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Associative monotonic commutative operator (AMC) UV

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Abstract: Uninorms and nullnorms are associative monotonic and commutative operators. In this study, an operator UV is defined by uninorm U and nullnorm V and showed that UV satisfies associativity, monotonicity and commutativity properties. And it is showed that UV does not have neutral element and have zero element if U is disjunctive uninorm on [0,k]. Moreover, it is show that if it is taken smallest uninorm and nulnorm, UV is also smallest AMC operator.

Keywords: Uninorm, nullnorm, chain, AMC operator.

1 Introduction

Aggregation functions have been a topic of study in recent years [7]. A lot of studies have been done on some special aggregation functions such as t-norms, t-conorms, uninorms and nullnorms. In addition to the relationships among these operators, getting new functions with these operators is also a very popular work. Uninorms were introduced on $[0,1$ by Yager and Rybalov [10] are special aggregation operators many working on them [3,4,5]. At the same time, they are generalization of t-norms and t-conorms [6]. Nullnorms are aggregation operators with zero element and are also generalizations of triangular norms and triangular conorms. In [8], smallest and strongest nullnorms were determined and some properties of these operators are studied. Uninorms and nullnorms are linked to each other because they are generalizations of t-norm and t-conorms. In this study, an operator UV is defined and studied some properties are investigated.

The paper is organized as follows: We shortly recall some basic notions and results in Section 2. In Section 3, a method to obtain associative, commutative, monotonic operator on chain L is given using uninorm on $[0,k]$ and nullnorm on $[e,1]$. Some properties of this construction method are also investigated in Section 3.

2 Notations, definitions and a review of previous results

A bounded lattice $(L,\leqslant)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, i.e., there exist two elements $1,0\in L$ such that $0\leqslant x\leqslant 1$, for all $x\in L$.

Definition 1. [1] *Given a bounded lattice $(L,\leq,0,1)$, and $a,b\in L$, if a and b are incomparable, in this case we use the notation $a\|b$. If all elements of bounded lattice L are comparable each other, L is called as chain.*

Definition 2. [1] *Given a bounded lattice $(L,\leq,0,1)$, and $a,b\in L$, $a\leq b$, a subinterval $[a,b]$ of L is a sublattice of L defined as*

$$[a,b]=\{x\in L\mid a\leq x\leq\ b\}.$$

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Similarly, $(a,b] = \{x \in L \mid a < x \leq b\}$, $[a,b) = \{x \in L \mid a \leq x < b\}$ *and* $(a,b) = \{x \in L \mid a < x < b\}$.

Definition 3. [9] *Let* $(L, \leq, 0, 1)$ *be a bounded lattice. An operation* $U : L^2 \to L$ *is called a uninorm on* L*, if it is commutative, associative, increasing with respect to the both variables and has a neutral element* $e \in L$.

In this study, the notation $\mathscr{U}(e)$ *will be used for the set of all uninorms with neutral element e. If* $U(0,1) = 0$, U *is called conjunctive uninorm and if* $U(0,1) = 1$, U *is called disjunctive uninorm.*

Consider the set $\mathscr{U}(e)$ of all uninorms on L with the following order:
For $U_1, U_2 \in \mathscr{U}(e)$,

$$U_1 \leqslant U_2 \Longleftrightarrow U_1(x,y) \leqslant U_2(x,y) \text{ for all } (x,y) \in L^2.$$

Corollary 1. [9] *Let* $(L, \leq, 0, 1)$ *be a chain and* $e \in L \setminus \{0,1\}$*. Then the following uninorm* $U_{S_\vee} : L^2 \to L$ *is the smallest uninorm on L with neutral element e.*

$$U_{S_\vee}(x,y) = \begin{cases} x \vee y, & if \ (x,y) \in [e,1]^2 \\ x \wedge y, & if \ (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \\ 0, & otherwise. \end{cases}$$

Definition 4. [3] *An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element* 1 *(0).*

Definition 5. [8] *Let* $(L, \leq, 0, 1)$ *be a bounded lattice. An operation* $V : L^2 \to L$ *is called a nullnorm on* L*, if it is commutative, associative, increasing with respect to the both variables and there is an element* $a \in L$ *such that* $V(x,0) = x$ *for all* $x \leq a$, $V(x,1) = x$ *for all* $x \geq a$.

