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Asymptotic Analysis of an Affine Transformation in the Supply of Missing Data

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ABSTRACT

Supply of missing data, also known as inpainting, is an important application of image processing. Wavelets are commonly used for inpainting algorithms. Shearlet transform which is an affine transformation is the improvement of the wavelet transform. An asymptotic analysis may help to evaluate the performance of an algorithm. In this article we compare the asymptotical analysis for wavelet and shearlet transforms in the case of inpainting where the missing data is shaped like a rectangle.

Keywords: Shearlet transform, wavelet transform, inpainting, asymptotic analysis

1. INTRODUCTION

Efficient representation of multidimensional data is an important issue which is an active research area [1-4]. Among these studies, Shearlet transform, introduced in 2006 by Guo et al., is a mathematical transform obtained as an extension of wavelets which is well-known as a good representation of one-dimensional data. [5,6] One of the most valuable properties of shearlets is that in order to control directional selectivity, it has the shearing parameter instead of the direction parameter in the curvelets. Due to this difference, shearlet transform can be represented by only one or a finite number of generator functions. That is why it presents optimal sparse representation for the multidimensional data. Besides, we can use shearlets for functions with finite support, and because of this transformation, we can obtain fast/superfast and effective algorithms. [10-14]

Inpainting problem is an inverse problem which is mainly concerned with finding some missing data in a signal or image. Missing data issue is a common problem in real life, and inpainting has many application areas: removing scratches or unwanted overlaid texts and graphics from old photos, or in general any image, etc.

Some of the recent publications are as following: In [15], Häuser and Ma uses a shearlet based algorithm to recover missing data from seismic data. In [16], King et al. studied data separation and reconstruction by using clustered sparsity. In [17], King, Kutyniok, and Zhuang examined inpainting problem via clustered sparsity and showed an asymptotic analysis for the issue presented in [15]. In [18], King et al. considered the inpainting problem with missing data having

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different shapes except for horizontally positioned rectangle shape; but they did not show an asymptotic analysis. In this article we consider the inpainting problem which has missing data in a horizontally positioned rectangle shape and presents an asymptotic analysis for wavelet and shearlet transforms for this particular case. [19]

In Section 2 we give brief information on the theory of wavelets and shearlets. In Section 3 we introduce basic definitions and theorems which will be used for asymptotic analysis along with reconstruction model. In Section 4 we do the asymptotic analysis of the shearlet transform used for the particular inpainting problem described in Section 3.

2. PRELIMINARIES

2.1. Wavelet Transform

For 2-D wavelets let $\gamma \in L^2(\mathbb{R}^2)$. Let the continuous affine systems of $L^2(\mathbb{R}^2)$ be defined as $\gamma_{N,s} = T_s D_N^{-1}\gamma = |\det N|^{\frac{1}{2}}\gamma\big(N(-s)\big)\colon (N,s) \in F \times \mathbb{R}^2$. $GL_2(\mathbb{R})$ is the group of invertible matrices and let F be a subset of it. Here D_N is the dilation operator on $L^2(\mathbb{R}^2)$ determined by $D_N\gamma(t)= |\det N|^{-\frac{1}{2}}\gamma(N^{-1}t),\ N \in GL_2(\mathbb{R})$. T_s is the translation operator on $L^2(\mathbb{R}^2)$, defined by $T_s\gamma(t) = \gamma(t-s), s \in \mathbb{R}^2$. Any $g \in L^2(\mathbb{R}^2)$ can be recovered from its coefficients $\big(\langle g,\gamma_{N,s}\rangle\big)_{N,s}$. Therefore, one needs to discover requirements on γ. We explain a group structure like (N,s). $(N',s') = (NN',\ s+Ns')$ to determine this. This group is said to be affine group on $\mathbb{R}^2$. It is denoted by A_2 [20].

Theorem 2.1. Let l_m be a left Haar measure of A_2 and l_i be a left invariant Haar measure on $F \subset GL_2(\mathbb{R})$. Moreover, suppose that $\gamma \in L^2(\mathbb{R}^2)$ satisfies the admissibility condition $\int_F |\hat{\gamma}(N^T\delta)|^2 |detN|\, l_i(N) = 1$. Then any function $g \in L^2(\mathbb{R}^2)$ can be recovered via the reproducing formula $g = \int_{A_2}\langle g,\gamma_{N,s}\rangle\gamma_{N,s}d\theta(N,s)$ explained weakly.

When the hypothesis of the above theorem are satisfied, $\gamma \in L^2(\mathbb{R}^2)$ is called a continuous wavelet. Thus, $L^2(\mathbb{R}^2) \ni g \to W_\gamma g(N,s) = \langle g,\gamma_{N,s}\rangle$ is defined to be the Continuous Wavelet Transform.

