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New Fractional Integral Inequalities for Convex and α-Star s-Convex Functions

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Abstract

In this study, new integral inequalities are obtained for convex functions and $\alpha -$star $s-$convex function classes by using the Caputo-Fabrizio fractional integral operator. In the results, some special cases are considered for $\alpha -$star $s-$convex functions. The well-known Hölder and Young inequalities are used to provide new estimations for convex and $\alpha -$star $s-$convex functions. While proving the integral inequalities, the definition of function classes and basic analysis methods are utilized.

Keywords

Convex Function,
$\alpha -$Star $s-$ Convex Function,
Caputo-Fabrizio Fractional İntegrals

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Konveks ve α-Star s-Konveks Fonksiyonlar için Yeni Kesirli İntegral Eşitsizlikler

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Özet

Bu çalışmada, Caputo-Fabrizio kesirli integral operatörü kullanılarak konveks ve α-star s-konveks fonksiyon sınıfları için yeni integral eşitsizlikler elde edilmiştir. Sonuçlarda, α-star s-konveks fonksiyonlar için bazı özel durumlar ele alınmıştır. konveks ve α-star s-konvveks fonksiyonlar için yeni tahminlemeler sağlamak amacıyla bilinen Hölder ve Young eşitsizlikleri kullanılmıştır. İntegral eşitsizlikleri ispatlanırken, fonksiyon sınıflarının tanımı ve temel analiz yöntemlerinden yararlanılmıştır.

Anahtar kelimeler

Konveks Fonksiyon, $\alpha -$ Star $s -$ Konveks Fonksiyon, Caputo-Fabrizio Kesirli İntegrali

1. INTRODUCTION

In the 21st century, inequalities have extended far beyond the realm of mathematics, establishing a significant presence across various scientific disciplines, particularly within engineering. This expansion has drawn the interest of numerous researchers who have explored inequalities from diverse perspectives, aiming to understand and apply them in new contexts. A key concept in this field is convexity, which plays a central role in inequality theory and has become an essential tool for researchers.
In this study, we will start by defining the concept of convexity as in [1], laying the foundation for further exploration and application in our analysis.

Definition 1. (See [1]). Let $\mathring{A} \subseteq \mathbb{R}$. Then, $\mathcal{F}: \mathring{A} \longrightarrow \mathbb{R}$ is said to be convex, if the following inequality

$$\mathcal{F}(kn + (1-k)m) \leq k\mathcal{F}(n) + (1-k)\mathcal{F}(m)$$

holds for all $n, m \in \mathring{A}$ and $k \in [0,1]$.

The primary objective of studying various types of convexity is to refine and broaden the bounds of certain well-known classical inequalities. Within this context, a significant category of convex functions has emerged, known as $\alpha - star\ s -$ convex functions which are defined specifically considering the optimization goal. The formal definition of alpha-star s-convex functions is presented as follows.

Definition 2. (See [2]). Let $\mathring{A} \subseteq \mathbb{R}^+$. Then, $\mathcal{F}: \mathring{A} \longrightarrow \mathbb{R}$ is said to be $\alpha - star\ s -$ convex functions, if

$$\mathcal{F}(kn + (1-k)m) \leq k^s\mathcal{F}(n) + \left((1-k)\alpha\right)^s\mathcal{F}(m)$$

holds for all $\forall n, m \in \mathring{A}$ and $k, \alpha \in [0,1]$

In [3], Akdemir defined the $\alpha - star\ s -$ convex function classes on the co-ordinates and brought new and original results to the literature (Akdemir [3]).

In [4], Aslan made a significant contribution to the mathematical literature by introducing innovative and original findings through the definition of exponentially α-star s-convex function classes on the coordinates. These new function classes extend the traditional concept of α-star s-convex functions by offering a more generalized approach in the context of the coordinate plane. This work not only broadens the scope of convex analysis but also provides a fresh perspective for researchers who explore advanced convexity properties in mathematical functions, potentially impacting areas like optimization, economics, and applied mathematics where convexity plays a crucial role (Aslan [4]).

For further research on convexity and its properties, we encourage interested readers to refer to references [5-11].

