Erciyes Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 33, Sayı 2 , 2017

Erciyes University
Journal of Natural and Applied Sciences
Volume 33, Issue 2, 2017

Oktoniyon Matrislerin Yapısı Üzerine

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(Alınış / Received: 28.02.2017, Kabul / Accepted: 29.05.2017, Online Yayınlanma / Published Online: 29.08.2017)

Anahtar Kelimeler
De-Moiver formülü,
Euler formülü,
Reel oktoniyon,
Hamilton Operatörü

Öz: Oktoniyonların string teorisi, özel relativite ve kuantum mantık gibi bir çok alanda uygulamaları vardır. Hamilton operatörleri yardımı ile, reel oktoniyonlar 8×8 matrisler ile gösterilebilir. Bu matrisler 8 boyutlu öklid uzayında dönmeleri ve homotetik hareketleri ifade etmek için kullanılır. Bu makalede, reel oktoniyonların bazı cebirsel özelliklerini inceledik ve De Moivre formülünü kullanarak oktoniyon matrislerinin her kuvvetini elde ettik. Ayrıca oktoniyon matrislerinin kuvvetleri arasındaki ilşkiyi bulduk. De Moivre formülünün A^n=I_8 , ∀n≥3 şartını sağlayan sayılamaz sonsuz çoklukta birim oktoniyon matrislerinin varlığını gerektirdiğini gösterdik.

On The Structure Of The Octonion Matrices

Keywords
De-Moiver's formula,
Euler's formula,
Real octonion,
Hamilton Operator

Abstract: Octonions have applications in fields such as string theory, special relativity, and quantum logic. With the aid of the Hamilton operators, real octonions have been expressed in terms of 8×8 matrices. These matrices are being used to describe the rotation and determine a homothetic motions in 8-dimensional Euclidean space E8. In this paper, we study some algebraic properties of real octonions, and by using De Moivre's formula, we obtain any power of such matrices. Also, a relation between the powers of matrices of octonions is given. The De Moivre's formula implies that there are uncountably many matrices of the unit octonions A satisfying $A^n = I_8$ for every integer $n \geq 3$.

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1. Introduction

The octonions were discovered in 1843 by John T. Graves. In mathematics, the real octonions are a normed division algebra over the real numbers, usually represented by $\mathbb{O}$ [1, 3]. A matrix corresponding to Hamilton operators which is defined for octonions has determined a homothetic motion in E^8 [2]. The algebraic properties and geometric applications of these matrices are studied by Tain [4].

In this paper, after reviewing some algebraic properties of the real octonions, we study the Euler's and De Moivre's formulas for the matrices associated with these octonions. With the aid of De-Moivre's formula, we obtain *n-th* roots of such matrices. Finally, the relations between the powers of these matrices are given. We give some example for the purpose of more clarification. We hope that this work will contribute to the study of kinematics and physics.

2. Materials and Methods

The real octonions are an eight-dimensional extension of the two-dimensional complex numbers. They form the largest normed division algebra over the real numbers.

Every octonion is a real linear combination of the unit octonions:

$$\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\},$$

where e_0 is the scalar or real element; and they satisfy the equalities is given the table below;

The Multiplication of unit octonions

1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Every octonion x can be written in the form

$$x = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7$$

with real coefficients $\{a_i\}$. By the Cayley-Dichson process, a octonion x can also be written as

$$x = a' + a'' e,$$

where $e^2 = -1$ and

$$a', a'' \in \mathbb{H} \equiv \left\{ q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \;\middle|\; e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1 \,, a_i \in \mathbb{R} \right\}$$

the real quaternion division algebra. The set of all real octonions is denoted by $\mathbb{O}$. Addition and subtraction of octonions is done by adding and subtracting corresponding terms and hence their coefficients, like quaternions. Multiplication is distributive over addition, so the product of two octonions can be calculated by summing the product of all the terms, again like quaternions. Multiplication of octonion can be described by a matrix-vector product as

$$x.\omega = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix},$$

where $x, \omega \in \mathbb{O}$. It is useful, therefore, to define the following terms:

