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A MESH-FREE TECHNIQUE OF NUMERICAL SOLUTION OF NEWLY DEFINED CONFORMABLE DIFFERENTIAL EQUATIONS

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Abstract. Motivated by the recently defined conformable derivatives proposed in [2], we introduced a new approach of solving the conformable ordinary differential equation with the mesh-free numerical method. Since radial basis function collocation technique has outstanding feature in comparison with the other numerical methods, we use it to solve non-integer order of differential equation. We subsequently present the results of numerical experimentation to show that our algorithm provide successful consequences.

1. Introduction

Until quite recently, the question of how to take non-integer order of derivative or integration was phenomenon among the mathematicans. However together with the development of mathematics knowledge, this question was answered via fractional differentiation and integration [8], [9], [11], [12]. Although there are a number of different type of definition of fractional derivatives or integrations, Riemann-Liouville and Caputo are the most popular ones among them. Then Abdeljawad [1] and Khalil et. al. [7] defined the limit based conformable derivative which is another type of fractional derivative and integrations. In more recent times, Anderson and Ulness [2] have described another precise definition of conformable derivatives motivated by a proportional derivative controller. As a result of this new definition of conformable derivatives, its differential equations need to be handled.

In this paper, we develop a meshless algorithm for the numerical solution of the conformable differential equations by taking advantageous of radial basis function (RBF) interpolation [3], [5], [10]. The goal of this approach is to acquire approximate solution of conformable differential equations with RBF collocation method. Of course this approach would provide an insight the solution of more complex cases.

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The remainder of this work is organized as follows: In Section 2, the conformable derivatives are summarised, along with the newly defined type. In Section 3, the RBF interpolation method is reviewed while in Section 4 the numerical scheme of solving conformable ordinary differential equation using mesh-free method is introduced and we also reviewed the RBF collocation technique. Numerical examples are given in Section 5, while some conclusions and further directions of research are discussed in Section 6.

2. A class of conformable derivatives

In [7] and [1], a new version of limit based fractional derivative called conformable derivative have been defined via

$$D^{\alpha}u(x) = \lim_{\xi \to 0} \frac{u(x + \xi x^{1-\alpha}) - u(x)}{\xi}, \tag{2.1}$$

on condition that limit exists. Another proposed limit based fractional derivative is

$$D^{\alpha}u(x) = \lim_{\xi \to 0} \frac{u(x e^{\xi x^{-\alpha}}) - u(x)}{\xi}, \tag{2.2}$$

in [6]. For both approaches the conformable derivative can be summarised via

$$D^{\alpha}u(x) = x^{1-\alpha} \frac{d}{dx} u(x), \tag{2.3}$$

where $\frac{d}{dx}$ denotes the classical derivative operators. In addition to this, Anderson and Ulness [2] introduced a new class of conformable derivatives via proportional-derivative controller.

Definition 2.1. [2] Let $\alpha \in [0,1]$. The conformable derivative operator D^{α} describe as

$$D^{\alpha}u(x) = \kappa_1(\alpha, x)u(x) + \kappa_0(\alpha, x)\frac{d}{dx}u(x) \tag{2.4}$$

where $\kappa_1, \kappa_0 : [0,1] \times \mathbb{R} \to [0, \infty)$ are continuous function such that

$$\begin{aligned}
&\lim_{\alpha \to 0^+} \kappa_1(\alpha, x) = 1, \qquad \lim_{\alpha \to 0^+} \kappa_0(\alpha, x) = 0, \qquad \text{for all x} \in \mathbb{R},\\
&\lim_{\alpha \to 1^-} \kappa_1(\alpha, x) = 0, \qquad \lim_{\alpha \to 1^-} \kappa_0(\alpha, x) = 1, \qquad \text{for all x} \in \mathbb{R},\\
&\kappa_1(\alpha, x), \kappa_0(\alpha, x) \neq 0, \qquad \alpha \in (0,1], \qquad \text{for all x} \in \mathbb{R}.
\end{aligned}$$

So, for instance, one can define the conformable derivative operator

$$D^{\alpha}u(x) = (1-\alpha)e^{\alpha}u(x) + \alpha e^{1-\alpha}\frac{d}{dx}u(x), \tag{2.5}$$

or

$$D^{\alpha}u(x) = \cos(\alpha\pi/2)e^{\alpha}u(x) + \sin(\alpha\pi/2)e^{1-\alpha}\frac{d}{dx}u(x). \tag{2.6}$$

This new definition of conformable derivative enables to compute the non-integer order of derivatives via classical derivative operator. Thus, conformable differential equations can be solved with the numerical methods after this transformation has been applied. In next section, we will summarised the RBF methods which is one of the mesh-free techniques and then applied it to solve conformable differential equations.