2.2. Shearlet Transform

Shearlets has arisen in late times by various powerful applications. [21-24] shows some of the associated work. For produce waveforms with anisotropic support is required the scaling operator. Suppose that dilation operator like in wavelets. We will use the dilation operators $D_{B_b}, b > 0$, related to parabolic scaling matrices $B_b = \begin{pmatrix} b & 0 \\ 0 & \sqrt{b}\end{pmatrix}$. The orientations of the waveforms can be changed by an orthogonal transformation. We select the shearing operator $D_{C_c},\ c \in \mathbb{R}$, where the shearing matrix C_c is given by $C_c = \begin{pmatrix} 1 & c \\ 0 & 1\end{pmatrix}$ for orthogonal transformation. The shearing matrix uses variable c associated with the slopes. Lastly, T_s is used for the translation operator. Thus, continuous shearlet system $SH(\gamma)$ can be defined by combining these 3 operators for $\gamma \in L^2(\mathbb{R}^2)$:
$SH(\gamma) = \{\gamma_{b,c,s} = T_s D_{B_b} D_{C_c}\gamma : b > 0, c \in \mathbb{R}, s \in \mathbb{R}^2\}$.

3. BASIC DEFINITIONS AND THEOREMS

In this section, we introduce some fundamental definitions and theorems which will be used later. Meyer wavelet function will be used for wavelet transformation. Auxilliary function $v \in C^\infty(\mathbb{R})$ which satisfies $v(\cdot) + v(1-\cdot) = \mathbb{1}_{\mathbb{R}}(\cdot)$ to form Meyer wavelet function is defined as

$$v(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}. \qquad (1)$$

Indicator function $\mathbb{1}_{\mathbb{R}}(\cdot)$ is defined as $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \in A^c \end{cases}$. Meyer wavelet $\psi(\omega)$ is defined as

$$\psi(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}}\sin\left(\frac{\pi}{2}v\left(\frac{3|\omega|}{2\pi}-1\right)\right)e^{j\omega/2}, & 2\pi/3 < |\omega| < 4\pi/3 \\ \frac{1}{\sqrt{2\pi}}\cos\left(\frac{\pi}{2}v\left(\frac{3|\omega|}{4\pi}-1\right)\right)e^{j\omega/2}, & 4\pi/3 < |\omega| < 8\pi/3 \\ 0, & \text{otherwise} \end{cases} \qquad (2)$$

Fourier transformation of 1-D Meyer wavelet function is then

$$W(\xi) = \begin{cases} e^{-\pi i\xi}\sin\left[\frac{\pi}{2}v(3|\xi|-1)\right], & 1/3 \le \xi \le 2/3 \\ e^{-\pi i\xi}\cos\left[\frac{\pi}{2}v\left(\frac{3}{2}|\xi|-1\right)\right], & 2/3 \le \xi \le 4/3 \\ 0, & \text{otherwise} \end{cases} \qquad (3)$$

1-D Meyer scaling function is

$$\phi(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & |\omega| < 2\pi/3 \\ \frac{1}{\sqrt{2\pi}}\cos\left(\frac{\pi}{2}v\left(\frac{3|\omega|}{2\pi}-1\right)\right)e^{j\omega/2}, & 2\pi/3 < |\omega| < 4\pi/3 \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

Fourier transformation of 1-D Meyer scaling function is

$$\hat{\phi}(\xi) = \begin{cases} 1, & |\xi| \le \frac{1}{3} \\ \cos\left[\frac{\pi}{2}v(3|\xi|-1)\right], & \frac{1}{3} \le |\xi| \le \frac{2}{3} \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Fourier transformation for $f \in L^1(\mathbb{R}^n)$ is $\mathcal{F}f := \hat{f} = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle\cdot,x\rangle}dx$. Here, $\langle\cdot,\cdot\rangle$ stands for standard Euclidian inner product. Inverse Fourier transformation is defined as $\mathcal{F}^{-1}f := \check{f} = \int_{\mathbb{R}^n} f(\xi)e^{-2\pi i\langle\cdot,\xi\rangle}d\xi$. When W^h stands for wavelet function to investigate horizontal mask case, function $W^h \in C^\infty \cap L^2(\mathbb{R}^2)$ is defined as $W^h(\xi) = W(\xi_1)\hat{\phi}(\xi_2)$. Orthonormal Meyer wavelet system is defined as $\{\psi_\lambda\}$: $\lambda = (\iota, j, k), \iota \in \{h, v, d\}, j \in \mathbb{Z},\ k \in \mathbb{Z}^2$ and function $\hat{\psi}_\lambda(\xi)$ is defined as $\hat{\psi}_\lambda(\xi) = 2^{-j}W^\iota(\xi/2^j)e^{-2\pi i k\xi/2^j},\ \ \lambda = (\iota, j, k)$. Parabolic scaling matrix A_a^h and shear matrix S_s^h are defined as $A_a^h = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and $S_s^h = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. Shearlet function $\hat{\sigma}^h$ is defined as $\hat{\sigma}^h(\xi_1, \xi_2) = W(\xi_1)V\left(\frac{\xi_2}{\xi_1}\right)$. Function $V \in L^2(\mathbb{R})$ satisfies $\hat{V} \in C^\infty(\mathbb{R})$, $supp\hat{V} \subseteq [-1,1]$ and $\sum_{k=-1}^{1}\left|\hat{V}(\xi + \pi k)\right|^2 = 1, \xi \in [-1,1]$. Notation $\hat{\sigma}_\eta$ is defined as $\hat{\sigma}_\eta = 2^{3j/4}\sigma^\iota\left(S_\ell^\iota A_{2^j}^\iota \cdot -k\right),\ \eta = (\iota, j, k, \ell)$. Here, $\iota \in \{h, v\},\ j \in \mathbb{Z},\ k \in \mathbb{Z}^2, \ell \in \mathbb{Z}$. In this case shearlet system can be defined as $\{\phi(\cdot - k): k \in \mathbb{Z}^2\} \cup \{\sigma_\eta: \iota \in \{h, v\}, j \in \mathbb{Z}, j \ge 0, k \in Z^2, \text{and } \ell \in \mathbb{Z}, |\ell| \le \lceil 2^{j/2} \rceil\}$. Here, $\lceil x \rceil$ stands for an integer larger than or equal to x.

