studied the sequence spaces \(r^*\ell(p)\), \(r^{*\infty}(p)\) in [1] and \(r^*_c(p), r^{*\infty}_0(p)\) in [2] which are derived by the Riesz means from the sequence spaces \(\ell(p)\), \(\ell_\infty(p)\), \(c(p)\) and \(c_0(p)\) of Maddox, respectively.
With the notation of (2.1), the spaces $\ell(p), bs(p), r^t(p), r_0^t(p), r_0^c(p)$ and $r_0^e(p)$ may be redefined by

$$
\ell(p) = [\ell(p)]_S, bs(p) = [\ell_\infty(p)]_S, r^t(p) = [\ell(p)]_T^r,
$$

$$
r_0^t(p) = [\ell_\infty(p)]_T^r, r_0^c(p) = [c(p)]_T^r, r_0^e(p) = [c_0(p)]_T^r.
$$

In [6], Demiriz and Çakan have defined the sequence spaces $c_0^r(u, p)$ and $c^r(u, p)$ which consists of all sequences such that $E^{r,u}$- transforms are in $c_0(p)$ and $c(p)$, respectively

$$
e_{nk}^r(u) = \begin{cases}
\binom{n}{k}(1 - r)^{n-k}r^k u_k & (0 \leq k \leq n), \\
0 & (k > n)
\end{cases}
$$

for all $k, n \in \mathbb{N}$ and $0 < r < 1$.

In [9], the Taylor sequence spaces $t_0^r$ and $t^r$ of non-absolute type, which are the matrix domains of Taylor mean $T^r$ of order $r$ in the sequence spaces $c_0$ and $c$, respectively, are introduced, some inclusion relations and Schauder basis for the spaces $t_0^r$ and $t^r$ are given, and the $\alpha-, \beta-$ and $\gamma-$ duals of those spaces are determined. The main purpose of this paper is to introduce the sequence spaces $t_0^r(p), t^r_c(p)$ and $t^r(p) of nonabsolute type which are the set of all sequences whose $T^r-$transforms are in the spaces $c_0(p), c(p)$ and $\ell(p)$, respectively; where $T^r$ denotes the matrix

$$
T^r = \{t_{nk}^r\}
$$

defined by

$$
t_{nk}^r = \begin{cases}
\binom{k}{n}(1 - r)^{n+1}r^k u_k & (k \geq n), \\
0 & (0 \leq k < n)
\end{cases}
$$

where $0 < r < 1$. Also, we have constructed the basis and computed the $\alpha-$, $\beta-$ and $\gamma-$duals and investigated some topological properties of the spaces $t_0^r(p), t^r_c(p)$ and $t^r(p)$.

Following Choudhary and Mishra [5], Başar and Altay [3], Altay and Başar [1, 2], Demiriz [6], Kirişçi [9], we define the sequence spaces $t_0^r(p), t^r_c(p)$ and $t^r(p)$, as the sets of all sequences such that $T^r-$transforms of them are in the spaces $c_0(p), c(p)$ and $\ell(p)$, respectively, that is,

$$
t_0^c(p) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left| \sum_{k=n}^\infty \binom{k}{n}(1 - r)^{n+1}r^k x_k \right|^p_n = 0 \right\},
$$

$$
t^c_c(p) = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \left| \sum_{k=n}^\infty \binom{k}{n}(1 - r)^{n+1}r^k x_k - l \right|^p_n = 0 \text{ for some } l \in \mathbb{R} \right\}
$$

and

$$
t^r(p) = \left\{ x = (x_k) \in w : \left| \sum_{k=n}^\infty \sum_{l=n}^\infty \binom{k}{n}(1 - r)^{n+1}r^k x_k \right|^p_n < \infty \right\}.
$$

In the case $(p_n) = e = (1, 1, 1, \ldots)$, the sequence spaces $t_0^c(p), t^c_c(p)$ and $t^r(p)$ are, respectively, reduced to the sequence spaces $t_0^c$, $t^c_c$ which are introduced by Kirişçi [9] and $t^r(p)$ is reduced to the sequence space $t^r_c$. With the notation of (2.1), we may redefine the spaces $t_0^c(p), t^c_c(p)$ and $t^r(p)$ as follows:

$$
t_0^c(p) = [c_0(p)]_{T^r}, t^c_c(p) = [c(p)]_{T^r} \text{ and } t^r(p) = [\ell(p)]_{T^r}.
$$
Define the sequence \( y_k = \{y_k(r)\} \), which will be frequently used, as the \( T_r^{\tau} \)-transform of a sequence \( x = (x_k) \), i.e.,
\[
y_k := \sum_{k=n}^{\infty} \binom{k}{n} (1 - r)^{n+1} r^{k-n} x_k \quad \text{for all } k \in \mathbb{N}.
\]

(2.3)

Now, we may begin with the following theorem which is essential in the text.

