Communication in Mathematical Modeling and Applications

http://ntmsci.com/cmma

Related Dynkin games for doubly reflected BSDEs under weak assumptions

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Received: 27 April 2016, Accepted: 4 August 2016
Published online: 18 November 2016.

Abstract: We take the results of existence and uniqueness of the solution for doubly reflected backward stochastic differential equations (BSDEs in short) proved recently by Hassairi in [7], we study its connection with Dynkin games problem under very weak assumptions. We show in the present paper that this differential game have a value function and a saddle point.

Keywords: Doubly reflected backward SDEs, Dynkin game, Mokobodzki's condition, local solution.

1 Introduction

The Dynkin game problem has been largely studied in the litterature. The first results ([1],[2]) were obtained under the Mokobodzki's condition which says that the payoffs are separated by the difference of two nonegative finite supermartingales. To verify Mokobodzki's condition in practice appears as a difficult question.

We recall that it is well known that BSDEs with two reflecting barriers are connected with zero-sum Dynkin games because the solution of such BSDE provides the value function and the saddle point of the associated game. However the known results, the data are supposed to be square integrable or $\mathbb{L}^p$-integrable with $p \in (1,2)$ (see for [4,6]), which is restrictive. So if we assume only the $\mathbb{L}^1$-integrability, and this is the novelty of this work, our result enlarge the class of data for which the Dynkin game has a value and a saddle point. Our work is strongly linked with the results obtained in [7] where the author assumed that the data are just integrable and that the solution of BSDE obtained is continuous and belongs to class $\mathbb{D}$.

This paper is organised as follows: In section 2, we introduce some notations used in the rest of the paper and recall the known results proved in [7]. In section 3, under the integrability on the data we prove that the Dynkin game associated has a value function which represents the solution of the doubly reflected BSDE and a saddle point .

2 Main assumptions and known results

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space such that $(\mathscr{F}_t^0 := \sigma\{B_s, s\leq t\})_{t\leq T}$ be the natural filtration of $B=(B_t)_{0\leq t\leq T}$ a standard d-dimensional Brownian motion, while $(\mathscr{F}_t)_{t\leq T}$ is the completed filtration of $(\mathscr{F}_t^0)_{t\leq T}$ with the $\mathbb{P}$-null sets of $\mathscr{F}$, therefore $(\mathscr{F}_t)_{t\leq T}$ is right continuous and complete. We denote by $\mathscr{P}$ the σ-algebra of $\mathscr{F}_t$-progressively measurable sets on $[0,T]\times\Omega$ and finally $\mathscr{T}$ is the space of all $\mathscr{F}_t$-stopping times $\tau\in[0,T]$, $\mathbb{P}-a.s.$

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A zero-sum Dynkin game on $[\tau,T]$, where τ a stopping time, is a differential game on stopping times between two players p_1 and p_2 whose advantages are opposed. In other words, when p_1 (resp. p_2) decides to stop the game at the stopping time ν (resp. σ), then p_1 pays to p_2 an amount denoted $J_\tau(\nu,\sigma)$. For more details, one can see [5].

Lemma 1. *The study of zero-sum Dynkin games contains two principles points.*

(i) *the Dynkin game have a value function on $[\tau,T]$ i.e.*

$$\sup_\sigma \inf_\nu J_\tau(\nu,\sigma) = \inf_\nu \sup_\sigma J_\tau(\nu,\sigma) \tag{1}$$

(ii) *the Dynkin game have a saddle point, that is, there exists a couple of stopping times (ν_τ,σ_τ) which verifies*

$$J_\tau(\nu_\tau,\sigma) \le J_\tau(\nu_\tau,\sigma_\tau) \le J_\tau(\nu,\sigma_\tau). \tag{2}$$

We will consider the same notations as in [7] for the coherence of the paper. Let $f(t,\omega,y,z),\xi,L$ and U be our set of data which satisfy the following assumptions.

An $\mathbb{R}$-valued process $f := (f_s)_{s\le T}$ which belongs to $\mathbb{L}^1([0,T]\times\Omega, dt\otimes d\mathbb{P})$ such that.