The *conjugate* of x is

$$\overline{x} = a' - e\,a''.$$

The *norm* of x is

$$N_x = x\overline{x} = \overline{x}\,x = \sum_{i=0}^{7} a_i^2.$$

If $N_x = 1$, then x is called a unit real octonion.
The *inverse* of x with $N_x \neq 0$, is

$$x^{-1} = \frac{1}{N_x}\overline{x}.$$

Every nonzero real octonion $x = a_0\,e_0 + a_1\,e_1 + a_2\,e_2 + a_3\,e_3 + a_4\,e_4 + a_5\,e_5 + a_6\,e_6 + a_7\,e_7$ (with $N_x \neq 0$) can be written in the polar form

$$x = \sqrt{N_x}(\cos\theta + \vec{w}\sin\theta)$$

where $\cos\theta = \dfrac{a_0}{\sqrt{N_x}}$ and $\sin\theta = \dfrac{\left(\sum_{i=1}^{7} a_i^2\right)^{1/2}}{\sqrt{N_x}}$. The unit vector $\vec{w}$ is given by

$$\vec{w} = (w_1, w_2, ..., w_7) = \frac{1}{(\sum_{i=1}^{7} a_i^2)^{1/2}}(a_1, a_2, ..., a_7).$$

Example 1. The polar form of the real octonions $x_1 = \dfrac{\sqrt{2}}{2} + \dfrac{1}{\sqrt{14}}(1,1,1,1,1,1,1)$ is $x_1 = \cos\dfrac{\pi}{4} + \vec{w}_1\sin\dfrac{\pi}{4}$ and $x_2 = \sqrt{3} + \dfrac{1}{\sqrt{7}}(1,1,1,1,1,1,1)$ is $x_2 = 2(\cos\dfrac{\pi}{6} + \vec{w}_2\sin\dfrac{\pi}{6})$.

Theorem 1. (De Moivre's formula) Let $x = e^{\vec{w}\theta} = \cos\theta + \vec{w}\sin\theta$ be a unit real octonion. Then for any integer n;

$$x^n = e^{n\vec{w}\theta} = \cos n\theta + \vec{w}\sin n\theta.$$

Proof: The proof is easily followed by induction on n [5]. ■

Example 2. Let $x = \dfrac{\sqrt{2}}{2} + \dfrac{1}{\sqrt{14}}(1,1,1,1,1,1,1)$ be a unit real octonion. Every power of this octonion is found with the aid of Theorem 1. For example, *20-th* and *53-th* powers are

$$\begin{aligned} x^{20} &= \cos 20\frac{\pi}{4} + \vec{w}\sin 20\frac{\pi}{4} \\ &= \cos 5\pi + 0 = -1, \end{aligned}$$

and

$$\begin{aligned} x^{53} &= \cos 53\frac{\pi}{4} + \vec{w}\sin 53\frac{\pi}{4} \\ &= -\frac{\sqrt{2}}{2} - \frac{1}{\sqrt{14}}(1,1,1,1,1,1,1). \end{aligned}$$

Theorem 2. De Moivre's formula implies that there are uncountably many unit real octonion x satisfying $x^n = 1$ for $n \geq 3$.

Proof: For every unit vector $\vec{w}$, the unit octonion

$$x = \cos\frac{2\pi}{n} + \vec{w}\sin\frac{2\pi}{n}$$

is of order n. For $n = 1$ or $n = 2$, the octonion x is independent of $\vec{w}$. ■

Example 3. $x = \dfrac{\sqrt{2}}{2} + \dfrac{1}{\sqrt{14}}(1,1,1,1,1,1,1)$ is of order 8 and $x = \dfrac{1}{2} + \dfrac{1}{2}(1,1,0,0,0,\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}})$ is of order 6.

Theorem 3. Let $x = \cos\theta + \vec{w}\sin\theta$ be a unit octonion. The equation $a^n = x$ has n roots, and they are

$$a_k = \cos(\frac{\theta+2k\pi}{n}) + \vec{w}\sin(\frac{\theta+2k\pi}{n}), \qquad k = 0,1,2,\ldots,n-1.$$

Proof: The proof can be found in [5]. ■

Example 4. Let $x = \frac{\sqrt{3}}{2} + \frac{1}{4}(1, \frac{-1}{\sqrt{2}}, 1, 0, \frac{1}{\sqrt{2}}, -1, 0) = \cos\frac{\pi}{6} + \vec{w}\sin\frac{\pi}{6}$ be a unit real octonion. The cube roots of the octonion x are

$$x_k^{1/3} = \cos(\frac{\pi/6+2k\pi}{3}) + \vec{w}\sin(\frac{\pi/6+2k\pi}{3}), \qquad k = 0,1,2.$$

For $k=0$, the first root is $x_0^{1/3} = \cos\frac{\pi}{18} + \vec{w}\sin\frac{\pi}{18} = 0.98 + 0.17\vec{w}$, and the second one for $k=1$ is $x_1^{1/3} = \cos\frac{13\pi}{18} + \vec{w}\sin\frac{13\pi}{18} = -0.64 + 0.76\vec{w}$ and third one is $x_2^{1/3} = \cos\frac{25\pi}{18} + \vec{w}\sin\frac{25\pi}{18} = -0.34 - 0.93\vec{w}$. Also, it is easy to see that $x_0^{\frac{1}{3}} + x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} = 0$.