3.1. Reconstruction Model

Modeling the reconstruction stage is highly important. To do so, let $\mathcal{H}$ stand for Hilbert space, $\mathcal{H}_M$ stand for the lost part and $\mathcal{H}_K$ stand for the known part. Then we can write $\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_K$. For a given signal $x^0 \in \mathcal{H}$, the unknown part of x^0 will be in the subspace $\mathcal{H}_M$ and the known part of x^0 will be in the subspace $\mathcal{H}_K$. P_M and P_K show corresponding orthogonal projection transformations for these subspaces. In this case recovery problem is formulated as recovering x^0 from the known $P_K x^0$. To do so, iterative thresholding will be used. During inpainting applications, recovered image sequences $(f_j)_j$ will be obtained by $(f_j)_j = \left(P_{\mathbf{R}^2/\mathcal{M}_k} w\mathcal{L}_j\right)_j$. Thresholding determination stage is done as follows: For thresholding value β_j at level j, we consider the set $\mathcal{T}_j = \{i: |\langle f_j, \phi_j\rangle| \ge \beta_j\}$ and apply iterative thresholding. In this case, recovered image at level j is obtained as $L_j = \Phi \mathbb{1}_{\mathcal{T}_j}\Phi^* w\mathcal{L}_j$. Vectors $\Phi = \{\varphi_i\}_{i\in I}$ in $\mathcal{H}$ generates a *Parseval frame* for $\mathcal{H}$ if for every $x \in \mathcal{H},\ \sum_{i\in I}|\langle x, \varphi_i\rangle|^2 = \|x\|^2$.

Definition 3.1. [17] If Φ is a Parseval frame and Λ is an index set of coefficients, then the concentration is defined on $\mathcal{H}_M$ via $\kappa = \kappa(\Lambda, \mathcal{H}_M) = \sup_{f\in\mathcal{H}_M} \frac{\|\mathbb{1}_\Lambda \Phi^* f\|_1}{\|\Phi^* f\|_1}$.

Definition 3.2. [17] Let $\Phi_1 = \{\varphi_{1i}\}_{i\in I}$ and $\Phi_2 = \{\varphi_{2j}\}_{j\in J}$ be in $\mathcal{H}$. Let $\Lambda \subseteq I$. Then the *cluster coherence* $\mu_c(\Lambda, \Phi_1; \Phi_2)$ of Φ_1 and Φ_2 with respect to Λ can be defined by $\mu_c(\Lambda, \Phi_1; \Phi_2) = \max_{j\in J} \sum_{i\in\Lambda}|\langle\varphi_{1i}, \varphi_{2j}\rangle|$.

Lemma 3.1. [17] The relation between the concentration $\kappa(\Lambda, \mathcal{H}_M)$ and cluster coherence μ_c can be obtained like that $\kappa(\Lambda, \mathcal{H}_M) \le \mu_c(\Lambda, P_M\Phi\,; P_M\Phi\,) = \mu_c(\Lambda, P_M\Phi\,; \Phi\,)$.

Lemma 3.2. [17] Let x^* and $\mathcal{T}$ be computed by the horizontal mask algorithm when $\delta > 0$. Consider that x^0 is relatively sparse in Φ with respect to $\mathcal{T}$. Then $\|x^* - x^0\|_2 \le c[\delta + \|\mathbb{1}_{\mathcal{T}}\Phi^* P_M x^0\|_1]$.

Let $w: \mathbb{R} \to [0,1]$ be a smooth function having finite support in $[-\rho, \rho]$. Let L show the real image and $w\mathcal{L}$ show the recovered image. We can use the following relation to see the effect of $w\mathcal{L}$ on the recovery model: $\langle w\mathcal{L}, f\rangle = \int_{-\rho}^{\rho} w(x_2)f(0, x_2)dx_2$. Here, 2ρ corresponds to the height of the horizontal rectangle (See Figure 4.1). Fourier transformation of $w\mathcal{L}$ is defined as $\langle\widehat{w\mathcal{L}}, f\rangle = \langle w\mathcal{L}, \hat{f}\rangle = \int_{\mathbb{R}} w(\xi_2)\int_{\mathbb{R}} f(\xi_1, \xi_2)d\xi_2\, d\xi_1$. Let $\check{F}_j$ be the filter corresponding to 2-D Meyer wavelet function and shearlet function at level j. Fourier transformation of this filter is defined as $F_j = \sum_{\iota\in\{h,v,d\}} W^\iota(2^{-j}\xi)$. The filter of $w\mathcal{L}$ is denoted as $w\mathcal{L}_j$. Thus, we obtain $w\mathcal{L}_j = w\mathcal{L} \star \check{F}_j = \int_{\mathbf{R}^2} w\mathcal{L}(\cdot - t)\,\check{F}_j(t)dt$. This equation corresponds to cros-correlation. The lemma below lets us to evaluate norm of $w\mathcal{L}_j$.