3. Radial basis function interpolation method

The history of the RBF approximation goes back to 1968 with Hardy who introduced the multiquadric RBFs in academia [4]. Thereafter RBF method become increasingly popular interpolation technique as it provides us delicately and accurately results with no mesh. Not only interpolation or quadrature of any function, but also solving partial differential equations is also an application area of RBFs technique.

One can define the RBF interpolation as follows:

Definition 3.1. Consider a given data set $\mathbf{f} = (f_1, ..., f_N)^T \in \mathbb{R}^N$ of function values, taken from an unknown function $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}$ at scattered data points $\mathbf{x}_k \in \mathbb{R}^d$, $k = 1, ..., N$ such that $\mathbf{f}_k = \mathbf{f}(x_k)$ and $d \geq 1$. The RBF interpolation is given by

$$P_f(\mathbf{x}) = \sum_{k=1}^{N} a_k \varphi(\|\mathbf{x} - \mathbf{x_k}\|), \tag{3.1}$$

where $\varphi(\cdot)$ is a radial function and $\|\cdot\|$ is the Euclidean distance. The coefficient a_j can be determined from interpolation requirements $P_f(\mathbf{x}_j) = \mathbf{f}_j$ by solving the following symmetric linear system:

$$\mathbf{A}\mathbf{a} = \mathbf{f}, \tag{3.2}$$

where the matrix $A_{(N\times N)}$ is constructed for φ_{jk} such that $\varphi_{jk} = \varphi(\|x_j - x_k\|)$, $j, k = 1, \ldots, N$.

Here the basis function φ must be choose as a positive definite function. Additionally, radial basis functions can be divided into two major groups: piecewise smooth and infinitely smooth which are given in Table 1 and Table 2. The rate of convergence in the infinitely smooth RBFs is quicker in comparison with the piecewise smooth RBFs which cause to an algebraical rate of convergence.

Piecewise Smooth RBFs	$\varphi(r)$
Piecewise Polynomial (R_n)	$\lvert r\rvert^n$, n odd
Thin Plate Spline (TPS_n)	$\lvert r\rvert^n ln\lvert r\rvert$, n even

Table 1. Piecewise Smooth

Additionally, RBFs can be expressed by using a scaling parameter named the shape parameter ε. This can be done in the manner that $\varphi(r)$ is replaced by $\varphi(\varepsilon r)$.

Infinitely Smooth RBFs	$\varphi(r)$
Multiquadric (MQ)	$\sqrt{1+r^2}$
Inverse Multiquadric (IMQ)	$\dfrac{1}{\sqrt{1+r^2}}$
Inverse Quadratic (IQ)	$\dfrac{1}{1+r^2}$
Gaussian (GA)	e^{-r^2}
Bessel (BE)	$J_0(2r)$

TABLE 2. Infinitely Smooth

In general shape parameter have been chosen arbitrarily since there are no exact results about how to choose best shape parameter.

4. NUMERICAL SCHEME USING MESH-FREE TECHNIQUE

Together with the development of derivative concept, the question of how to solve non-integer order differential equations have arisen in the scientific area. One of the similar problem has been faced for the conformable differential equations since it contains the non-integer order derivative terms. However through the definition of conformable derivative operator one can transform it to classical ordinary differential equations that there are huge amount of literature about it. Thus by applying the mesh-free numerical methods, we can find an approximation results of conformable differential equations. The conformable ordinary differential equation can be expressed via

$$D^{\alpha}u(x) + \vartheta(x)u(x) = v(x), \qquad u_0(x) = u(x_0). \tag{4.1}$$

Then by substituting of equation (2.4) into equation (4.1), we get

$$\kappa_1(\alpha, x)u(x) + \kappa_0(\alpha, x)\frac{d}{dx}u(x) + \vartheta(x)u(x) = v(x). \tag{4.2}$$

Then by rearranging of equation (4.2), we obtain the below classical ordinary differential equation, that is

$$\frac{d}{dx}u(x) + A(\alpha, x)u(x) = B(\alpha, x), \qquad u_0(x) = u(x_0), \tag{4.3}$$

where

$$A(\alpha, x)u(x) = \frac{\kappa_1(\alpha, x) + \vartheta(x)}{\kappa_0(\alpha, x)} \qquad \text{and} \qquad B(\alpha, x) = \frac{v(x)}{\kappa_0(\alpha, x)}. \tag{4.4}$$

Now the above equation can be solved easily by applying the RBF collocation method which will present next section.