**Theorem 2.1.** \( t_0^r(p) \) and \( t_c^r(p) \) are the complete linear metric space paranormed by \( g \), defined by
\[
g(x) = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} x_j \right|^{p_k/L}.
\]

Also, \( t_p^r(p) \) is the complete linear metric space paranormed by \( h \), defined by
\[
h(x) = \left( \left| \sum_{j=0}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M}.
\]

**Proof.** Since the proof is similar for \( t_0^r(p) \) and \( t_c^r(p) \), we give the proof only for the space \( t_0^r(p) \). The linearity of \( t_0^r(p) \) with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for \( x, z \in t_0^r(p) \) (see Maddox [11, p.30])
\[
\sup_{n \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} (x_j + z_j) \right|^{p_k/L} \leq \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} x_j \right|^{p_k/L} + \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} z_j \right|^{p_k/L}
\]

and for any \( \alpha \in \mathbb{R} \) (see [14])
\[
|\alpha|^{p_k} \leq \max \{1, |\alpha|^{L} \}.
\]

(2.6)

It is clear that \( g(\theta) = 0 \) and \( g(x) = g(-x) \) for all \( x \in t_0^r(p) \). Again the inequalities (2.5) and (2.6) yield the subadditivity of \( g \) and
\[
g(\alpha x) \leq \max \{1, |\alpha|^{L} \} g(x).
\]

Let \( \{x^n\} \) be any sequence of the points \( x^n \in t_0^r(p) \) such that \( g(x^n - x) \to 0 \) and \( (\alpha_n) \) also be any sequence of scalars such that \( \alpha_n \to \alpha \). Then, since the inequality \( g(x^n) \leq g(x) + g(x^n - x) \)
holds by the subadditivity of \( g \), \( \{g(x^n)\} \) is bounded and we thus have
\[
g(\alpha^n x^n - \alpha x) = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} (\alpha^n x^n_j - \alpha x_j) \right|^{p_k/L}
\]
\[
\leq |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x),
\]
which tends to zero as \( n \to \infty \). This means that the scalar multiplication is continuous. Hence, \( g \) is paranorm on the space \( t_0^r(p) \).
It remains to prove the completeness of the space $t_0^p(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $t_0^p(p)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \frac{\epsilon}{2}$$

for all $i, j > n_0(\epsilon)$. Using the definition of $g$ we obtain for each fixed $k \in \mathbb{N}$ that

$$(2.7) \quad |(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} \leq \sup_{k \in \mathbb{N}} |(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} < \frac{\epsilon}{2}$$

for every $i, j > n_0(\epsilon)$ which leads to the fact that $\{(T^r x^i)_k, (T^r x^j)_k, (T^r x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(T^r x)_k \to (T^r x)_k$ as $i \to \infty$. Using these infinitely many limits $(T^r x)_0, (T^r x)_1, \ldots$, we define the sequence $\{(T^r x)_0, (T^r x)_1, \ldots\}$. From (2.7) with $j \to \infty$, we have

$$(2.8) \quad |(T^r x^i)_k - (T^r x)_k|^{p_k/L} \leq \frac{\epsilon}{2} (i, j > n_0(\epsilon))$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in t_0^p(p)$ for each $i \in \mathbb{N}$, there exists $k_0(\epsilon) \in \mathbb{N}$ such that

$$(2.9) \quad |(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2}$$

for every $k \geq k_0(\epsilon)$ and for each fixed $i \in \mathbb{N}$. Therefore, taking a fixed $i > n_0(\epsilon)$ we obtain by (2.8) and (2.9) that

$$|(T^r x)_k|^{p_k/L} \leq |(T^r x)_k - (T^r x^i)_k|^{p_k/L} + |(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2}$$

for every $k > k_0(\epsilon)$. This shows that $x \in t_0^p(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $t_0^p(p)$ is complete and this concludes the proof.

Now, $t'(p)$ is the complete linear metric space paranormed by $h$ defined by (2.4). It is easy to see that the space $t'(p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm $h$ defined by (2.4).

It is clear that $h(\theta) = 0$ where $\theta = (0, 0, 0, \ldots)$ and $h(x) = h(-x)$ for all $x \in t'(p)$. 
Let $x, y \in t'(p)$; then by Minkowski’s inequality we have

$$h(x + y) = \left( \sum_{k=0}^{\infty} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} (x_j + y_j) \right]^{p_k} \right)^{1/M}$$

$$= \left( \sum_{k=0}^{\infty} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right]^{p_k} \right)^{1/M}$$

$$\leq \left( \sum_{k=0}^{\infty} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right]^{p_k} \right)^{1/M}$$

$$+ \left( \sum_{k=0}^{\infty} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} y_j \right]^{p_k} \right)^{1/M}$$

(2.10)

$$= h(x) + h(y)$$

Let $\{x^n\}$ be any sequence of the points $x^n \in t'(p)$ such that $h(x^n - x) \to 0$ and $(\lambda_n)$ also be any sequence of scalars such that $\lambda_n \to \lambda$. We observe that

$$h(\lambda^n x^n - \lambda x) \leq h[(\lambda^n - \lambda)(x^n - x)]$$

(2.11)

$$+ h[\lambda(x^n - x)]$$

$$+ h[(\lambda^n - \lambda)x].$$

It follows from $\lambda_n \to \lambda (n \to \infty)$ that $|\lambda^n - \lambda| < 1$ for all sufficiently large $n$; hence

$$\lim_{n \to \infty} h[(\lambda_n - \lambda)(x^n - x)] \leq \lim_{n \to \infty} h(x^n - x) = 0.$$  

(2.12)

Furthermore, we have

$$\lim_{n \to \infty} h[\lambda(x^n - x)] \leq \max\{1, |\lambda|^M\} \lim_{n \to \infty} h(x^n - x) = 0.$$  

(2.13)

Also, we have

$$\lim_{n \to \infty} h[(\lambda_n - \lambda)x] \leq |\lambda_n - \lambda| h(x) = 0.$$  

(2.14)

Then, we obtain from (2.11), (2.12), (2.13) and (2.14) that $h(\lambda^n x^n - \lambda x) \to 0$, as $n \to \infty$. This shows that $h$ is a paranorm on $t'(p)$.

Furthermore, if $h(x) = 0$, then $\left( \sum_{k=0}^{\infty} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right]^{p_k} \right)^{1/M} = 0$.

Therefore $\left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} = 0$ for each $k \in \mathbb{N}$. Since $0 < r < 1$, we have $\binom{j}{k} (1-r)^{k+1} r^{j-k} > 0$. Then, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. That is, $x = \theta$. This shows that $h$ is a total paranorm.