The relation between the powers of octonions can be found in the following theorem.

Theorem 4. Let x be a unit real octonion with the polar form $x = \cos\theta + \vec{u}\sin\theta$. If $m = \frac{2\pi}{\theta} \in Z^+ - \{1\}$, then $x^n = x^p$ if and only if $n \equiv p \pmod{m}$.

Proof: Let $n \equiv p \pmod{m}$. Then we have $n = am + p$, where $a \in Z$.

$$\begin{aligned} x^n &= \cos n\theta + \vec{u}\sin n\theta \\ &= \cos(am+p)\theta + \vec{u}\sin(am+p)\theta \\ &= \cos(a\frac{2\pi}{\theta}+p)\theta + \vec{u}\sin(a\frac{2\pi}{\theta}+p)\theta \\ &= \cos(p\theta + a2\pi) + \vec{u}\sin(p\theta + a2\pi) \\ &= \cos p\theta + \vec{u}\sin p\theta \\ &= x^p. \end{aligned}$$

Now suppose $x^n = \cos n\theta + \vec{u}\sin n\theta$ and $x^p = \cos p\theta + \vec{u}\sin p\theta$. If $x^n = x^p$ then we get

$$\cos n\theta = \cos p\theta$$

and

$$\sin n\theta = \sin p\theta,$$

which means $n\theta = p\theta + 2\pi a,\ a \in Z$. Thus $n = p + \frac{2\pi}{\theta}a$ or $n \equiv p \pmod{m}$. ■

Example 5. Let $x = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{14}}(1,1,1,1,1,1,1)$ be a unit real octonion. From Theorem 4, $m = \frac{2\pi}{\pi/4} = 8$, so we have

$$\begin{aligned} &x = x^9 = x^{17} = \ldots \\ &x^2 = x^{10} = x^{18} = \ldots \\ &x^3 = x^{11} = x^{19} = \ldots \\ &x^4 = x^{12} = x^{20} = \ldots = -1 \\ &\vdots \\ &x^8 = x^{16} = x^{24} = \ldots = 1. \end{aligned}$$

3. Results and Discussion

3.1. 8×8 real matrix representations of octonions

We introduce the $\mathbb{R}$-linear transformations representing left multiplication in $\mathbb{O}$ and look for also the De-Moiver's formula for corresponding matrix representation. Let

$$x = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 = a' + e a'',$$

be an octonion and $\varphi_q : \mathrm{O} \to \mathrm{O}$ defined as follows:

$$\varphi_x(\omega) = x\omega, \qquad \omega \in \mathrm{O}.$$

The Hamilton's operator φ_x, could be represented as the matrix;

$$\varphi_x = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{1}$$

or equality

$$\varphi_x = \begin{bmatrix} H^+(a') & -N^T \\ N & H^-(a') \end{bmatrix},$$

where H^+, H^- are Hamilton operators for quaternions and N is a 4×4 matrix [2]. Some algebraic properties of Hamilton operators of real octonion are given by Tain [3]. The polar form of matrix (1) is

$$\begin{bmatrix} \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta & -w_4\sin\theta & -w_5\sin\theta & -w_6\sin\theta & -w_7\sin\theta \\ w_1\sin\theta & \cos\theta & -w_3\sin\theta & w_2\sin\theta & -w_5\sin\theta & w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta \\ w_2\sin\theta & w_3\sin\theta & \cos\theta & -w_1\sin\theta & -w_6\sin\theta & -w_7\sin\theta & w_4\sin\theta & w_5\sin\theta \\ w_3\sin\theta & -w_2\sin\theta & w_1\sin\theta & \cos\theta & -w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & w_4\sin\theta \\ w_4\sin\theta & w_5\sin\theta & w_6\sin\theta & w_7\sin\theta & \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta \\ w_5\sin\theta & -w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta & w_1\sin\theta & \cos\theta & w_3\sin\theta & -w_2\sin\theta \\ w_6\sin\theta & -w_7\sin\theta & -w_4\sin\theta & w_5\sin\theta & w_2\sin\theta & -w_3\sin\theta & \cos\theta & w_1\sin\theta \\ w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & -w_4\sin\theta & w_3\sin\theta & w_2\sin\theta & -w_1\sin\theta & \cos\theta \end{bmatrix}.$$