4.1. **RBF collocation technique.** In order to solve equation (4.3) by numerically we use the RBF collocation method which is quite popular method in the engineering and applied mathematics. Let $\mathbf{x}_{k=1}^N$ be the collocation points for interior and boundary region. Then by using definition of RBF interpolation, we get

$$\sum_{k=1}^{N} a_k \left[\frac{d}{dx}u(x) + A(\alpha, x)\right] \varphi(\|\mathbf{x} - \mathbf{x_k}\|) = B(\alpha, x), \tag{4.5}$$

with the boundary condition

$$\sum_{k=1}^{N} a_k \varphi(\|\mathbf{x_0} - \mathbf{x_k}\|) = u(x_0). \tag{4.6}$$

Then by using the points $\mathbf{x}_{k=1}^N$, we can collocate the equations (4.5) and (4.6) to determine the unknown coefficients a_k's. Thus the unknown function value $u(x)$ can be calculated by using the determined coefficients with collocation method.

An algorithm for RBF collocation of conformable differential equation is as follows:

Algorithm 1: RBF collocation method for conformable differential equation

Require: Equally spaced grid data decomposition for $0, M$.

1: Initialize the matrix $\mathbf{A}$ and $\mathbf{f}$ via collocation points $\mathbf{x}_{k=1}^N$.
2: Construct and solve the matrix equality $\mathbf{Aa} = \mathbf{f}$ to determine the unknown values of a_k's.
3: By using the value of a_k's, calculate the solution of equation for each collocation points.
4: **return** Approximation value

5. Numerical experiments

In this section, we presents some numerical results to verify proposed algorithm. To do that, we take the first order conformable ODE which is solved by RBF collocation technique.

5.1. **Numerical solution of conformable ODE.** For this example, we take the below conformable ODE [2] to solve it via RBF method,

$$D^{\alpha}u(x) + u(x) = v(x) \tag{5.1}$$

with the boundary condition

$$u_0(x) = u(x_0) \tag{5.2}$$

Let x_i be equally spaced grid points in the interval $0 \leq x_i \leq M$ such that $1 \leq i \leq N$, $x_1 = 0$ and $x_N = M$. Additionally, because collocation approach has been used we not only require an expression for the value of the function

$$u(x) = \sum_{k=1}^{N} a_j \varphi(\|x - x_k\|) \tag{5.3}$$

but also for the conformal derivative given in (5.1). Thus, by conformal differentiating (5.3), we get

$$D^{\alpha} u(x) = \sum_{k=1}^{N} a_j D^{\alpha} \varphi(\|x - x_k\|) \tag{5.4}$$

where D^{α}denotes the conformable derivative the with respect to x. In a particular case of Multiquadric and Gaussian basis functions, we have

$$D^{\alpha}\varphi(\|x - x_k\|) = \quad \kappa_1(\alpha, x)\sqrt{\|x - x_k\|^2 + \varepsilon^2} + \kappa_0(\alpha, x)\frac{x - x_k}{\sqrt{\|x - x_k\|^2 + \varepsilon^2}}$$

$$D^{\alpha}\varphi(\|x - x_k\|) = \quad \kappa_1(\alpha, x)e^{-\|x-x_k\|^2/\varepsilon^2} - \kappa_0(\alpha, x)\frac{2(x - x_k)}{\varepsilon^2}e^{-\|x-x_k\|^2/\varepsilon^2} \tag{5.5}$$

where κ_0 and κ_1 are given in Definition 3.1. So in order to determine the value of a_j's in equation (5.3), we need to solve

$$\sum_{k=1}^{N} a_j D^{\alpha}\varphi(\|x_j - x_k\|) + \sum_{k=1}^{N} a_j \varphi(\|x_j - x_k\|) = v(x) \tag{5.6}$$

by using

$$\sum_{k=1}^{N} a_j \varphi(\|x_1 - x_k\|) = u(x_0) \tag{5.7}$$

where $j = 2, \ldots, N$. If we put the equations (5.5) into equation (5.6), we get the classical ODE which can be solved easily. In other words, one need to solve below algebraic systems

$$\phi_{[N\times N]} a_{[N\times 1]} = \nu_{[N\times 1]} \tag{5.8}$$

where

$$\phi = \begin{pmatrix} D^\alpha\varphi_{1,1}+\varphi_{1,1} & \dots & D^\alpha\varphi_{1,N}+\varphi_{1,N} \\ D^\alpha\varphi_{2,1}+\varphi_{2,1} & \dots & D^\alpha\varphi_{2,N}+\varphi_{2,N} \\ \vdots & \ddots & \vdots \\ D^\alpha\varphi_{N,1}+\varphi_{N,1} & \dots & D^\alpha\varphi_{N,N}+\varphi_{N,N} \end{pmatrix}, a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \nu = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

to determine a_i's. Then one can obtain the numerical solution using a_i's into RBF method. The numerical experiment results has been presented for different left hand side functions such as $v_1(x) = x\sqrt{x} + 1/2x^2\sqrt{x} + x^2$, $v_2(x) = e^{-x}(x + \sqrt{x}/2)$ and $v_3(x) = (1 - \sqrt{x}/2)\cos(4\sqrt{x}) - \sin(4\sqrt{x})$ in Figures 1, 2 and 3 respectively. These results confirm that RBF method converge the solution of ordinary conformable differential equations.