Now, we show that $t'(p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in the space $t'(p)$, where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}$. Then, for a given $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that $h(x^n - x^m) < \epsilon$ for all $n, m > n_0(\epsilon)$. Since for
each fixed \( k \in \mathbb{N} \) that

\[
| (T^r x^n)_k - (T^r x^m)_k | \leq \left[ \sum_k | (T^r x^n)_k - (T^r x^m)_k |^p \right]^{1/p} \\
= h(x^n - x^m) < \epsilon
\]

(2.15)

for every \( n, m > n_0(\epsilon) \), \( \{(T^r x^0)_k, (T^r x^1)_k, (T^r x^2)_k, \ldots\} \) is a Cauchy sequence of real numbers for every fixed \( k \in \mathbb{N} \). Since \( \mathbb{R} \) is complete, it converges, say \( (T^r x^n)_k \to (T^r x)_k \) as \( n \to \infty \). Using these infinitely many limits \( (T^r x)_0, (T^r x)_1, \ldots \), we define the sequence \( \{(T^r x)_0, (T^r x)_1, \ldots\} \). For each \( K \in \mathbb{N} \) and \( n, m > n_0(\epsilon) \)

(2.16)

\[
\left[ \sum_{k=0}^{K} | (T^r x^n)_k - (T^r x^m)_k |^p \right]^{1/p} \leq h(x^n - x^m) < \epsilon.
\]

By letting \( m, K \to \infty \), we have for \( n > n_0(\epsilon) \) that

(2.17)

\[
h(x^n - x) = \left[ \sum_k | (T^r x^n)_k - (T^r x)_k |^p \right]^{1/p} < \epsilon.
\]

This shows that \( x^n - x \in t^r(p) \). Since \( t^r(p) \) is a linear space, we conclude that \( x \in t^r(p) \); it follows that \( x^n \to x \), as \( n \to \infty \) in \( t^r(p) \), thus we have shown that \( t^r(p) \) is complete.

Note that the absolute property does not hold on the spaces \( t_0^r(p) \), \( t_c^r(p) \) and \( t^r(p) \), since there exists at least one sequence in the spaces \( t_0^r(p) \), \( t_c^r(p) \) and \( t^r(p) \) and such that \( g(x) \neq g(|x|) \), where \( |x| = (|x_k|) \). This says that \( t_0^r(p) \), \( t_c^r(p) \) and \( t^r(p) \) are the sequence spaces of non-absolute type.

**Theorem 2.2.** The sequence spaces \( t_0^r(p) \), \( t_c^r(p) \) and \( t^r(p) \) of non-absolute type are linearly isomorphic to the spaces \( c_0(p) \), \( c(p) \) and \( \ell(p) \), respectively, where \( 0 < p_k \leq H < \infty \).

**Proof.** To avoid repetition of similar statements, we give the proof only for \( t_0^r(p) \). We should show the existence of a linear bijection between the spaces \( t_0^r(p) \) and \( c_0(p) \). With the notation of (2.3), define the transformation \( T \) from \( t_0^r(p) \) and \( c_0(p) \) by \( x \to y = Tx \). The linearity of \( T \) is trivial. Furthermore, it is obvious that \( x = \theta \) whenever \( Tx = \theta \), and hence \( T \) is injective.

Let \( y \in c_0(p) \) and define the sequence

\[
x_k(r) := \sum_{j=k}^{\infty} \binom{j}{k} (-r)^{-j} (1 - r)^{-j+1} y_j; \quad k \in \mathbb{N}.
\]

Then, we have

\[
g(x) = \sup_{k \in \mathbb{N}} \left[ \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{j+1} r^{-j} x_j \right]^{p_k/L} = \sup_{k \in \mathbb{N}} |y_k|^{p_k/L} = g_1(y) < \infty.
\]

Thus, we have that \( x \in t_0^r(p) \) and consequently \( T \) is surjective. Hence, \( T \) is a linear bijection and this says that the spaces \( t_0^r(p) \) and \( c_0(p) \) are linearly isomorphic, as was desired. \( \square \)

**Theorem 2.3.** Convergence in \( t^r(p) \) is stronger than coordinate-wise convergence.
Proof. First we show that \( h(x^n - x) \to 0 \), as \( n \to \infty \) implies \( x^n_k \to x_k \); as \( n \to \infty \) for every \( k \in \mathbb{N} \). We fix \( k \), then we have

\[
\lim_{n \to \infty} \left| \sum_{n=k}^{\infty} \binom{n}{k} (1 - r)^{k+1} r^{n-k} [x^{(n)}_k - x_k] \right|^p \\
\leq \lim_{n \to \infty} \sum_k \sum_{n=k}^{\infty} \binom{n}{k} (1 - r)^{k+1} r^{n-k} [x^{(n)}_k - x_k] \\
\leq \lim_{n \to \infty} [h(x^n - x)]^M = 0.
\]

(2.18)

Hence, we have for \( k = 0 \) that

\[
\lim_{n \to \infty} \left| \sum_{n=0}^{\infty} (1 - r)r^n [x^{(n)}_0 - x_0] \right| = 0
\]

which gives the fact that \( |x^{(n)}_0 - x_0| \to 0 \), as \( n \to \infty \). Similarly, for each \( k \in \mathbb{N} \), we have \( x^n_k \to x_k \); as \( n \to \infty \).