Theorem 5. (De Moivre's formula) For an integer n and matrix

$$A = \begin{bmatrix} \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta & -w_4\sin\theta & -w_5\sin\theta & -w_6\sin\theta & -w_7\sin\theta \\ w_1\sin\theta & \cos\theta & -w_3\sin\theta & w_2\sin\theta & -w_5\sin\theta & w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta \\ w_2\sin\theta & w_3\sin\theta & \cos\theta & -w_1\sin\theta & -w_6\sin\theta & -w_7\sin\theta & w_4\sin\theta & w_5\sin\theta \\ w_3\sin\theta & -w_2\sin\theta & w_1\sin\theta & \cos\theta & -w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & w_4\sin\theta \\ w_4\sin\theta & w_5\sin\theta & w_6\sin\theta & w_7\sin\theta & \cos\theta & -w_1\sin\theta & -w_2\sin\theta & -w_3\sin\theta \\ w_5\sin\theta & -w_4\sin\theta & w_7\sin\theta & -w_6\sin\theta & w_1\sin\theta & \cos\theta & w_3\sin\theta & -w_2\sin\theta \\ w_6\sin\theta & -w_7\sin\theta & -w_4\sin\theta & w_5\sin\theta & w_2\sin\theta & -w_3\sin\theta & \cos\theta & w_1\sin\theta \\ w_7\sin\theta & w_6\sin\theta & -w_5\sin\theta & -w_4\sin\theta & w_3\sin\theta & w_2\sin\theta & -w_1\sin\theta & \cos\theta \end{bmatrix}, \tag{2}$$

n-*th* power of the matrix A reads as

$$A^{n}=\begin{bmatrix} \cos n\theta & -w_1\sin n\theta & -w_2\sin n\theta & -w_3\sin n\theta & -w_4\sin n\theta & -w_5\sin n\theta & -w_6\sin n\theta & -w_7\sin n\theta \\ w_1\sin n\theta & \cos n\theta & -w_3\sin n\theta & w_2\sin n\theta & -w_5\sin n\theta & w_4\sin n\theta & w_7\sin n\theta & -w_6\sin n\theta \\ w_2\sin n\theta & w_3\sin n\theta & \cos n\theta & -w_1\sin n\theta & -w_6\sin n\theta & -w_7\sin n\theta & w_4\sin n\theta & w_5\sin n\theta \\ w_3\sin n\theta & -w_2\sin n\theta & w_1\sin n\theta & \cos n\theta & -w_7\sin n\theta & w_6\sin n\theta & -w_5\,\mathrm{sinn}\,\theta & w_4\,\mathrm{sinn}\,\theta \\ w_4\sin n\theta & w_5\sin n\theta & w_6\sin n\theta & w_7\sin n\theta & \cos n\theta & -w_1\sin n\theta & -w_2\sin n\theta & -w_3\sin n\theta \\ w_5\sin n\theta & -w_4\sin n\theta & w_7\sin n\theta & -w_6\sin n\theta & w_1\sin n\theta & \cos n\theta & w_3\sin n\theta & -w_2\sin\theta \\ w_6\sin n\theta & -w_7\sin n\theta & -w_4\sin n\theta & w_5\sin n\theta & w_2\sin n\theta & -w_3\sin n\theta & \cos n\theta & w_1\sin n\theta \\ w_7\sin n\theta & w_6\sin n\theta & -w_5\sin n\theta & -w_4\sin n\theta & w_3\sin n\theta & w_2\sin n\theta & -w_1\sin n\theta & \cos n\theta \end{bmatrix}.$$