Lemma 3.3. [17] For any $c > 0$, $\|w\mathcal{L}_j\|_2 \ge c2^{j/2}$ is obtained as $j \to \infty$.

4. ASYMPTOTIC ANALYSIS OF HORIZONTAL MASK APPLICATION

To get precise error analysis, we first apply a mask function to an image so that some parts of the data is missing.
Let the function $\mathcal{M}_h(x_1, x_2) = \mathbb{1}_{\{|x_2|\le h\}}$ be horizontal mask function with height $2h$. The mask function $\mathcal{M}_h$ is shown in Figure 4.1.

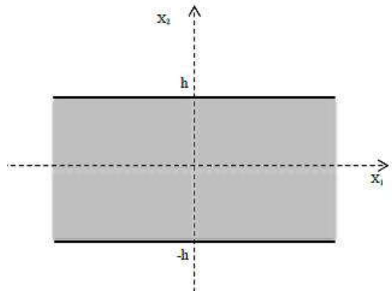

Figure 4.1 The image of horizontal mask function $\mathcal{M}_h$

Lemma 4.1. Fourier transformation $\widehat{\mathcal{M}}_h$ of the horizontal mask function $\mathcal{M}_h$ can be written as $\widehat{\mathcal{M}}_h = 2h\mathrm{sinc}(2h\xi_2)\delta(\xi_1)$.

Proof. The horizontal mask function $\mathcal{M}_h$ can be described by the Heaviside function as $\mathcal{M}_h = H\big((x_1,x_2)+(0,h)\big) - H\big((x_1,x_2)-(0,h)\big)$. From this equality, Fourier transformation $\widehat{\mathcal{M}}_h$ of the horizontal mask function $\mathcal{M}_h$ can be written as explicitly

$$\widehat{\mathcal{M}}_h = \left(e^{2\pi i h\xi_2} - e^{-2\pi i h\xi_2}\right)(2\pi i\xi_2)^{-1}\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1}dx_1$$
$$= 2\sin(2\pi h\xi_2)/(2\pi\xi_2)\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1}dx_1$$
$$= 2h\mathrm{sinc}(2h\xi_2)\int_{-\infty}^{\infty} e^{-2\pi i x_1\xi_1}dx_1$$
$$= 2h\mathrm{sinc}(2h\xi_2)\delta(\xi_1). \qquad (6)$$

We can represent the optimal δ-clustered sparsity by δ_j for filtered coefficients. Thresholding schemes will analyzed by $\delta_j = \sum_{\lambda\in\mathcal{T}_j^c}\left|\langle w\mathcal{L}_j,\Psi_\lambda\rangle\right|$ where the $\mathcal{T}_j$ coefficients are obtained in the thresholding algorithm. The inpainting achieved on the filtered levels j will be denoted by L_j. Here, we will denote the real filtered image by $w\mathcal{L}_j$; that is, $w\mathcal{L}\star\check{F}_j$, where we will denote the original image by $\mathcal{L}$. Thus, the basic theorems will show that $\frac{\|L_j - w\mathcal{L}_j\|_2}{\|w\mathcal{L}_j\|_2}\to 0,\ j\to\infty$. Here, the asymptotic behavior of the gap h_j is important for these results.

Lemma 4.2. [17] For $j\to\infty$ and $h_j = o(2^{-j})$, thresholding values $\{\beta_j\}_j$ exist such that for $j\geq j_0$, $\{k\colon |k_1|\leq \rho 2^{j(1+n_1)}, |k_2|\leq \rho 2^{jn_1}\}\subseteq \mathcal{T}_j$ holds.

Lemma 4.3. [17] For $j\to\infty$ we obtain $\delta_j = \sum_{k\in\mathcal{T}_j{}^c}\left|\langle w\mathcal{L}_j,\Psi_\lambda\rangle\right| = o\left(\|w\mathcal{L}_j\|_2\right)$.

Lemma 4.4. For $j\to\infty$ and $h_j = o(2^{-j})$ we obtain $\sum_{k\in\mathcal{T}_j}\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j,\Psi_\lambda\rangle\right| = o(2^{j/2})$.