A sequence space \( \lambda \) with a linear topology is called a \( K \)-space provided each of the maps \( p_i : \lambda \to \mathbb{C} \) defined by \( p_i(x) = x_i \) is continuous for all \( i \in \mathbb{N} \), where \( \mathbb{C} \) denotes the complex field. A \( K \)-space \( \lambda \) is called an \( FK \)-space provided \( \lambda \) is complete linear metric space. An \( FK \)-space whose topology is normable is called a \( BK \)-space. Given a \( BK \)-space \( \lambda \supset \phi \), we denote the \( n \) th section of a sequence \( x = (x_k) \in \lambda \) by \( x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)} \), and we say that \( x = (x_k) \) has the property \( AK \) if \( \lim_{n \to \infty} ||x - x^{[n]}||_\lambda = 0 \). If \( AK \) property holds for every \( x \in \lambda \), then we say that the space \( \lambda \) is called \( AK \)-space (cf. [7]). Now, we may give the following. \( \square \)

**Theorem 2.4.** The space \( t'(p) \) has \( AK \).

Proof. For each \( x = (x_k) \in t'(p) \), we put

\[
x^{<m>} = \sum_{k=0}^{m} x_k e^{(k)}, \forall m \in \{1, 2, \ldots \}.
\]

(2.19)

Let \( \epsilon > 0 \) and \( x \in t'(p) \) be given. Then, there is \( N = N(\epsilon) \in \mathbb{N} \) such that

\[
\sum_{k=N}^{\infty} \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} x_j \left| \right|^p < \epsilon^M.
\]

(2.20)

Then we have for all \( m \geq N \),

\[
\begin{aligned}
\sum_{k=m}^{\infty} x_k e^{(k)} &= \sum_{k=0}^{m} x_k e^{(k)} \\
&= h \left( x - \sum_{k=0}^{m} x_k e^{(k)} \right) \\
&= \left( \sum_{k=m+1}^{\infty} \sum_{j=k}^{\infty} \binom{j}{k} (1 - r)^{k+1} r^{j-k} x_j \right)^{p_k} \frac{1}{M} < \epsilon.
\end{aligned}
\]

(2.21)

This shows that \( x = \sum_k x_k e^{(k)} \).
Now, we have to show that this representation is unique. We assume that $x = \sum_k \lambda_k e^{(k)}$. Then for each $k$,

$$
\left(\sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} \lambda_j - \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right)^{p_k} \frac{1}{M} \leq \left(\sum_k \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} \lambda_j - \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right)^{p_k} \frac{1}{M} $$

(2.22)

Hence, $\sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j = \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j$ for each $j$. Then, $\lambda_j = x_j$ for each $j$. Therefore, the representation is unique. □

3. The Basis for the Spaces $t_0^r(p)$, $t_r^c(p)$ and $t^r(p)$

Let $(\lambda, h)$ be a paranormed space. Recall that a sequence $(b_k)$ of the elements of $\lambda$ is called a basis for $\lambda$ if and only if, for each $x \in \lambda$, there exists a unique sequence $(\alpha_k)$ of scalars such that

$$
h \left( x - \sum_{k=0}^{n} \alpha_k b_k \right) \to 0 \text{ as } n \to \infty.
$$

The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$, and written as $x = \sum \alpha_k b_k$. Since it is known that the matrix domain $\lambda_A$ of a sequence space $\lambda$ has a basis if and only if $\lambda$ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [8, Remark 2.4]), we have the following. Because of the isomorphism $T$ is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of those spaces $\ell_0^r(p)$, $c(p)$ and $\ell(p)$ are the basis of the new spaces $t_0^r(p)$, $t_r^c(p)$ and $t^r(p)$, respectively. Therefore, we have the following:

**Theorem 3.1.** Let $\lambda_k(r) = (T^r x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(r) = \{b^{(k)}(r)\}_{k \in \mathbb{N}}$ of the elements of the space $t_0^r(p)$, $t_r^c(p)$ and $t^r(p)$ by

$$
b^{(k)}(r) = \left\{ \begin{array}{ll}
\binom{k}{n}(1-r)^{-(k+1)}(-r)^{k-n} & , \quad k \geq n \\
0 & , \quad 0 \leq k < n
\end{array} \right.
$$

for every fixed $k \in \mathbb{N}$. Then

(a): The sequence $\{b^{(k)}(r)\}_{k \in \mathbb{N}}$ is a basis for the space $t_0^r(p)$, and any $x \in t_0^r(p)$ has a unique representation of the form

$$
x = \sum_k \lambda_k(r) b^{(k)}(r),
$$

(b): The set $e, b^{(1)}(r), b^{(2)}(r), \ldots$ is a basis for the space $t_r^c(p)$, and any $x \in t_r^c(p)$ has a unique representation of the form

$$
x = le + \sum_k [\lambda_k(r) - l] b^{(k)}(r),
$$

where $l = \lim_{k \to \infty} (T^r x)_k$. 
(c): The sequence \( \{b^{(k)}(r)\}_{k \in \mathbb{N}} \) is a basis for the space \( t^r(p) \), and any \( x \in t^r(p) \) has a unique representation of the form
\[
x = \sum_{k} \lambda_k(r)b^{(k)}(r).
\]

4. The \( \alpha-, \beta- \) and \( \gamma- \) Duals of the Spaces \( t_0^r(p), t_0^r(p) \) and \( t^r(p) \)

In this section, we state and prove the theorems determining the \( \alpha-, \beta- \) and \( \gamma- \) duals of the sequence spaces \( t_0^r(p), t_0^r(p) \) and \( t^r(p) \) of non-absolute type.

We shall firstly give the definition of \( \alpha-, \beta- \) and \( \gamma- \) duals of sequence spaces and after quoting the lemmas which are needed in proving the theorems given in Section 4.