Proof*:* The proof is easily followed by induction on *n*. ■

Example 6. Let $x=\sqrt{3}+\frac{1}{\sqrt{7}}(1,1,1,1,1,1,1)$ be a real octonion. The matrix corresponding to this octonion is

$$A=\frac{1}{\sqrt{7}}\begin{bmatrix} \sqrt{21} & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & \sqrt{21} & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & \sqrt{21} & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & \sqrt{21} & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & \sqrt{21} & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & \sqrt{21} & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & \sqrt{21} & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & \sqrt{21} \end{bmatrix},$$

every power of this matrix with the aid of Theorem 5 is found to be expressible similarly, for example, *52-th* and *90-th* powers are

$$A^{52}=2^{52}\times\frac{\sqrt{3}}{\sqrt{7}}\begin{bmatrix} -\frac{\sqrt{7}}{2\sqrt{3}} & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -\frac{\sqrt{7}}{2\sqrt{3}} & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -\frac{\sqrt{7}}{2\sqrt{3}} & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -\frac{\sqrt{7}}{2\sqrt{3}} & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -\frac{\sqrt{7}}{2\sqrt{3}} & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -\frac{\sqrt{7}}{2\sqrt{3}} & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -\frac{\sqrt{7}}{2\sqrt{3}} & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -\frac{\sqrt{7}}{2\sqrt{3}} \end{bmatrix},$$

$$A^{90}=2^{90}\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}=-2^{90}I_8.$$

For more details we refer the reader to the Appendix.

Theorem 6. De Moivre's formula implies that there are uncountably many matrices of the unit octonions A satisfying $A^n = I_8$ for every integer $n \geq 3$.

Proof: For every unit vector $\vec{w}$, the matrix A of unit octonion $x = \cos\frac{2\pi}{n} + \vec{w}\sin\frac{2\pi}{n}$ is

$$A = \begin{bmatrix} \cos\frac{2\pi}{n} & -w_1\sin\frac{2\pi}{n} & -w_2\sin\frac{2\pi}{n} & -w_3\sin\frac{2\pi}{n} & -w_4\sin\frac{2\pi}{n} & -w_5\sin\frac{2\pi}{n} & -w_6\sin\frac{2\pi}{n} & -w_7\sin\frac{2\pi}{n} \\ w_1\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & -w_3\sin\frac{2\pi}{n} & w_2\sin\frac{2\pi}{n} & -w_5\sin\frac{2\pi}{n} & w_4\sin\frac{2\pi}{n} & w_7\sin\frac{2\pi}{n} & -w_6\sin\frac{2\pi}{n} \\ w_2\sin\frac{2\pi}{n} & w_3\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & -w_1\sin\frac{2\pi}{n} & -w_6\sin\frac{2\pi}{n} & -w_7\sin\frac{2\pi}{n} & w_4\sin\frac{2\pi}{n} & w_5\sin\frac{2\pi}{n} \\ w_3\sin\frac{2\pi}{n} & -w_2\sin\frac{2\pi}{n} & w_1\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & -w_7\sin\frac{2\pi}{n} & w_6\sin\frac{2\pi}{n} & -w_5\sin\frac{2\pi}{n} & w_4\sin\frac{2\pi}{n} \\ w_4\sin\frac{2\pi}{n} & w_5\sin\frac{2\pi}{n} & w_6\sin\frac{2\pi}{n} & w_7\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & -w_1\sin\frac{2\pi}{n} & -w_2\sin\frac{2\pi}{n} & -w_3\sin\frac{2\pi}{n} \\ w_5\sin\frac{2\pi}{n} & -w_4\sin\frac{2\pi}{n} & w_7\sin\frac{2\pi}{n} & -w_6\sin\frac{2\pi}{n} & w_1\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & w_3\sin\frac{2\pi}{n} & -w_2\sin\frac{2\pi}{n} \\ w_6\sin\frac{2\pi}{n} & -w_7\sin\frac{2\pi}{n} & -w_4\sin\frac{2\pi}{n} & w_5\sin\frac{2\pi}{n} & w_2\sin\frac{2\pi}{n} & -w_3\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} & w_1\sin\frac{2\pi}{n} \\ w_7\sin\frac{2\pi}{n} & w_6\sin\frac{2\pi}{n} & -w_5\sin\frac{2\pi}{n} & -w_4\sin\frac{2\pi}{n} & w_3\sin\frac{2\pi}{n} & w_2\sin\frac{2\pi}{n} & -w_1\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{bmatrix}$$

A is of order n. For $n=1$ or $n=2$, the octonion x is independent of $\vec{w}$. ■