Proof. First let us evaluate the term $\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j,\Psi_\lambda\rangle\right|$. Let G_j be the inverse Fourier transformation of F_j. From the cross-correlation theorem and the property $G_j(x) = G_j(-x)$, we have $\langle \widehat{\mathcal{M}}_h \star \widehat{w}\mathcal{L}_j, \widehat{\Psi}_\lambda\rangle = \langle \mathcal{M}_h w\mathcal{L}_j,\Psi_\lambda\rangle$. Thus we obtain $\langle \mathcal{M}_h w\mathcal{L}_j,\Psi_\lambda\rangle = \langle \mathcal{M}_h G_j \star w\mathcal{L},\Psi_\lambda\rangle = \langle \mathcal{M}_h w\mathcal{L}, G_j\star\Psi_\lambda\rangle = \langle \widehat{\mathcal{M}}_h\star\widehat{w}\mathcal{L}, F_j\widehat{\Psi}_\lambda\rangle$. For filter functions we get $\widehat{w}\mathcal{L}_j(\xi) = \widehat{w}\mathcal{L}(\xi)F_j(\xi) = \widehat{w}\mathcal{L}(\xi)\,F(\xi/2^j)$.
Function $\widehat{\Psi}_\lambda$ can be written as $\widehat{\Psi}_\lambda = 2^{-j}W^h(\xi/2^j)e^{-2\pi i\langle k,\xi/2^j\rangle}$.
We then obtain

$$\langle \mathcal{M}_{h_j} w\mathcal{L}_j,\Psi_\lambda\rangle = \langle \widehat{\mathcal{M}}_{h_j}\star\widehat{w}\mathcal{L}_j,\widehat{\Psi}_\lambda\rangle$$
$$= 2h_j\int_{\mathbb{R}^2}\mathrm{sinc}(2h_j\tau_2)\int_{\mathbb{R}^2}\widehat{w}(\xi_2)\,(\widehat{\Psi}_\lambda F_j)\big((0,\tau_2) + (\xi_1,\xi_2)\big)d\xi d\tau_2$$
$$= 2h_j\int_{\mathbb{R}}\left[\widehat{w}(\xi_2)\int_{\mathbb{R}}\mathrm{sinc}(2h_j\tau_2)F(\xi_1,(\xi_2+\tau_2)/2^j)\right.$$
$$\left.\times\ W^h(\xi_1,(\xi_2+\tau_2)/2^j)\,e^{-2\pi i\langle k_2,(\xi_2+\tau_2)/2^j\rangle}d\tau_2 d\xi_2\right]$$
$$x\ e^{-2\pi i\langle k_1,\xi_1\rangle}d\xi_1. \qquad (7)$$

From here, we get

$$\hat{G}(\xi_1) = \int_{\mathbb{R}}\widehat{w}(\xi_2)2h_j\int_{\mathbb{R}}\mathrm{sinc}(2h_j\tau_2)F(\xi_1,(\xi_2+\tau_2)/2^j)\times W^h\big(\xi_1,(\xi_2+\tau_2)/2^j\big)e^{-2\pi i\langle k_2/2^j,(\xi_2+\tau_2)\rangle}d\tau_2 d\xi_2 = \int_{\mathbb{R}}\widehat{w}(\xi_2)\widehat{H}_{\xi_1}(\xi_2)e^{-2\pi i\langle k_2/2^j,\xi_2\rangle}d\xi_2 \qquad (8)$$

and

$$\widehat{H}_{\xi_1}(\xi_2) = 2h_j\int_{\mathbb{R}}\mathrm{sinc}(2h_j\tau_2)F\big(\xi_1,(\xi_2+\tau_2)/2^j\big)$$
$$\times W^h\big(\xi_1,(\xi_2+\tau_2)/2^j\big)e^{-2\pi i\langle k_2,(\xi_2+\tau_2)/2^j\rangle}d\tau_2. \qquad (9)$$

Finite support of function $\hat{G}$ will be the set $[1/2\,,2]$. From this, we obtain $\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j,\Psi_\lambda\rangle\right| \leq cN_1\|\hat{G}\|_\infty\langle|k_1|\rangle^{-N_1}$. By Plancherel theorem and w having finite support, we obtain

$$\left|\int_R \widehat{w}(\xi_2)\,\widehat{H}_{\xi_1}(\xi_2)e^{-2\pi i\langle k_2/2^j,\xi_2\rangle}d\xi_2\right| = \left|(\widehat{w}\widehat{H}_{\xi_1})^V\right|(-k_2/2^j) = \left|(w\star H_{\xi_1})(-k_2/2^j)\right| = \left|\int w(-k_2/2^j - x)H_{\xi_1}(x)\,dx\right| \approx c\left|\int_{-k_2/2^j-\rho}^{-k_2/2^j+\rho} H_{\xi_1}(x)dx\right| \qquad (10)$$

By using basic properties of Fourier transformation and cross-correlation theorem, we can write H_{ξ_1} as follows:

$$H_{\xi_1}(x) = \left(\left(2h_j\mathrm{sinc}(2h_j\,\cdot)e^{-2\pi i k_2/2^j\cdot}\right)\star\left(FW^h(\xi_1,\cdot/2^j)\right)\right)^V(-x)$$