The set \( S(\lambda, \mu) \) defined by
\[
(4.1) \quad S(\lambda, \mu) = \{ z = (z_k) \in w : xz = (x_kz_k) \in \mu \text{ for all } x = (x_k) \in \lambda \}
\]
is called the multiplier space of the sequence spaces \( \lambda \) and \( \mu \). One can easily observe for a sequence space \( \nu \) with \( \lambda \supset \nu \supset \mu \) that the inclusions
\[
S(\lambda, \mu) \subset S(\nu, \mu) \text{ and } S(\lambda, \mu) \subset S(\lambda, \nu)
\]
hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space \( \lambda \), which are respectively denoted by \( \lambda^\alpha \), \( \lambda^\beta \) and \( \lambda^\gamma \) are defined by
\[
\lambda^\alpha = S(\lambda, \ell_1), \lambda^\beta = S(\lambda, \ell_2) \text{ and } \lambda^\gamma = S(\lambda, bs).
\]
The alpha-, beta- and gamma-duals of a sequence space are also referred as Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively.

For to give the alpha-, beta- and gamma-duals of the spaces \( t_0^r(p), t_0^r(p) \) and \( t^r(p) \) of non-absolute type, we need the following Lemma:

**Lemma 4.1.** [7] Let \( A = (a_{nk}) \) be an infinite matrix. Then, the following statements hold

(i): \( A \in (c_0(p) : \ell(q)) \) if and only if
\[
(4.2) \quad \sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K} a_{nk} M^{-1/p_k} \right|^{q_n} < \infty, \ \exists M \in \mathbb{N}_2.
\]

(ii): \( A \in (c(p) : \ell(q)) \) if and only if \( (4.2) \) holds and
\[
(4.3) \quad \left| \sum_{n} \left| \sum_{k} a_{nk} \right|^{q_n} < \infty.
\]

(iii): \( A \in (c_0(p) : c(q)) \) if and only if
\[
(4.4) \quad \sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2,
\]
\[
(4.5) \quad \exists (a_{nk}) \subset \mathbb{R} \ni \lim_{n \to \infty} |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k \in \mathbb{N},
\]
\[
(4.6) \quad \exists (a_{nk}) \subset \mathbb{R} \ni \sup_{n \in \mathbb{N}} N^{1/q_n} \sum_{k} |a_{nk} - \alpha_k| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2 \text{ and } \forall N \in \mathbb{N}_1.
\]
Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold

(i): $A \in (c(p) : c(q))$ if and only if $(4.4), (4.5), (4.6)$ hold and

$$\exists \alpha \in \mathbb{R} \exists \lim_{n \to \infty} \sum_{k} a_{nk} - \alpha^{q_{n}} = 0. \tag{4.7}$$

(ii): $A \in (c_{0}(p) : \ell_{1})$ if and only if

$$\sup_{n \in \mathbb{N}} \left( \sum_{k} |a_{nk}| M^{-1/p_{k}} \right)^{q_{n}} < \infty, \exists M \in \mathbb{N}_{2}. \tag{4.8}$$

(iii): $A \in (c_{0}(p) : \ell_{\infty})$ if and only if

$$\sup_{a, \beta} \left( \sum_{k} |a_{nk}| M^{-1/p_{k}} \right) < \infty. \tag{4.9}$$

(iv): $A \in (c_{0}(p) : \ell_{1})$ if and only if

$$\sup_{n \in \mathbb{N}} \left( \sum_{k} |a_{nk}| M^{-1/p_{k}} \right)^{q_{n}} < \infty, \exists M \in \mathbb{N}_{2}. \tag{4.10}$$

Lemma 4.2. [10] Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold

(i): $A \in (\ell(p) : \ell_{\infty})$ if and only if

(a): $0 < p_{k} \leq 1$ for all $k \in \mathbb{N}$. Then,

$$\sup_{n, k \in \mathbb{N}} |a_{nk}| p_{k} < \infty. \tag{4.11}$$

(b): $1 < p_{k} \leq H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that

$$\sup_{n \in \mathbb{N}} \sum_{k} |a_{nk} M^{-1/p_{k}}|^{p_{k}'} < \infty. \tag{4.12}$$

(ii): $0 < p_{k} \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.11) and (4.12) hold, and

$$\lim_{n \to \infty} a_{nk} = \beta_{k}, \forall k \in \mathbb{N}. \tag{4.13}$$

Theorem 4.1. Let $K \in \mathcal{F}$ and $K^{*} = \{ k \in \mathbb{N} : n \geq k \} \cap K$ for $K \in \mathcal{F}$. Define the sets $T_{1}^{1}(p), T_{2}^{1}, T_{3}(p)$ and $T_{4}(p)$ as follows:

$$T_{1}^{1}(p) = \bigcup_{M > 1} \left\{ a = (a_{k}) \in w : \sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K^{*}} c_{nk} M^{-1/p_{k}} \right|^{q_{n}} < \infty \right\},$$

$$T_{2}^{1} = \left\{ a = (a_{k}) \in w : \sum_{n} \left| \sum_{k=0}^{n} c_{nk} \right| \text{ exists for each } n \in \mathbb{N} \right\},$$

$$T_{3}(p) = \bigcup_{M > 1} \left\{ a = (a_{k}) \in w : \sup_{N \in \mathcal{F}} \sum_{k} \left| \sum_{n \in N} c_{nk} M^{-1/p_{k}'} \right|^{p_{k}'} < \infty \right\},$$

$$T_{4}(p) = \left\{ a = (a_{k}) \in w : \sup_{N \in \mathcal{F}} \sum_{k \in \mathbb{N}} \left| \sum_{n \in N} c_{nk} \right|^{p_{k}} < \infty \right\},$$
where the matrix $C(r) = (c_{nk}^r)$ defined by

$$
(4.14) \quad c_{nk}^r = \begin{cases} 
\binom{n}{k} (-r)^{k-n} (1-r)^{-(k+1)} a_n, & (k \geq n), \\
0, & (0 \leq k < n).
\end{cases}
$$