(i) f is Lipschitz in (y,z) uniformly in (t,ω), i.e., there exists a constant κ such that for any t,y,y',z,z',

$$\mathbb{P}-a.s,\ \left|f(t,y,z)-f(t,y',z')\right| \le \kappa(\left|y-y'\right|+\left|z-z'\right|);$$

(ii) $\mathbb{P}-a.s.$, $\forall r>0$, $\int_0^T \sup_{|y|\le r}|f(s,y,0)-f(s,0,0)|\,ds < +\infty$.

(iii) There exist two constants $\gamma\ge 0$, $\alpha\in[0,1[$ and a nonnegative progressively measurable process g such that $\mathbb{E}[\int_0^T g_s ds]<\infty$ and

$$|f(t,y,z)-f(t,y,0)| \le \gamma(g_t+|y|+|z|)^\alpha, t\in[0,T], y\in\mathbb{R} \textit{ and } z\in\mathbb{R}^d.$$

ξ is an $\mathbb{R}$-valued, $\mathscr{F}_T$-measurable random variable such that $\mathbb{E}[|\xi|]<\infty$,

L and U are continuous, $\mathscr{P}$-measurable, of class $\mathbb{D}$ and satisfying.

$$\mathbb{P}-a.s., \forall t<T,\ L_t<U_t \text{ and } L_T\le\xi\le U_T.$$

Let us now recall the definition of the notion of a $\mathbb{D}$-solution of the doubly reflected BSDE associated with (f,ξ,L,U) considered in [7] in the following result.

Lemma 2. *Under the above assumptions on (f,ξ,L,U), a solution for such an equation is a quadruple of processes $(Y,Z,K^+,K^-) := (Y_t,Z_t,K_t^+,K_t^-)_{t\le T}$ $\mathscr{P}$-measurable with values in $\mathbb{R}^{1+d+1+1}$ such that.*

$$\begin{cases} Y\in\mathbb{D} \textit{ and continuous}, Z\in\mathscr{M}^0 \textit{ and } K^\pm\in\mathscr{C}_i, \\ Y_t = \xi+\int_t^T f(s,Y_s,Z_s)ds + (K_T^+-K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s,\ \forall t\in[0,T] \\ L\le Y\le U \ \textit{ and } \ \int_0^T (Y_s-L_s)dK_s^+ = \int_0^T (U_s-Y_s)dK_s^- = 0,\ \mathbb{P}-a.s. \end{cases} \tag{3}$$

where for $\beta\in(0,1]$, $\mathscr{M}^\beta$ be the set of $\mathscr{F}_t$-predictable processes $(X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$ such that

$$\|X\|_{\mathscr{M}^\beta} = \mathbb{E}[(\int_0^T |X_s|^2 ds)^{\beta/2}] < +\infty.$$

If $\beta = 1$, $\mathscr{M}^\beta$ (i.e. $\mathscr{M}^1$) is a Banach space endowed with this norm and for $\beta \in (0,1)$, $\mathscr{M}^\beta$ is a complete metric space with the resulting distance.

$\mathscr{M}^0$ is the set of $\mathscr{P}$-measurable processes $Z := (Z_t)_{t\leq T}$ with values in $\mathbb{R}^d$ such that $\int_0^T |Z_s(\omega)|^2 ds < \infty$, $\mathbb{P}-a.s.$ and $\mathscr{C}_i$ is the set of $\mathscr{F}_t$-adapted continuous non-decreasing processes $K := (K_t)_{t\in[0,T]}$ such that $K_0 = 0$ and $K_T < \infty$, $\mathbb{P}-a.s.$.