Function	**Alpha**	ε	**Number of Nodes**	**Max-Error**	**RMS-Error**
$v_1(x)$	0.5	5	500	3.829195e-006	5.527372e-008
$v_2(x)$	0.5	5	500	2.352912e-005	3.757950e-007
$v_3(x)$	0.5	5	500	2.267579e-004	3.500312e-006

TABLE 3. Numerical results of conformable ordinary differential equation via RBF using Multiquadric on the domain $[0, 10]$.

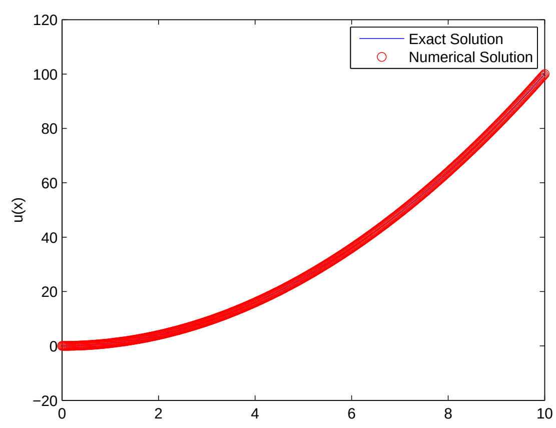

FIGURE 1. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_1(x) = x\sqrt{x} + 1/2x^2\sqrt{x} + x^2$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

In the numerical experiments, *Max-Error* represents the maximum modulus error, i.e., $\|f - g\|_\infty$ and *Rms-Error* represents the standard root mean squared error, i.e.

$$\sqrt{\frac{\sum_{i=1}^{Neval}|f_i - g_i|^2}{Neval}}, \tag{5.9}$$

where f is the exact solution, g is the approximate solution, and $Neval$ is the number of the test points.

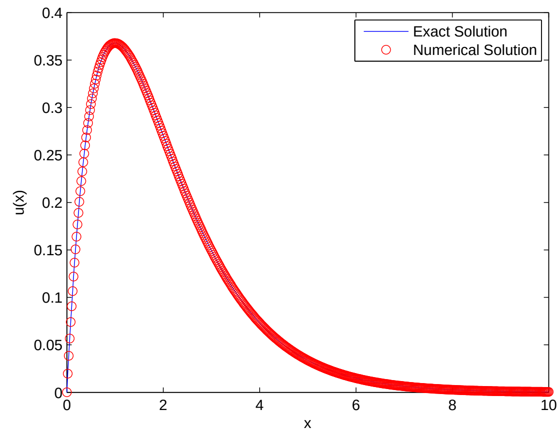

FIGURE 2. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_2(x) = e^{-x}(x + \sqrt{x}/2)$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

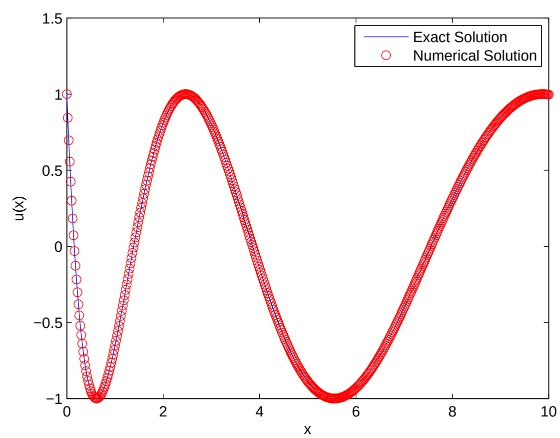

FIGURE 3. $u(x)$ versus x using Multiquadric basis function with $\varepsilon = 5$ for $v_3(x) = (1 - \sqrt{x}/2)\cos(4\sqrt{x}) - \sin(4\sqrt{x})$: Exact solution (Blue) and Numerical solution (Red circle) on equally spaced evaluation grid.

6. Concluding remark

A new radial basis function collocation technique to solve conformable ordinary differential equation is proposed and tested in this paper. To do that Gaussian or Multiquadric basis functions can be used. In order to verify this methods stability, we have presented some numerical results. Thus this study would help to solve modelled non-integer order of differential equations.

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