Then, $[t_0^r(p)]^\alpha = T_1^r(p)$, $[t_0^r(p)]^\alpha = T_1^r(p) \cap T_2^r$ and

$$
(4.15) \quad [t^r(p)]^\alpha = \begin{cases} 
T_3(p), & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\
T_4(p), & 0 < p_k \leq 1, \forall k \in \mathbb{N}.
\end{cases}
$$

**Proof.** We chose the sequence $a = (a_k) \in w$. We can easily derive that with the (2.3) that

$$
(4.16) \quad a_n x_n = \sum_{k=n}^{\infty} \binom{n}{k} (-r)^{k-n} (1-r)^{-(k+1)} a_n y_k = (C^r y)_n, \quad (n \in \mathbb{N}).
$$

for all $k, n \in \mathbb{N}$, where $C^r = (c_{nk}^r)$ defined by (4.14). It follows from (4.16) that $ax = (a_n x_n) \in \ell_2$ whenever $x \in t_0^r(p)$ if and only if $Cy \in \ell_1$ whenever $y \in c_0(p)$. This means that $a = (a_n) \in [t_0^r(p)]^\alpha$ if and only if $C \in (c_0(p) : \ell_1)$. Then, we derive by (4.2) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[t_0^r(p)]^\alpha = T_1^r(p)$.

Using the (4.3) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.16), the proof of the $[t_0^r(p)]^\alpha = T_1^r(p) \cap T_2$ can also be obtained in a similar way. Also, using the (4.9),(4.10) and (4.16), the proof of the

$$
[t^r(p)]^\alpha = \begin{cases} 
T_3(p), & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\
T_4(p), & 0 < p_k \leq 1, \forall k \in \mathbb{N},
\end{cases}
$$

can also be obtained in a similar way. \hfill \square

**Theorem 4.2.** The matrix $D(r) = (d_{nk}^r)$ is defined by

$$
(4.17) \quad d_{nk}^r = \begin{cases} 
\sum_{k=0}^{n} \binom{n}{k} (-r)^{n-k} (1-r)^{-(n+1)} a_k, & (0 \leq k \leq n) \\
0, & (k > n)
\end{cases}
$$

for all $k, n \in \mathbb{N}$. Define the sets $T_5^r(p)$, $T_6^r$, $T_7^r$, $T_8(p)$, $T_9(p)$ and $T_{10}(p)$ as follows:

$$
T_5^r(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sum_{k=M}^{\infty} \left| d_{nk}^r M^{-1/p_k} \right| < \infty \right\},
$$

$$
T_6^r = \left\{ a = (a_k) \in w : \lim_{n \to \infty} |d_{nk}^r| \text{ exists for each } k \in \mathbb{N} \right\},
$$

$$
T_7^r = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} |d_{nk}^r| \text{ exists} \right\},
$$

$$
T_8(p) = \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}^r M^{-1/p_k} | < \infty \right\},
$$

$$
T_9(p) = \left\{ a = (a_k) \in w : d_{nk} < \infty \right\},
$$

$$
T_{10}(p) = \left\{ a = (a_k) \in w : \sup_{n, k \in \mathbb{N}} |d_{nk}^r|^{p_k} < \infty \right\}.
$$

Then, $[t_0^r(p)]^\beta = T_5^r(p) \cap T_6^r$, $[t_0^r(p)]^\beta = [t_0^r(p)]^\beta \cap T_7^r$ and

$$
(4.18) \quad [t^r(p)]^\beta = \begin{cases} 
T_8(p) \cap T_9(p), & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\
T_9(p) \cap T_{10}(p), & 0 < p_k \leq 1, \forall k \in \mathbb{N}.
\end{cases}
$$
Proof. We give the proof again only for the space $t_0^c(p)$. Consider the equation

$$
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \sum_{j=0}^{\infty} \binom{k}{j} (r)^{k-j}(1-r)^{-(k+1)} y_k \right] a_k
$$

(4.19)

$$
= \sum_{k=0}^{n} \left[ \sum_{j=0}^{k} \binom{k}{j} (r)^{k-j}(1-r)^{-(k+1)} a_j \right] y_k = (D^r y)_n,
$$

where $D^r = (d_{nk}^r)$ defined by (4.17). Thus, we deduce from (4.19) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in t_0^c(p)$ if and only if $D^r y \in c$ whenever $y = (y_k) \in c_0(p)$. That is to say that $a = (a_k) \in [t_0^c(p)]^{\beta}$ if and only if $D^r \in (c_0(p) : c)$. Therefore, we derive from (4.4), (4.5) and (4.6) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[t_0^c(p)]^{\beta} = T_0^c(u,p) \cap T_0^c(u)$.