Definition 3. (See [12]) Consider $\mathcal{F} \in H^1(0, m)$ with $m > n$ and $\alpha \in [0,1]$. Under these conditions, the left-hand and right-hand formulations of the Caputo-Fabrizio fractional integral are defined as follows:

$$\left({}_{n}^{CF}I^{\varrho}\right)(k) = \frac{1-\varrho}{B(\varrho)}\mathcal{F}(k) + \frac{\varrho}{B(\varrho)}\int_{n}^{k}\mathcal{F}(y)dy,$$

and

$$\left({}^{CF}I_{m}^{\varrho}\right)(k) = \frac{1-\varrho}{B(\varrho)}\mathcal{F}(k) + \frac{\varrho}{B(\varrho)}\int_{k}^{m}\mathcal{F}(y)dy$$

where $B(\varrho) > 0$ denotes the normalization function.

In the remainder of the paper, we will denote the normalization function by $B(\varrho)$, where $B(0) = B(1) = 1$.

In [13], Tariq et al. obtained Hermite-Hadamard type integral inequality for convex functions by presenting an inequality based on the Caputo-Fabrizio fractional integral. This approach provides a new perspective in estimating the integral means of convex functions by using the properties provided by the Caputo-Fabrizio fractional operator.

Theorem 1. (See [13]) Let $\mathcal{F}: I = [b_1, b_1 + \mu(b_2, b_1)] \rightarrow \mathbb{R}^+$ be a preinvex function on I° and $\mathcal{F} \in L[b_1, b_1 + \mu(b_2, b_1)]$. If $\varrho \in [0,1]$, then the following inequality holds:

$$\mathcal{F}\left(\frac{2b_1 + \mu(b_2, b_1)}{2}\right) \leq \frac{B(\varrho)}{\varrho\mu(b_2, b_1)} \times \left[{}_{b_1}^{CF}I^{\varrho}\{\mathcal{F}(b)\} + {}^{CF}I_{b_1+\mu(b_2,b_1)}^{\varrho}\{\mathcal{F}(b)\} - \frac{2(1-\varrho)}{B(\varrho)}\mathcal{F}(b)\right] \leq \frac{\mathcal{F}(b_1) + \mathcal{F}(b_2)}{2}$$

where $b \in [b_1, b_1 + \mu(b_2, b_1)]$.

We recommend the readers who want to gain in-depth knowledge on different types of fractional operators and gain a comprehensive perspective on the subject to have a look to the papers ([14-29]). These resources provide a wide range of information, from the theoretical foundations of fractional operators to various application areas.

2. MAIN RESULTS

Theorem 2. Let $\mathrm{I} \subseteq \mathbb{R}$. Suppose that $\mathcal{F}: [n, m] \subseteq \mathrm{I} \longrightarrow \mathbb{R}$ is a convex function on $[n, m]$ such that $\mathcal{F} \in L_1[n, m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}_{n}^{CF}I^{\varrho}\mathcal{F}\right)(b) + \left({}^{CF}I_{m}^{\varrho}\mathcal{F}\right)(b) \leq \frac{4(1-\varrho)\mathcal{F}(b) + \varrho(m-n)\left(\mathcal{F}(n) + \mathcal{F}(m)\right)}{2B(\varrho)}.$$

Where $B(\varrho) > 0$ denotes the normalization function and $\varrho \in [0,1]$.

Proof: By using the definition of convex function, we can write

$$\mathcal{F}(\&n + (1-\&)m) \leq \&\mathcal{F}(n) + (1-\&)\mathcal{F}(m).$$

By integrating each side of the above inequality over the interval [0,1] with respect to $\&$, we get

$$\int_0^1 \mathcal{F}(\&n + (1-\&)m)\, d\& \leq \int_0^1 \&\mathcal{F}(n)\, d\& + \int_0^1 (1-\&)\mathcal{F}(m)\, d\&.$$

By making the substitution $x = (\&n + (1-\&)m)$, and then evaluating the left-hand and right-hand side, we have

$$\frac{1}{m-n}\int_n^m \mathcal{F}(x)\, dx \leq \frac{\mathcal{F}(n) + \mathcal{F}(m)}{2}.$$

By multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}\mathcal{F}(\&)$, we obtain

$$\frac{2(1-\varrho)}{B(\varrho)}\mathcal{F}(\&) + \frac{\varrho}{B(\varrho)}\int_n^m \mathcal{F}(x)\, dx \leq \frac{2(1-\varrho)}{B(\varrho)}\mathcal{F}(\&) + \frac{\varrho(m-n)}{B(\varrho)}\frac{\mathcal{F}(n) + \mathcal{F}(m)}{2}.$$

By simplifying the above inequality, it follows

$$\left(\frac{1-\varrho}{B(\varrho)}\mathcal{F}(\&) + \frac{\varrho}{B(\varrho)}\int_n^{\&} \mathcal{F}(x)\, dx\right) + \left(\frac{1-\varrho}{B(\varrho)}\mathcal{F}(\&) + \frac{\varrho}{B(\varrho)}\int_k^m \mathcal{F}(x)\, dx\right) \leq \frac{2(1-\varrho)}{B(\varrho)}\mathcal{F}(\&) + \frac{\varrho(m-n)}{B(\varrho)}\frac{\mathcal{F}(n) + \mathcal{F}(m)}{2}$$

After necessary operations, the result will be as follows

$$\left({}_{n}^{CF}I^{\varrho}\mathcal{F}\right)(\&) + \left({}^{CF}I_m^{\varrho}\mathcal{F}\right)(\&) \leq \frac{4(1-\varrho)\mathcal{F}(\&) + \varrho(m-n)\big(\mathcal{F}(n) + \mathcal{F}(m)\big)}{2B(\varrho)}.$$

The proof is completed.

Theorem 3. Let I ⊆ ℝ. Suppose that $|\mathcal{F}|: [n,m] \subseteq I \longrightarrow \mathbb{R}$ is a convex function on $[n,m]$ such that $\mathcal{F} \in L_1[n,m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\&) + \left({}^{CF}I_m^{\varrho}|\mathcal{F}|\right)(\&) \leq \frac{2(1-\varrho)|\mathcal{F}(\&)|(p+1)^{\frac{1}{p}} + \varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)(p+1)^{\frac{1}{p}}}$$

where $B(\varrho) > 0$ denotes the normalization function, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\varrho \in [0,1]$.

Proof: By using the definition of convex function with absolute value of each side, we can write

$$|\mathcal{F}(\&n + (1-\&)m)| \leq \&|\mathcal{F}(n)| + (1-\&)|\mathcal{F}(m)|.$$

By integrating each side of the inequality over the interval [0,1] with respect to $\&$, we get

$$\int_0^1 |\mathcal{F}(\&n + (1-\&)m)|\, d\& \leq \int_0^1 \&\,|\mathcal{F}(n)| d\& + \int_0^1 (1-\&)\,|\mathcal{F}(m)| d\&.$$

By applying Hölder's inequality to the right side of the inequality, the following result is yielded

$$\int_0^1 |\mathcal{F}(\&n + (1-\&)m)|\, dt \leq |\mathcal{F}(n)|\left(\int_0^1 \&^p\, d\&\right)^{\frac{1}{p}}\left(\int_0^1 d\&\right)^{\frac{1}{q}} + |\mathcal{F}(m)|\left(\int_0^1 (1-\&)^p\, d\&\right)^{\frac{1}{p}}\left(\int_0^1 d\&\right)^{\frac{1}{q}}.$$

By making the substitution $x = (\&n + (1-\&)m)$, and then calculating the left-hand and right-hand side, we get

$$\frac{1}{m-n}\int_n^m |\mathcal{F}(x)|\, dx \leq |\mathcal{F}(n)|\left(\frac{1}{p+1}\right)^{\frac{1}{p}} + |\mathcal{F}(m)|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

By multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\&)|$, we obtain

$$\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\&)| + \frac{\varrho}{B(\varrho)}\int_n^m |\mathcal{F}(x)|\, dx \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\&)| + \frac{\varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

and by simplifying the inequality, the result follows

$$\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(\&)| + \frac{\varrho}{B(\varrho)}\int_n^{\&} |\mathcal{F}(x)|\, dx\right) + \left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(\&)| + \frac{\varrho}{B(\varrho)}\int_{\&}^m |\mathcal{F}(x)|\, dx\right) \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\&)| + \frac{\varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

After necessary computations, one can obtain

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\&) + \left({}^{CF}I_m^{\varrho}|\mathcal{F}|\right)(\&) \leq \frac{2(1-\varrho)|\mathcal{F}(\&)|(p+1)^{\frac{1}{p}} + \varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)(p+1)^{\frac{1}{p}}}.$$

The desired result is obtained.