$= \left(2h_j \mathrm{sinc}(2h_j \cdot) e^{-2\pi i k_2/2^j \cdot}\right)^V (-x)$
$\left((FW^h)(\xi_1, \cdot/2^j)\right)^V (-x)$
$= \mathbb{1}_{[-h_j,h_j]}(-x - k_2/2^j)\left((FW^h)(\xi_1, \cdot/2^j)\right)^V (-x) \quad (11)$

From here, when $h_j < \rho$, we obtain

$$\left|\int_{-k_2/2^j-\rho}^{-k_2/2^j+\rho} H_{\xi_1}(x)dx\right| = c\left|\int_{k_2/2^j-h_j}^{k_2/2^j+h_j}\left((FW^h)(\xi_1, \cdot/2^j)\right)^V (x)dx\right| = c\left|\int_{k_2-2^j h_j}^{k_2+2^j h_j}\left((FW^h)((\xi_1, \cdot))\right)^V (x)dx\right|. \quad (12)$$

Thus by considering $|(FW^h)(|(\cdot,\cdot)|)^V(x)| \le c\langle|x|\rangle^{-N_2}$, we obtain

$$\|\hat{G}\|_\infty \le c\,\langle\min\{|k_2 - 2^j h_j|, |k_2 + 2^j h_j|\}\rangle^{-N_2}. \quad (13)$$

Combining all these considerations, we obtain $\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \Psi_\lambda\rangle\right| \le c\langle|k_1|\rangle^{-N_1}\langle\min\{|k_2 - 2^j h_j|, |k_2 + 2^j h_j|\}\rangle^{-N_2}$. By using Lemma 4.2, as a result we obtain

$$\sum_{k\in\mathcal{T}_j}\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \Psi_\lambda\rangle\right| \le c\sum_{k\in\mathcal{T}_j}\langle|k_1|\rangle^{-N_1}\langle\min\{|k_2 - 2^j h_j|, |k_2 + 2^j h_j|\}\rangle^{-N_2} \le c \quad (14)$$

Expected convergence for normalized error ℓ_2 of reconstructed filter L_j can be obtained using iterative method via wavelet transformation by the following theorem.

Theorem 4.1. Consider 2-D Meyer orthonormal system Φ with the filter L_j for $h_j = o(2^{-j})$. Then $\frac{\|L_j - w\mathcal{L}_j\|_2}{\|w\mathcal{L}_j\|_2} \to 0,\ j \to \infty$ holds.

Proof. By letting $x^* = L_j$ and $x^0 = w\mathcal{L}_j$ in the Lemma 3.2, we obtain

$$\|L_j - w\mathcal{L}_j\|_2 = \|\Phi\mathbb{1}_{\mathcal{T}^c}\Phi^* P_K w\mathcal{L}_j - \Phi\mathbb{1}_{\mathcal{T}}\Phi^* P_M w\mathcal{L}_j\|_2 = \left\|\sum_{k\in\mathcal{T}_j^c}|\langle w\mathcal{L}_j, \Psi_\lambda\rangle| - \sum_{k\in\mathcal{T}_j}\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \Psi_\lambda\rangle\right|\right\|_2 \le \left\|\sum_{k\in\mathcal{T}_j^c}|\langle w\mathcal{L}_j, \Psi_\lambda\rangle|\right\|_2 + \left\|\sum_{k\in\mathcal{T}_j}\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \Psi_\lambda\rangle\right|\right\|_2 < \underbrace{o\left(\|w\mathcal{L}_j\|_2\right)}_{\text{Lemma 4.3}} + \underbrace{c2^{j/2}}_{\text{Lemma 4.4}} \quad (15)$$

From Lemma 3.3, we obtain

$$\frac{\|L_j - w\mathcal{L}_j\|_2}{\|w\mathcal{L}_j\|_2} < \frac{o\left(\|w\mathcal{L}_j\|_2\right)}{\|w\mathcal{L}_j\|_2} + c_1 2^{-j/2} \to 0,\ j \to \infty \quad (16)$$

Similar to the wavelet transformation case above, expected convergence for normalized error ℓ_2 of reconstructed filter L_j can be obtained by the following steps: We consider the set $\mathcal{T}_j := \{\eta = (\iota; j, \ell, k) : |\langle w\mathcal{L}_j, \sigma_\eta\rangle| \ge \beta_j\}$ of coefficients of thresholding values for $\beta_j > 0$.

Lemma 4.5. [17] For all $j \ge j_0$ and for some values j_0, v_1 and $v_2 < 1/4$, thresholding coefficients $\{\beta_j\}_j$ exist as follows: $\{(\iota; j, \ell, k) : |k_1| \le \rho 2^{j(1+v_1)},\ |k_2| \le \rho 2^{jv_1}, \ell = 0; \iota = v\} \subseteq \mathcal{T}_j$. For $h_j = o(2^{-j/2})$ when $j \to \infty$.

Lemma 4.6. [17] When $j \to \infty$, we obtain $\sum_{\eta\in\mathcal{T}_j^c}|\langle w\mathcal{L}_j, \sigma_\eta\rangle| = o(2^{j/2})$

Lemma 4.7. When $j \to \infty$ for $h_j = o(2^{-j/2})$, we obtain $\sum_{\eta\in\mathcal{T}_j^c}\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \sigma_\eta\rangle\right| = o(2^{j/2}), j \to \infty$.