Using the (4.4), (4.5), (4.6) and (4.7) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.19), the proofs of the $[t_0^c(p)]^{\beta} = [t_0^c(p)]^{\beta} \cap T_0^c$ can also be obtained in a similar way. Also, using the (4.11), (4.12), (4.13) and (4.19), the proofs of the

$$
[t^c(p)]^{\beta} = \begin{cases} 
T_0^c(p) \cap T_0^c(p), & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\
T_0^c(p) \cap T_10(p), & 0 < p_k \leq 1, \forall k \in \mathbb{N}
\end{cases}
$$

can also be obtained in a similar way. \hfill \Box

Theorem 4.3. Define the set $T_0^c(u)$ by

$$
T_0^c(u) = \left\{ a = (a_k) \in w : \left\{ \sum_{j=0}^{k} \binom{k}{j} (r)^{k-j}(1-r)^{-(k+1)} a_j \right\} \in bs \right\}.
$$

Then, $[t_0^c(p)]^{\gamma} = T_0^c(p) \cap T_0^c, [t_0^c(p)]^{\gamma} = [t_0^c(p)]^{\gamma} \cap T_{11}$ and

$$
[t^c(p)]^{\gamma} = \begin{cases} 
T_0^c(p), & 1 < p_k \leq H < \infty, \forall k \in \mathbb{N}, \\
T_{10}(p), & 0 < p_k \leq 1, \forall k \in \mathbb{N}
\end{cases}
$$

Proof. This is obtained in the similar way used in the proof of Theorem 4.2. \hfill \Box

5. Certain Matrix Mappings on the Sequence Spaces $t_0^c(p)$, $t_0^c(p)$ and $t^c(p)$

In this section, we characterize some matrix mappings on the spaces $t_0^c(p), t_0^c(p)$ and $t^c(p)$.

We known that, if $t_0^c(p) \cong c_0(p), t_0^c(p) \cong c(p)$ and $t^c(p) \cong l(p)$, we can say: The equivalence “$x \in t_0^c(p), t_0^c(p)$ or $t^c(p)$ if and only if $y \in c_0(p), c(p)$ or $l(p)$” holds.

In what follows, for brevity, we write,

$$
\tilde{a}_{nk} := \sum_{k=0}^{n} \binom{n}{k} (r)^{n-k}(1-r)^{-(n+1)} a_{nk}
$$

for all $k, n \in \mathbb{N}$.

Theorem 5.1. Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation

(5.1)

$$
e_{nk} := \tilde{a}_{nk}
$$

for all $k, n \in \mathbb{N}$ and $\mu$ be any given sequence space. Then,
(i): $A \in (t^*_0(p) : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t^*_0(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$.

(ii): $A \in (t^*_c(p) : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t^*_c(0)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c(p) : \mu)$.

(iii): $A \in (t^*(p) : \mu)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t^*(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell(p) : \mu)$.

**Proof.** We prove only part of (i). Let $\mu$ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $t^*_0(p)$ and $c_0(p)$ are linearly isomorphic.

Let $A \in (t^*_0(p) : \mu)$ and take any $y = (y_k) \in c_0(p)$. Then $ET(r)$ exists and $\{a_{nk}\}_{k \in \mathbb{N}} \in T^*_0(p) \cap T^*_0(p)$ which yields that $\{e_{nk}\}_{k \in \mathbb{N}} \in c_0(p)$ for each $n \in \mathbb{N}$. Hence, $Ey$ exists and thus

$$\sum_k e_{nk}y_k = \sum_k a_{nk}x_k$$

for all $n \in \mathbb{N}$.

We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0(p) : \mu)$.

Conversely, let $\{a_{nk}\}_{k \in \mathbb{N}} \in \{t^*_0(p)\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$ hold, and take any $x = (x_k) \in t^*_0(p)$. Then, $Ax$ exists. Therefore, we obtain from the equality

$$\sum_{k=0}^\infty a_{nk}x_k = \sum_{k=0}^\infty \left[ \sum_{j=0}^k \binom{j}{k} (-r)^{j-k}(1-r)^{-(j+1)}a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, so $Ey = Ax$ and this shows that $A \in (t^*_0(p) : \mu)$. This completes the proof of part of (i). \hfill $\Box$

**Theorem 5.2.** Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation

$$(5.2) \quad b_{nk} := \sum_{j=n}^\infty \binom{j}{n} (1-r)^{n+1} r^{j-n} a_{jk} \quad \text{for all } k, n \in \mathbb{N}.$$ 

Let $\mu$ be any given sequence space. Then,

(i): $A \in (\mu : t^*_0(p))$ if and only if $B \in (\mu : c_0(p))$.

(ii): $A \in (\mu : t^*_c(p))$ if and only if $B \in (\mu : c(p))$.

(iii): $A \in (\mu : t^*(p))$ if and only if $B \in (\mu : \ell(p))$.

**Proof.** We prove only part of (i). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^m b_{nk}z_k = \sum_{j=n}^\infty \binom{j}{n} (1-r)^{n+1} r^{j-n} \left( \sum_{k=0}^m a_{jk}z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{T(r)(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in t^*_0(p)$ whenever $z \in \mu$ if and only if $Bz \in c_0(p)$ whenever $z \in \mu$. This completes the proof of part of (i). \hfill $\Box$

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space $\mu$. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(t^*_0(p) : \mu), (\mu : t^*_0(p)), (t^*_c(p) : \mu), (\mu : t^*_c(p))$ and $(t^*(p) : \mu), (\mu : t^*(p))$ may be derived by replacing the entries of $C$ and $A$ by those of the entries of $E = C(T(r))^{-1}$ and $B = T(r)A$, respectively; where
the necessary and sufficient conditions on the matrices \( E \) and \( B \) are read from the concerning results in the existing literature.