Theorem 4. Let $I \subseteq \mathbb{R}$. Suppose that $|\mathcal{F}|: [n, m] \subseteq I \longrightarrow \mathbb{R}$ is a convex function on $[n, m]$ such that $\mathcal{F} \in L_1 [n, m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}_{\ n}I^{\varrho}|\mathcal{F}|\right)(b) + \left({}^{CF}I^{\varrho}_{m}|\mathcal{F}|\right)(b) \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

where $B(\varrho) > 0$ denotes the normalization function, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $\varrho \in [0,1]$.

Proof 4. By using the definition of convex function, we can write

$$|\mathcal{F}(kn + (1-k)m)| \leq k|\mathcal{F}(n)| + (1-k)|\mathcal{F}(m)|.$$

By integrating each side of the inequality over the interval $[0,1]$ with respect to k, we get

$$\int_0^1 |\mathcal{F}(kn + (1-k)m)|\, dk \leq |\mathcal{F}(n)| \int_0^1 k\, dk + |\mathcal{F}(m)| \int_0^1 (1-k)\, dk.$$

By applying Young's inequality to the right side of the inequality, the following result is yielded

$$\int_0^1 |\mathcal{F}(kn + (1-k)m)|\, dk \leq |\mathcal{F}(n)| \left(\frac{1}{p}\left(\int_0^1 k^p\, dt\right) + \frac{1}{q}\left(\int_0^1 1^q dk\right)\right) + |\mathcal{F}(m)| \left(\frac{1}{p}\left(\int_0^1 (1-k)^p\, dk\right) + \frac{1}{q}\left(\int_0^1 1^q dk\right)\right).$$

By making necessary operations, we get

$$\frac{1}{m-n}\int_n^m |\mathcal{F}(x)|\, dx \leq |\mathcal{F}(n)| \left(\frac{1}{p(p+1)} + \frac{1}{q}\right) + |\mathcal{F}(m)| \left(\frac{1}{p(p+1)} + \frac{1}{q}\right).$$

By multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)|$, we obtain

$$\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho}{B(\varrho)}\int_n^m |\mathcal{F}(x)|\, dx \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)}\left(\frac{1}{p(p+1)} + \frac{1}{q}\right).$$

By changing the notation of the Caputo-Fabrizio fractional integral in the inequality, the result can be obtained as follows:

$$\left({}^{CF}_{\ n}I^{\varrho}|\mathcal{F}|\right)(b) + \left({}^{CF}I^{\varrho}_{m}|\mathcal{F}|\right)(b) \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho(m-n)(|\mathcal{F}(n)| + |\mathcal{F}(m)|)}{B(\varrho)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

This completes the proof.

Theorem 5. Let $I \subseteq \mathbb{R}$. Suppose that $|\mathcal{F}|: [n, m] \subseteq I \longrightarrow \mathbb{R}$ is a $\alpha - star\ s -$convex function on $[n, m]$ such that $\mathcal{F} \in L_1 [n, m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}_{\ n}I^{\varrho}|\mathcal{F}|\right)(b) + \left({}^{CF}I^{\varrho}_{m}|\mathcal{F}|\right)(b) \leq \frac{2(1-\varrho)|\mathcal{F}(b)|(s+1) + \varrho(m-n)(|\mathcal{F}(n)| + \alpha^s|\mathcal{F}(m)|)}{B(\varrho)(s+1)}$$

where $B(\varrho) > 0$ denotes the normalization function, $s \in (0,1]$ and $\alpha, \varrho \in [0,1]$.