We say that a $\mathscr{P}$-measurable process $X := (X_t)_{t\in[0,T]}$ belongs to class $\mathbb{D}$ if the family of random variables $\{X_\tau,\ \tau \in \mathscr{T}\}$ is uniformly integrable. In ([3], pp.90) it is observed that the space of continuous (càdlàg) $\mathscr{F}_t$-adapted processes from class $\mathbb{D}$ is complete under the norm

$$\|X\|_1 = \sup_{\tau\in\mathscr{T}} \mathbb{E}[|X_\tau|]. \tag{4}$$

3 Connection with BSDEs

Let us consider the zero-sum Dynkin game on $[\tau, T]$ associated with (f, ξ, L, U) where the payoff after τ is given by:

$$J_\tau(\nu,\sigma) = \mathbb{E}[\int_\tau^{\nu\wedge\sigma} f(s,Y_s,Z_s)ds + L_\sigma \mathbf{1}_{[\sigma\leq\nu<T]} + U_\nu \mathbf{1}_{[\nu<\sigma]} + \xi \mathbf{1}_{[\nu=\sigma=T]} | \mathscr{F}_\tau], \tag{5}$$

where ν and σ are two stopping times.The value function of such game is an $\mathscr{F}_t$-adapted process Y_t such that $\mathbb{P}-a.s$, $\forall t \in [\tau, T]$,

$$Y_t = ess\ \inf_{\nu\in\mathscr{T}_t} ess\ \sup_{\sigma\in\mathscr{T}_t} J_t(\nu,\sigma) = ess\ \sup_{\sigma\in\mathscr{T}_t} ess\ \inf_{\nu\in\mathscr{T}_t} J_t(\nu,\sigma), \tag{6}$$

where $\mathscr{T}_t$ is the set of stopping times that take values in $[t, T]$. The random variable Y_τ is called the value of the zero-sum Dynkin game on $[\tau, T]$.

Using the notion of local solution mentionned in [7], let $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ be a local solution on $[\tau, \gamma]$ for the doubly reflected BSDE. Let (ν_τ, σ_τ) the couple of stopping times defined as follows:

$$\nu_\tau = \inf\{s \geq \tau, Y_s = U_s\} \wedge T \text{ and } \sigma_\tau = \inf\{s \geq \tau, Y_s = L_s\} \wedge T. \tag{7}$$

In the following result we give some feature, which is fairly not known, of the value function of Dynkin game.

Theorem 1. *If $max\{\nu_\tau, \sigma_\tau\} \leq \gamma$ then.*

$$J_\tau(\nu_\tau, \sigma) \leq J_\tau(\nu_\tau, \sigma_\tau) = Y_\tau \leq J_\tau(\nu, \sigma_\tau),$$

for any two stopping times ν and σ. Therefore Y_τ is the value function of the zero-sum Dynkin game on $[\tau, T]$ and (ν_τ, σ_τ) is a saddle point for the game.

Proof. Thanks to the continuity of the process Y on $[\tau, \gamma]$ and $max\{\nu_\tau, \sigma_\tau\} \leq \gamma$ then $Y_{\nu_\tau} = U_{\nu_\tau}$ on the interval $[\nu_\tau < T]$ and $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on the interval $[\sigma_\tau < T]$. The rest of the proof is classical (one can see [5]) and based on the existence of the solution for the doubly reflected BSDE then our desired result is a direct consequence of theorem 4.1 in [7]. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] M. Alario-Nazaret, J.P. Lepeltier and B. Marchal, Dynkin games. In Stochastic differential systems, *Lecture Notes in Control and Inform. Sci.*, **43** (1982), 23-32, Bad Honnef, Springer, Berlin.

[2] J.M. Bismut, Sur un problème de Dynkin, *Z. Warsch. V. Geb.*, **39**, 31-53.

[3] C. Dellacherie and P.A. Meyer, *Probabilités et Potentiel V-VIII*, Hermann, Paris, 1980.

[4] B. El Asri, S. Hamadène and H. Wang, L^p solutions for doubly reflected backward stochastic differential equations, *Stochastic Analysis and Applications*, **29** (2011), 907-932.

[5] S. Hamadène, Mixed zero-sum differential game and American game options, *SIAM J. Control Optim.*, **45** (2006), 496-518.

[6] S. Hamadène and M. Hassani, BSDEs with two reflecting barriers: the general result, *Probability theory and related fields*, **132** (2005), 237–264.

[7] I. Hassairi, Existence and uniqueness for D-solutions of reflected BSDEs with two barriers without Mokobodzki's condition, *Communications on Pure and Applied Analysis*, **volume 4** (2016), 1139–1156.