Proof. First, evaluation of $\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \sigma_{j,\ell,k}^\iota\rangle\right|$ is needed. Similar to the proof of Lemma 4.4, by using definitions of $\mathcal{M}_{h_j}$, $w\mathcal{L}_j$, $\sigma_{j,\ell,k}^\iota$, and cross-correlation theorem, we obtain

$$\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \sigma_{j,\ell,k}^h\rangle = 2^{j/4}\int\left[\int \hat{w}(\xi_2) 2h_j \int \mathrm{sinc}(2h_j\tau_2)\, F\left(\xi_1, \frac{\xi_2}{2^j}\right) \times W(\xi_1, \xi_2/2^j) V\left(\ell + 2^{-j/2}\frac{\tau_2 + \xi_2}{\xi_1}\right) \times e^{-2\pi i b_2(\tau_2+\xi_2)} d\tau_2 d\xi_2\right] \times e^{-2\pi i\langle\xi_1, 2^j b_1\rangle} d\xi_1 \quad (17)$$

The function $\hat{G}$ is defined as

$$\hat{G}(\xi_1) = \int \hat{w}(\xi_2) 2h_j \int \mathrm{sinc}(2h_j\tau_2)\, F\left(\xi_1, \frac{\xi_2}{2^j}\right) W(\xi_1, \xi_2/2^j)\, V\left(\ell + 2^{-j/2}\frac{\tau_2+\xi_2}{\xi_1}\right) \times e^{-2\pi i\langle b_2, \tau_2+\xi_2\rangle} d\tau_2 d\xi_2 \quad (18)$$

This function has finite support on the set $[1/2, 2]$. From here, we obtain $\left|\langle \mathcal{M}_{h_j} w\mathcal{L}_j, \sigma_{j,\ell,k}^h\rangle\right| \le c_{N_1} 2^{j/4}\|\hat{G}\|_\infty\langle|k_1|\rangle^{-N_1}$. The function $\hat{H}_{\xi_1}$ is defined as

$$\hat{H}_{\xi_1}(\xi_2) = 2h_j \int \mathrm{sinc}(2h_j\tau_2)\, F(\xi_1, \xi_2/2^j) W(\xi_1, \xi_2/2^j)\, V\left(\ell + 2^{-j/2}\frac{\tau_2+\xi_2}{\xi_1}\right) \times e^{-2\pi i\langle b_2, \tau_2+\xi_2\rangle} d\tau_2 \quad (19)$$

Let us investigate the norm $\|\hat{G}\|_\infty$:

$$\|\hat{G}\|_\infty = \left|\int \hat{w}(\xi_2)\hat{H}_{\xi_1}(\xi_2) e^{-2\pi i\langle b_2, \xi_2\rangle} d\xi_2\right| \quad (20)$$

By using Plancherel theorem and w having finite support, we obtain

$$\left|\int \hat{w}(\xi_2)\hat{H}_{\xi_1}(\xi_2)e^{-2\pi i\langle b_2,\xi_2\rangle}d\xi_2\right| = \left|\left(\hat{w}\hat{H}_{\xi_1}\right)^V(-b_2)\right| \approx c\left|\int_{-b_2-\rho}^{-b_2+\rho} H_{\xi_1}(x)dx\right| \quad (21)$$

From here, we obtain

$$H_{\xi_1}(x) = \Big(\left(2h_j\,\mathrm{sinc}(2h_j\,\cdot)\,e^{-2\pi i b_2}\right) \star \left(F(\xi_1,\cdot/2^j)W(\xi_1,\cdot/2^j)V\left(\ell + 2^{-j/2}(\cdot/\xi_1)\right)\right)\Big)^V(-x) = \left(2h_j\,\mathrm{sinc}(2h_j\,\cdot)\,e^{-2\pi i b_2}\right)^V(-x) \times \left(F(\xi_1,\cdot/2^j)W(\xi_1,\cdot/2^j)V\left(+2^{-j/2}(\cdot/\xi_1)\right)\right)^V(-x) = \mathbb{1}_{[-h_j,h_j]}(-x-b_2) \times \left(F(\xi_1,\cdot/2^j)W(\xi_1,\cdot/2^j)V\left(\ell + 2^{-j/2}(\cdot/\xi_1)\right)\right)^V(-x) \quad (22)$$