Proof 5. By using the definition of $\alpha - star\ s -$convex function, we can write

$$|\mathcal{F}(kn + (1-k)m)| \leq k^s|\mathcal{F}(n)| + \left((1-k)\alpha\right)^s|\mathcal{F}(m)|.$$

By integrating each side of the inequality over the interval $[0,1]$ with respect to k, we get

$$\int_0^1 |\mathcal{F}(kn + (1-k)m)|\, dk \leq \int_0^1 k^s|\mathcal{F}(n)|\, dk + \int_0^1 \left((1-k)\alpha\right)^s|\mathcal{F}(m)|\, dk.$$

By making the substitution $x = (kn + (1-k)m)$, and then evaluating the left-hand and right-hand side, we get

$$\frac{1}{m-n}\int_n^m |\mathcal{F}(x)|\, dx \leq \frac{|\mathcal{F}(n)| + \alpha^s|\mathcal{F}(m)|}{s+1}.$$

By multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)|$, we obtain:

$$\frac{2(1-\varrho)}{B(\alpha)}|\mathcal{F}(b)| + \frac{\varrho}{B(\varrho)}\int_n^m |\mathcal{F}(x)|\, dx \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho(m-n)}{B(\varrho)}\frac{|\mathcal{F}(n)| + \alpha^s|\mathcal{F}(n)|}{s+1}.$$

By simplifying the inequality, we have

$$\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho}{B(\varrho)}\int_n^b |\mathcal{F}(x)|\, dx\right) + \left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho}{B(\varrho)}\int_b^m |\mathcal{F}(x)|\, dx\right) \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| + \frac{\varrho(m-n)}{B(\varrho)}\frac{|\mathcal{F}(n)| + \alpha^s|\mathcal{F}(m)|}{s+1}$$

Therefore, we can write

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}|\mathcal{F}|\right)(\mathscr{b}) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(s+1)}{B(\varrho)(s+1)} + \frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|)}{B(\varrho)(s+1)}.$$

The proof is completed.

Remark 1. (See [24]) If we choose $\alpha^s = 1$ under the assumptions of the Theorem 5, we have

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}|\mathcal{F}|\right)(\mathscr{b}) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(s+1)}{B(\varrho)(s+1)} + \frac{\varrho(m-n)(|\mathcal{F}(n)|+|\mathcal{F}(m)|)}{B(\varrho)(s+1)}.$$

Theorem 6. Let I ⊆ ℝ. Suppose that $|\mathcal{F}|:[n,m]\subseteq I \longrightarrow \mathbb{R}$ is $\alpha-$star s-convex function on $[n,m]$ such that $\mathcal{F}\in L_1[n,m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}|\mathcal{F}|\right)(\mathscr{b}) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(ps+1)^{\frac{1}{p}}}{B(\varrho)(ps+1)^{\frac{1}{p}}} + \frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|)}{B(\varrho)(ps+1)^{\frac{1}{p}}}$$

where $B(\varrho)>0$ is normalization function, $s\in(0,1]$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha,\varrho\in[0,1]$.

Proof 6. By using the definition of $\alpha-$star s-convex function, we can write

$$|\mathcal{F}(\mathscr{k}n+(1-\mathscr{k})m)| \leq \mathscr{k}^{s}|\mathcal{F}(n)|+\left((1-\mathscr{k})\alpha\right)^{s}|\mathcal{F}(m)|.$$

By integrating each side of the inequality over the interval [0,1] with respect to $\mathscr{k}$, we get

$$\int_0^1|\mathcal{F}(\mathscr{k}n+(1-\mathscr{k})m)|\,d\mathscr{k} \leq \int_0^1\mathscr{k}^{s}|\mathcal{F}(n)|\,d\mathscr{k}+\int_0^1\left((1-\mathscr{k})\alpha\right)^{s}|\mathcal{F}(m)|\,d\mathscr{k}.$$

By applying Hölder's inequality to the right side of the inequality, we have

$$\int_0^1|\mathcal{F}(\mathscr{k}n+(1-\mathscr{k})m)|\,d\mathscr{k} \leq |\mathcal{F}(n)|\left(\int_0^1\mathscr{k}^{ps}\,d\mathscr{k}\right)^{\frac{1}{p}}\left(\int_0^1 1^{q}d\mathscr{k}\right)^{\frac{1}{q}} +\alpha^{s}|\mathcal{F}(m)|\left(\int_0^1(1-\mathscr{k})^{ps}\,d\mathscr{k}\right)^{\frac{1}{p}}\left(\int_0^1 1^{q}d\mathscr{k}\right)^{\frac{1}{q}}$$

$$=|\mathcal{F}(n)|\left(\frac{1}{ps+1}\right)^{\frac{1}{p}}+\alpha^{s}|\mathcal{F}(m)|\left(\frac{1}{ps+1}\right)^{\frac{1}{p}}.$$