3.2. Euler's formula for matrices associated with octonions

Let

$$W = \begin{bmatrix} 0 & -w_1 & -w_2 & -w_3 & -w_4 & -w_5 & -w_6 & -w_7 \\ w_1 & 0 & -w_3 & w_2 & -w_5 & w_4 & w_7 & -w_6 \\ w_2 & w_3 & 0 & -w_1 & -w_6 & -w_7 & w_4 & w_5 \\ w_3 & -w_2 & w_1 & 0 & -w_7 & w_6 & -w_5 & w_4 \\ w_4 & w_5 & w_6 & w_7 & 0 & -w_1 & -w_2 & -w_3 \\ w_5 & -w_4 & w_7 & -w_6 & w_1 & 0 & w_3 & -w_2 \\ w_6 & -w_7 & -w_4 & w_5 & w_2 & -w_3 & 0 & w_1 \\ w_7 & w_6 & -w_5 & -w_4 & w_3 & w_2 & -w_1 & 0 \end{bmatrix},$$

be a real matrix. One immediately finds $W^2 = -I_8$. We have a natural generalization of Euler's formula for matrix W;

$$\begin{aligned} e^{W\theta} &= I_8 + W\theta + \frac{(W\theta)^2}{2!} + \frac{(W\theta)^3}{3!} + \frac{(W\theta)^4}{4!} + \ldots \\ &= I_4(1 - \frac{\theta^2}{2!} + \frac{\theta 4}{4!} - \ldots) + W(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots) \\ &= I_4\cos\theta + W.\sin\theta \end{aligned}.$$

3.3. *n-th R*oots of matrices of real octonions

Let $x=\cos\theta+\vec{w}\sin\theta$ be a unit octonion. The matrix associated with this octonion x is of the form (1). The equation $x^n = A$ has n roots, and they are as follows

$$A_k^{\frac{1}{n}}=\begin{bmatrix}
\cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & -w_6\sin(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{\theta+2k\pi}{n}) \\
w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & w_4\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & -w_6\sin(\frac{\theta+2k\pi}{n}) \\
w_2\sin(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & -w_6\sin(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{\theta+2k\pi}{n}) & w_4\sin(\frac{\theta+2k\pi}{n}) & w_5\sin(\frac{\theta+2k\pi}{n}) \\
w_3\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & w_4\sin(\frac{\theta+2k\pi}{n}) \\
w_4\sin(\frac{\theta+2k\pi}{n}) & w_5\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) \\
w_5\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & -w_6\sin(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{\theta+2k\pi}{n}) \\
w_6\sin(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_5\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & -w_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{\theta+2k\pi}{n}) \\
w_7\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & -w_4\sin(\frac{\theta+2k\pi}{n}) & w_3\sin(\frac{\theta+2k\pi}{n}) & w_2\sin(\frac{\theta+2k\pi}{n}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n})
\end{bmatrix}$$

For $k=0$, the first root is

$$A_0^{\frac{1}{n}}=\begin{bmatrix}
\cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & -w_5\sin\frac{\theta}{n} & -w_6\sin\frac{\theta}{n} & -w_7\sin\frac{\theta}{n} \\
w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & -w_5\sin\frac{\theta}{n} & w_4\sin\frac{\theta}{n} & w_7\sin\frac{\theta}{n} & -w_6\sin\frac{\theta}{n} \\
w_2\sin\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & -w_6\sin\frac{\theta}{n} & -w_7\sin\frac{\theta}{n} & w_4\sin\frac{\theta}{n} & w_5\sin\frac{\theta}{n} \\
w_3\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_7\sin\frac{\theta}{n} & w_6\sin\frac{\theta}{n} & -w_5\sin\frac{\theta}{n} & w_4\sin\frac{\theta}{n} \\
w_4\sin\frac{\theta}{n} & w_5\sin\frac{\theta}{n} & w_6\sin\frac{\theta}{n} & w_7\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} \\
w_5\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_7\sin\frac{\theta}{n} & -w_6\sin\frac{\theta}{n} & w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & -w_2\sin\frac{\theta}{n} \\
w_6\sin\frac{\theta}{n} & -w_7\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_5\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & -w_3\sin\frac{\theta}{n} & \cos\frac{\theta}{n} & w_1\sin\frac{\theta}{n} \\
w_7\sin\frac{\theta}{n} & w_6\sin\frac{\theta}{n} & -w_5\sin\frac{\theta}{n} & -w_4\sin\frac{\theta}{n} & w_3\sin\frac{\theta}{n} & w_2\sin\frac{\theta}{n} & -w_1\sin\frac{\theta}{n} & \cos\frac{\theta}{n}
\end{bmatrix}.$$

Similarly, for $k=n-1$, we obtain the *n-th* root.

3.4. Relations between powers of octonions matrices

Some relations between the powers of matrices associated with a real octonion is sketched in the following theorem.