Thus when $h_j < \rho$, we obtain

$$\left|\int_{-b_2-\rho}^{-b_2+\rho} H_{\xi_1}(x)dx\right| = \left|\int_{b_2-h_j}^{b_2+h_j} \left(F(\xi_1,\cdot/2^j)W(\xi_1,\cdot/2^{j/2})V\left(\ell + 2^{-j/2}(\cdot/\xi_1)\right)\right)^V(-x)dx\right| = \left|\int_{2^{j/2}(b_2-h_j)}^{2^{j/2}(b_2+h_j)} \left(F(\xi_1,\cdot/2^{j/2})W(\xi_1,\cdot/2^{j/2}) \times V(\ell + (\cdot/\xi_1))\right)^V(-x)dx\right| \quad (23)$$

and when $(k,\ell) \in \mathcal{T}_j$, we obtain

$$\left|\int_{-b_2-\rho}^{-b_2+\rho} H_{\xi_1}(x)dx\right| = \left|\int_{k_2-2^{j/2}h_j}^{k_2+2^{j/2}h_j} \left(F(\xi_1,\cdot/2^{j/2})W(\xi_1,\cdot/2^{j/2}) \times V(\ell + (\cdot/\xi_1))\right)^V(-x)dx\right| \quad (24)$$

From here considering the evaluation

$$\left|\left(F(\xi_1,\cdot/2^{j/2})W(\xi_1,\cdot/2^{j/2})V(\ell + (\cdot/\xi_1))\right)^V(-x)\right| \leq c\langle|x|\rangle^{-N_2} \quad (25)$$

from previous calculations, we obtain

$$\|\hat{G}\|_\infty \leq c\,\langle\min\{|k_2 - 2^{j/2}h_j|, |k_2 + 2^{j/2}h_j|\}\rangle^{-N_2} \quad (26)$$

By combining all these evaluations, we obtain

$$\left|\langle\mathcal{M}_{h_j}w\mathcal{L}_j, \sigma_{j,\ell,k}^h\rangle\right| \leq c2^{j/4}\langle|k_1|\rangle^{-N_1}\,\langle\min\left\{\left|k_2 - 2^{j/2}h_j\right|, \left|k_2 + 2^{j/2}h_j\right|\right\}\rangle^{-N_2} \quad (27)$$

Thus finally when $v_2 < 1/4$ from Lemma 4.5, we obtain

$$\frac{1}{c}\sum_{\eta\in\mathcal{T}_j}\left|\langle\mathcal{M}_{h_j}w\mathcal{L}_j, \sigma_\eta\rangle\right| \leq 2^{j/4}\sum_{\eta\in\mathcal{T}_j}\langle|k_1|\rangle^{-N_1}\,\langle\min\left\{\left|k_2 - 2^{j/2}h_j\right|, \left|k_2 + 2^{j/2}h_j\right|\right\}\rangle^{-N_2} \leq 2^{j(1/4+v_2)} \quad (28)$$

As in the wavelet transformation case above, expected convergence for normalized error ℓ_2 of reconstructed filter L_j can be obtained using iterative method via shearlet transformation by the following theorem.

Teorem 4.2. Consider filter L_j for $h_j = o(2^{-j/2})$ with 2-D shearlet system Φ. Then

$\frac{\|L_j - w\mathcal{L}_j\|_2}{\|w\mathcal{L}_j\|_2} \to 0,\ j \to \infty$ holds.

Proof. Similarly by letting $x^* = L_j$ and $x^0 = w\mathcal{L}_j$ in the Lemma 3.2, we obtain

$$\|L_j - w\mathcal{L}_j\|_2 = \|\Phi\mathbb{1}_{\mathcal{T}^c}\Phi^*\,P_K w\mathcal{L}_j - \Phi\mathbb{1}_{\mathcal{T}}\Phi^*\,P_M w\mathcal{L}_j\|_2 = \left\|\sum_{\eta\in\mathcal{T}_j^c}|\langle w\mathcal{L}_j, \sigma_\eta\rangle| - \sum_{\eta\in\mathcal{T}_j^c}\left|\langle\mathcal{M}_{h_j}w\mathcal{L}_j, \sigma_\eta\rangle\right|\right\|_2 \leq \left\|\sum_{\eta\in\mathcal{T}_j^c}|\langle w\mathcal{L}_j, \sigma_\eta\rangle|\right\|_2 + \left\|\sum_{\eta\in\mathcal{T}_j^c}\left|\langle\mathcal{M}_{h_j}w\mathcal{L}_j, \sigma_\eta\rangle\right|\right\|_2 < \underbrace{o(2^{j/2})}_{\text{Lemma 4.6}} + \underbrace{o(2^{j/2})}_{\text{Lemma 4.7}} \quad (29)$$

From Lemma 3.3, we obtain

$$\frac{\|L_j - w\mathcal{L}_j\|_2}{\|w\mathcal{L}_j\|_2} < \frac{o(2^{j/2})}{\|w\mathcal{L}_j\|_2} + \frac{o(2^{j/2})}{\|w\mathcal{L}_j\|_2} \to 0, \quad j \to \infty \quad (30)$$

Thus, we prove that the image can be reconstructed well asymptotically, when the height of the horizontal mask decays faster than $2^{-j/2}$.

5. CONCLUSION

In this paper, we show the asymptotic analysis of wavelet and shearlet transforms used for the inpainting where the missing data have a horizontal rectangle shape. As a conclusion, we found out that the shearlet transformation is more effective for the problem discussed than the wavelet transform. If the height of the horizontal mask decays faster than $2^{-j/2}$, we proved that the image can be reconstructed asymptotically.

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