By making the substitution $x=(\mathscr{k}n+(1-\mathscr{k})m)$, and then evaluating the left-hand and right-hand side, we obtain

$$\frac{1}{m-n}\int_n^m|\mathcal{F}(x)|\,dx \leq \frac{|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|}{s+1}$$

multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\mathscr{b})|$, we obtain

$$\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\mathscr{b})|+\frac{\varrho}{B(\varrho)}\int_n^m|\mathcal{F}(x)|\,dx \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(\mathscr{b})|+\frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|)}{B(\varrho)(ps+1)^{\frac{1}{p}}}.$$

By simplifying the inequality, the result is as follows

$$\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(\mathscr{b})|+\frac{\varrho}{B(\varrho)}\int_n^{\mathscr{b}}|\mathcal{F}(x)|\,dx\right)+\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(\mathscr{b})|+\frac{\varrho}{B(\varrho)}\int_{\mathscr{b}}^m|\mathcal{F}(x)|\,dx\right) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(ps+1)^{\frac{1}{p}}}{B(\varrho)(ps+1)^{\frac{1}{p}}} \frac{+\varrho(m-n)(|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|)}{B(\varrho)(ps+1)^{\frac{1}{p}}}.$$

After making necessary computations, the result will be as follows

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}|\mathcal{F}|\right)(\mathscr{b}) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(ps+1)^{\frac{1}{p}}+\varrho(m-n)(|\mathcal{F}(n)|+\alpha^{s}|\mathcal{F}(m)|)}{B(\varrho)(ps+1)^{\frac{1}{p}}}.$$

This completes the proof.

Remark 2. (See [24]) If we choose $\alpha^s = 1$ with the assumptions in Theorem 6, the result will be as follows

$$\left({}_{n}^{CF}I^{\varrho}|\mathcal{F}|\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}|\mathcal{F}|\right)(\mathscr{b}) \leq \frac{2(1-\varrho)|\mathcal{F}(\mathscr{b})|(ps+1)^{\frac{1}{p}}+\varrho(m-n)(|\mathcal{F}(n)|+|\mathcal{F}(m)|)}{B(\varrho)(ps+1)^{\frac{1}{p}}}.$$

Theorem 7. Let I ⊆ ℝ. Suppose that $|\mathcal{F}|:[n,m]\subseteq I \longrightarrow \mathbb{R}$ is $\alpha-$star s-convex function on $[n,m]$ such that $\mathcal{F}\in L_1[n,m]$. Thus, the following inequality holds for Caputo-Fabrizio fractional integrals:

$$\left({}_{n}^{CF}I^{\varrho}\mathcal{F}\right)(\mathscr{b})+\left({}^{CF}I_{m}^{\varrho}\mathcal{F}\right)(\mathscr{b}) \leq \frac{2(1-\varrho)\mathcal{F}(k)\left(pq(ps+1)\right)}{B(\varrho)pq(ps+1)}$$

$$+\frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^s|\mathcal{F}(m)|)(q+p(ps+1))}{B(\varrho)pq(ps+1)}$$

where $B(\varrho)>0$ denotes the normalization function, $s\in(0,1]$, $q>1, \frac{1}{p}+\frac{1}{q}=1$ and $\varrho,\alpha\in[0,1]$.