The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann \([7]\). Let \( N \) and \( K \) denote the finite subset of \( N \), \( L \) and \( M \) also denote the natural numbers. Prior to giving the theorems, let us suppose that \((q_n)\) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

\[
\begin{align*}
(5.3) \quad & \lim_{n} |a_{nk}|^{q_n} = 0, \text{ for all } k \\
(5.4) \quad & \forall L, \exists M \ni \sup_{n} L^{1/q_n} \sum_{k} |a_{nk}| M^{-1/p_k} < \infty, \\
(5.5) \quad & \sup_{n} \sum_{k} a_{nk}^{q_n} < \infty, \\
(5.6) \quad & \lim_{n} \sum_{k} a_{nk}^{q_n} = 0, \\
(5.7) \quad & \forall L, \sup_{n \in K_1} |a_{nk} L^{1/q_n} p_k < \infty, \\
(5.8) \quad & \forall L, \exists M \ni \sum_{k \in K_2} |a_{nk} L^{1/q_n} M^{-1/p_k} | < \infty, \\
(5.9) \quad & \forall M, \lim_{n} \sum_{k} |a_{nk} M^{1/p_k} q_n = 0, \\
(5.10) \quad & \forall M, \sup_{n \in K_1} |a_{nk} M^{1/p_k} | < \infty, \\
(5.11) \quad & \forall M, \exists (\alpha_k) \ni \lim_{n} \sum_{k} |a_{nk} - \alpha_k| M^{1/p_k} q_n = 0, \\
(5.12) \quad & \forall M, \sup_{K} \sum_{n \in K} |\sum_{k \in K} a_{nk} M^{1/p_k} q_n < \infty.
\end{align*}
\]

**Lemma 5.1.** Let \( A = (a_{nk}) \) be an infinite matrix. Then

(i): \( A = (a_{nk}) \in (c_0(p) : \ell_\infty(q)) \) if and only if (4.8) holds.

(ii): \( A = (a_{nk}) \in (c(p) : \ell_\infty(q)) \) if and only if (4.8) and (5.5) hold.

(iii): \( A = \ell(p) \ni (a_{nk}) \) if and only if (4.11) and (4.12) hold.

(iv): \( A = (a_{nk}) \in (c_0(p) : c(q)) \) if and only if (4.4), (4.5) and (4.6) hold.

(v): \( A = (a_{nk}) \in (c(p) : c(q)) \) if and only if (4.4), (4.5), (4.6) and (4.7) hold.

(vi): \( A = (a_{nk}) \in (\ell(p) : c) \) if and only if (4.11), (4.12) and (4.13) hold.

(vii): \( A = (a_{nk}) \in (c_0(p) : c_0(q)) \) if and only if (5.3) and (5.4) hold.

(viii): \( A = (a_{nk}) \in (c(p) : c_0(q)) \) if and only if (5.3), (5.4) and (5.6) hold.

(ix): \( A = (a_{nk}) \in (\ell_\infty(p) : c_0(q)) \) if and only if (5.3), (5.7) and (5.8) hold.

(x): \( A = (a_{nk}) \in (\ell_\infty(p) : c_0(q)) \) if and only if (5.9) holds.

(xi): \( A = (a_{nk}) \in (\ell_\infty(p) : \ell(q)) \) if and only if (5.10) and (5.11) hold.

(xii): \( A = (a_{nk}) \in (\ell_\infty(p) : \ell(q)) \) if and only if (5.12) holds.

(xiii): \( A = (a_{nk}) \in (c(p) : \ell(q)) \) if and only if (4.2) holds.

(xiv): \( A = (a_{nk}) \in (c(p) : \ell(q)) \) if and only if (4.2) and (4.4) hold.

**Corollary 5.1.** Let \( A = (a_{nk}) \) be an infinite matrix. The following statements hold:
Corollary 5.2. Let \( A = (a_{nk}) \) be an infinite matrix. The following statements hold:

(i): \( A \in (t^r_c(p) : \ell_\infty(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r_c(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (4.8) holds with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(ii): \( A \in (t^r_c(p) : c_0(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r_c(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (5.3) and (5.4) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(iii): \( A \in (t^r_c(p) : c(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r_c(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (4.4), (4.5) and (4.6) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

Corollary 5.3. Let \( A = (a_{nk}) \) be an infinite matrix. The following statements hold:

(i): \( A \in (t^r(p) : \ell_\infty(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (4.8) and (5.5) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(ii): \( A \in (t^r(p) : c_0(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (5.3), (5.4) and (5.6) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(iii): \( A \in (t^r(p) : c(q)) \) if and only if \( \{a_{nk}\}_{k \in \mathbb{N}} \subseteq \{t^r(p)\}^\beta \) for all \( n \in \mathbb{N} \) and (4.4), (4.5), (4.6) and (4.7) hold with \( \tilde{a}_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

Corollary 5.4. Let \( A = (a_{nk}) \) be an infinite matrix and \( b_{nk} \) be defined by (5.2). Then, following statements hold:

(i): \( A \in (\ell_\infty(q) : t^r_0(p)) \) if and only if (5.9) holds with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(ii): \( A \in (c_0(q) : t^r_0(p)) \) if and only if (5.3) and (5.4) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(iii): \( A \in (c(q) : t^r_0(p)) \) if and only if (5.3), (5.4) and (5.6) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

Corollary 5.5. Let \( A = (a_{nk}) \) be an infinite matrix and \( b_{nk} \) be defined by (5.2). Then, following statements hold:

(i): \( A \in (\ell_\infty(q) : t^r_0(p)) \) if and only if (5.10) and (5.11) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(ii): \( A \in (c_0(q) : t^r_0(p)) \) if and only if (4.4), (4.5) and (4.6) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(iii): \( A \in (c(q) : t^r_0(p)) \) if and only if (4.4), (4.5), (4.6) and (4.7) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

Corollary 5.6. Let \( A = (a_{nk}) \) be an infinite matrix and \( b_{nk} \) be defined by (5.2). Then, following statements hold:

(i): \( A \in (\ell_\infty(q) : t^r(p)) \) if and only if (5.12) holds with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(ii): \( A \in (c_0(q) : t^r(p)) \) if and only if (4.2) holds with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).

(iii): \( A \in (c(q) : t^r(p)) \) if and only if (4.2) and (4.4) hold with \( b_{nk} \) instead of \( a_{nk} \) with \( q = 1 \).
References


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