Teorem 7. Let x be a unit real octonion with the polar form $x=\cos\theta+\vec{u}\sin\theta.$ And let $m=\frac{2\pi}{\theta}\in Z^{+}-\{1\}$ and the matrix A correspond to x. Then $n\equiv p(\operatorname{mod} m)$ is true if and only if $A^n = A^p$.

Proof: The proof is similar proof of Theorem 4. ■

Example 7. Let $x=-\frac{1}{2}+\frac{1}{\sqrt{8}}(1,-1,1,0,1,-1,1)$ be a unit real octonion. The matrix corresponding to this octonion is

$$A=\frac{1}{2}\begin{bmatrix} -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{bmatrix},$$

The square roots of the matrix A are:

i. The first root for $k=0$ is

$$A_0^{\frac{1}{2}}=\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{2} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{2} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{2} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{2} \end{bmatrix},$$

ii. and the second one for $k=1$ is

$$A_1^{\frac{1}{2}}=\begin{bmatrix} -\frac{1}{2} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\ 0 & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{2} & \frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} & -\frac{1}{2} & -\frac{1}{\sqrt{8}} & -\frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} & -\frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & -\frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & -\frac{1}{2} \end{bmatrix}.$$

See appendix for more clarification. Also, it is easy to see that $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

From the Theorem 7, with $m = \frac{2\pi}{2\pi/3} = 3$, we get

$$A = A^4 = A^7 = A^{10} = \dots$$
$$A^2 = A^5 = A^8 = A^{11} = \dots$$
$$A^3 = A^6 = A^9 = A^{12} = \dots = I_4.$$

4. Results and discussion

In this paper, we defined and gave some of algebraic properties of real octonions and investigated the De Moivre's formulas for the matrices associated with octonions. The relation between the powers of these matrices is given in Theorem 7. We also showed that the equation $A^n = 1$ has uncountably many solutions for any general unit real octonions (Theorem 6). In the next work, we will repeat this study for split octonions [6].

REFERENCES

[1] Octonion, http://en.wikipedia.org/wiki/Octonion.

[2] Yayli, Y., Bukuc, B. 1995. Homothetic motions at E^8 with Cayley numbers, Mechanism and Machine Theory, 27(1995), 417-420.

[3] Dray, T., Manogue, C. A. 2014. The geometry of the octonions, World scientific publishing Co., USA.

[4] Tian, Y. 2000. Matrix representation of octonions and their applications, Advance in application Clifford algebras 10(2000) 61-90.

[5] Flaut, C., Shpakivskyi, V. 2014. De Moivre's Formula and Euler's Formula for Octonions, arXiv: 1405.5655v1 (2104).

[6] Jafari, M. Split Octonion Analysis, Representation Theory and Geometry, https://www.researchgate.net /publication/261507762_Split_Octonion_Analysis_Representation_Theory_and_Geometry

APPENDIX

In example 6, the polar form of matrix A is

$$A=2\begin{bmatrix}
\cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6}
\end{bmatrix},$$

52-th and *90-th* powers are

$$A^{52}=2^{52}\begin{bmatrix}
\cos 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 52\frac{\pi}{6} & \cos 52\frac{\pi}{6}
\end{bmatrix}$$

$$=2^{52}\begin{bmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2} & \frac{\sqrt{3}}{\sqrt{7}} \\
\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & \frac{\sqrt{3}}{\sqrt{7}} & -\frac{\sqrt{3}}{\sqrt{7}} & -\frac{1}{2}
\end{bmatrix}$$

and

$$A^{90}=2^{90}\begin{bmatrix}
\cos 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin 90\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} \\
\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & -\frac{2}{\sqrt{7}}\sin\frac{\pi}{6} & \cos\frac{\pi}{6}
\end{bmatrix}$$

$$=2^{90}\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.$$

In example 7, the square roots of the matrix A can be calculated as follows:

$$A_k^{\frac{1}{2}} =$$

$$\begin{bmatrix}
\cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & -w_5\sin(\frac{2k\pi+2\pi/3}{2}) & w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_7\sin(\frac{2k\pi+2\pi/3}{2}) \\
w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{\theta+2k\pi}{n}) & -w_5\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & w_7\sin(\frac{2k\pi+2\pi/3}{2}) & w_6\sin(\frac{2k\pi+2\pi/3}{2}) \\
-w_2\sin(\frac{2k\pi+2\pi/3}{2}) & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_7\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & w_5\sin(\frac{2k\pi+2\pi/3}{2}) \\
w_3\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & w_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -w_7\sin(\frac{2k\pi+2\pi/3}{2}) & -w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_5\sin(\frac{2k\pi+2\pi/3}{2}) & 0 \\
0 & w_5\sin(\frac{2k\pi+2\pi/3}{2}) & -w_6\sin(\frac{\theta+2k\pi}{n}) & w_7\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) \\
w_5\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & w_7\sin(\frac{\theta+2k\pi}{n}) & w_6\sin(\frac{\theta+2k\pi}{n}) & w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & w_2\sin(\frac{2k\pi+2\pi/3}{2}) \\
-w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_7\sin(\frac{2k\pi+2\pi/3}{2}) & 0 & w_5\sin(\frac{\theta+2k\pi}{n}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_3\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2}) & w_1\sin(\frac{2k\pi+2\pi/3}{2}) \\
w_7\sin(\frac{2k\pi+2\pi/3}{2}) & -w_6\sin(\frac{2k\pi+2\pi/3}{2}) & -w_5\sin(\frac{\theta+2k\pi}{n}) & 0 & w_3\sin(\frac{2k\pi+2\pi/3}{2}) & -w_2\sin(\frac{2k\pi+2\pi/3}{2}) & -w_1\sin(\frac{2k\pi+2\pi/3}{2}) & \cos(\frac{2k\pi+2\pi/3}{2})
\end{bmatrix}$$

If $k=0$, then

$$A_0^{\frac{1}{2}}=\begin{bmatrix}
\cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & 0 & -w_5\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} & -w_7\sin\frac{\pi}{3} \\
w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & -w_5\sin\frac{\pi}{3} & 0 & w_7\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} \\
-w_2\sin\frac{\pi}{3} & w_3\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} & -w_7\sin\frac{\pi}{3} & 0 & w_5\sin\frac{\pi}{3} \\
w_3\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_7\sin\frac{\pi}{3} & -w_6\sin\frac{\pi}{3} & -w_5\sin\frac{\pi}{3} & 0 \\
0 & w_5\sin\frac{\pi}{3} & -w_6\sin\frac{\pi}{3} & w_7\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} \\
w_5\sin\frac{\pi}{3} & 0 & w_7\sin\frac{\pi}{3} & w_6\sin\frac{\pi}{3} & w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & w_3\sin\frac{\pi}{3} & w_2\sin\frac{\pi}{3} \\
-w_6\sin\frac{\pi}{3} & -w_7\sin\frac{\pi}{3} & 0 & w_5\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & -w_3\sin\frac{\pi}{3} & \cos\frac{\pi}{3} & w_1\sin\frac{\pi}{3} \\
w_7\sin\frac{\pi}{3} & -w_6\sin\frac{\pi}{3} & -w_5\sin\frac{\pi}{3} & 0 & w_3\sin\frac{\pi}{3} & -w_2\sin\frac{\pi}{3} & -w_1\sin\frac{\pi}{3} & \cos\frac{\pi}{3}
\end{bmatrix},$$

If $k=1$, then

$$A_0^{\frac{1}{2}}=\begin{bmatrix}
\cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & 0 & -w_5\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} & -w_7\sin\frac{4\pi}{3} \\
w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & -w_5\sin\frac{4\pi}{3} & 0 & w_7\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} \\
-w_2\sin\frac{4\pi}{3} & w_3\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} & -w_7\sin\frac{4\pi}{3} & 0 & w_5\sin\frac{4\pi}{3} \\
w_3\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_7\sin\frac{4\pi}{3} & -w_6\sin\frac{4\pi}{3} & -w_5\sin\frac{4\pi}{3} & 0 \\
0 & w_5\sin\frac{4\pi}{3} & -w_6\sin\frac{4\pi}{3} & w_7\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} \\
w_5\sin\frac{4\pi}{3} & 0 & w_7\sin\frac{4\pi}{3} & w_6\sin\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & w_3\sin\frac{4\pi}{3} & w_2\sin\frac{4\pi}{3} \\
-w_6\sin\frac{4\pi}{3} & -w_7\sin\frac{4\pi}{3} & 0 & w_5\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & -w_3\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3} & w_1\sin\frac{4\pi}{3} \\
w_7\sin\frac{4\pi}{3} & -w_6\sin\frac{4\pi}{3} & -w_5\sin\frac{4\pi}{3} & 0 & w_3\sin\frac{4\pi}{3} & -w_2\sin\frac{4\pi}{3} & -w_1\sin\frac{4\pi}{3} & \cos\frac{4\pi}{3}
\end{bmatrix}.$$