Proof 7. By using the definition of $\alpha-$ star $s-$convex function, we can write

$$|\mathcal{F}(kn+(1-k)m)| \leq k^s|\mathcal{F}(n)|+\left((1-k)\alpha\right)^s|\mathcal{F}(m)|$$

integrating each side of the inequality over the interval $[0,1]$ with respect to k, we get

$$\int_0^1|\mathcal{F}(kn+(1-k)m)|\,dk \leq |\mathcal{F}(n)|\int_0^1 k^s\,dk+\alpha^s|\mathcal{F}(m)|\int_0^1(1-k)^s\,dk.$$

By applying Young's inequality to the right side of the inequality, the following result is yielded

$$\int_0^1|\mathcal{F}(kn+(1-k)m)|\,dk \leq |\mathcal{F}(n)|\left(\frac{1}{p}\left(\int_0^1 k^{ps}\,dk\right)+\frac{1}{q}\left(\int_0^1 1^q dk\right)\right) +\alpha^s|\mathcal{F}(m)|\left(\frac{1}{p}\left(\int_0^1(1-k)^{ps}\,dk\right)+\frac{1}{q}\left(\int_0^1 1^q dk\right)\right).$$

By making the substitution $x=(kn+(1-k)m)$, and then evaluating the left-hand and right-hand side, we get

$$\frac{1}{m-n}\int_n^m|\mathcal{F}(x)|\,dx \leq \frac{(|\mathcal{F}(n)|+\alpha^s|\mathcal{F}(m)|)\left(q+p(ps+1)\right)}{pq(ps+1)}.$$

By multiplying both sides of the inequality by $\frac{\varrho(m-n)}{B(\varrho)}$ and adding $\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)|$, we obtain

$$\frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)|+\frac{\varrho}{B(\varrho)}\int_n^m|\mathcal{F}(x)|\,dx \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| +\frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^s|\mathcal{F}(m)|)(q+p(ps+1))}{B(\varrho)pq(ps+1)}.$$

By simplifying the above inequality, the result is as follows

$$\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(b)|+\frac{\varrho}{B(\varrho)}\int_n^b|\mathcal{F}(x)|\,dx\right) +\left(\frac{1-\varrho}{B(\varrho)}|\mathcal{F}(b)|+\frac{\varrho}{B(\varrho)}\int_b^m|\mathcal{F}(x)|\,dx\right) \leq \frac{2(1-\varrho)}{B(\varrho)}|\mathcal{F}(b)| +\frac{\varrho(m-n)(|\mathcal{F}(n)|+\alpha^s|\mathcal{F}(m)|)\left(q+p(ps+1)\right)}{B(\varrho)pq(ps+1)}.$$

After making use of the necessary calculations, the proof is completed.

Remark 3. (See [24]) If we choose $\alpha^s=1$ with the assumptions of Theorem 7, the result will be as follows

$$\left({}^{CF}_{\ n}I^{\varrho}|\mathcal{F}|\right)(b)+\left({}^{CF}I^{\varrho}_{m}|\mathcal{F}|\right)(b) \leq \frac{2(1-\varrho)|\mathcal{F}(b)|\left(pq(ps+1)\right)}{B(\varrho)pq(ps+1)} +\frac{\varrho(m-n)(|\mathcal{F}(n)|+|\mathcal{F}(m)|)(q+p(ps+1))}{B(\varrho)pq(ps+1)}.$$

3. DISCUSSION AND CONCLUSION

In this study, new integral inequalities are obtained for classes of convex functions and α-star s-convex functions using the structural properties of Caputo–Fabrizio fractional integral operators. Reconsidering Hermite–Hadamard type inequalities in the classical sense within the context of fractional analysis, combined with the advantages provided by the kernel-less structure of the Caputo–Fabrizio operator, has allowed us to reach more general and comprehensive results. Accordingly, both the fractional versions of known inequalities have been reinterpreted and new limits not previously found in the literature have been established.

One of the significant contributions of this study is the revelation of the interaction between the structure of α-star s-convex functions and Caputo–Fabrizio integrals, and the presentation of such integral inequalities for these function classes for the first time. The obtained results enrich both convexity theory and the analysis of fractional integral operators; they offer potential applications in many different fields such as analytic inequalities, optimization theory, and differential equation solutions.

In terms of future studies, the application of the inequalities we obtained to more general classes of functions, fractional operators of variable order, and stochastic fractional models can be focused on. Additionally, the extension of Caputo–Fabrizio-type integrals and derivatives to operator inequalities and their counterparts in matrix analysis can be considered an important research avenue. In this context, it is expected that the presented findings will shed light on new studies in both theoretical and applied mathematics.

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