It can be easily obtained that $V(x,a) = a$ *for all* $x \in L$*. So, the element* $a \in L$ *that provide* $V(x,a) = a$ *for all* $x \in L$ *is called (absorbing) zero element for operator* V *on* L.

In this study, the notation $\mathscr{V}(k)$ *will be used for the set of all nullnorms with zero element* a.

Corollary 2. [8] *Let* $(L, \leq, 0, 1)$ *be a chain and* $a \in L \setminus \{0,1\}$*. Then the following nullnorm* $V_a^{\vee} : L^2 \to L$ *is the smallest nullnorm on L with zero element* a.

$$V_a^{\vee}(x,y) = \begin{cases} x \vee y, & if \ (x,y) \in [0,a]^2 \\ x \wedge y, & if \ (x,y) \in [a,1)^2 \cup [0,a) \times (a,1] \cup (a,1] \times [0,a) \\ x \wedge y, & otherwise. \end{cases}$$

3 Associative monotonic commutative operator (AMC) UV

In this section, an operator UV has been defined on chain L via $U \in \mathscr{U}(e)$ uninorm on sub-interval $[0,k]$ of L and $V \in \mathscr{V}(k)$ nullnorm on subinterval $[e,1]$ of L such that $0 \leq e \leq k \leq 1$ and $U \downarrow [e,k]^2 = V \downarrow [e,k]^2$. For the operator UV defined in this way, it is showed that UV satisfies that the properties of associativity, monotonicity, commutativity.

Definition 6. *Let* $(L, \leq, 0, 1)$ *be a chain,* $U : [0,k]^2 \to [0,k]$ *be an uninorm with neutral element e and* $V : [e,1]^2 \to [e,1]$ *be a nullnorm with zero element k such that* $0 \leq e \leq k \leq 1$ *and* $U \downarrow [e,k]^2 = V \downarrow [e,k]^2$*. Define the following operator on L, for* $x, y \in L$*, as*

$$UV(x,y) = \begin{cases} U(x,y), & \text{if } (x,y) \in [0,k]^2 \\ V(x,y), & \text{if } (x,y) \in [e,1]^2 \\ k, & \text{otherwise.} \end{cases} \tag{1}$$

Proposition 1. *Let $(L, \leq, 0, 1)$ be a chain, UV be an operator as (1) on L under constraints as given in Definition 6. Then, UV satisfies that the properties of associativity, monotonicity, commutativity, namely UV is AMC operator on L.*

Proof. (i) Monotonicity: We prove that if $x \leqslant y$ then for all $z \in L$, $UV(x,z) \leqslant UV(y,z)$. The proof is split into all possible cases.

1. Let $x \in [0,e]$.

1.1. $y \in [0,e]$,

1.1.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,z) = U(x,z) \leqslant U(y,z) = UV(y,z)$$

1.1.2. $z \in [k,1]$,

$$UV(x,z) = k = UV(y,z)$$

1.2. $y \in [e,k]$,

1.2.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,z) = U(x,z) \leqslant U(y,z) = k = UV(y,z)$$

1.2.2. $z \in [k,1]$,

$$UV(x,z) = k = UV(y,z)$$

1.3. $y \in [k,1]$,

1.3.1. $z \in [0,e]$,

$$UV(x,z) = U(x,z) \leqslant k = UV(y,z)$$

1.3.2. $z \in [e,k]$,

$$UV(x,z) = U(x,z) \leqslant k = UV(y,z)$$

1.3.3. $z \in [k,1]$,

$$UV(x,z) = k \leqslant V(y,z) = UV(y,z)$$

2. Let $x \in [e,k]$ Then $y \geq e$.

2.1. $y \in [e,k]$,

2.1.1. $z \in [0,k]$,

$$UV(x,z) = U(x,z) \leq U(y,z) = UV(y,z)$$

2.1.2. $z \in [k,1]$,

$$UV(x,z) = k = UV(y,z)$$

2.2. $y \in [k,1]$,

2.2.1. $z \in [0,e]$,

$$UV(x,z) = U(x,z) \leq k = UV(y,z)$$

2.2.2. $z \in [e,1]$,

$$UV(x,z) = V(x,z) \leq V(y,z) = UV(y,z)$$

3. Let $x \in [k,1]$. Then $y \geq k$.

3.1. $y \in [k,1]$,

3.1.1. $z \in [0,e]$,

$$UV(x,z) = k = UV(y,z)$$

3.1.2. $z \in [e,1]$,

$$UV(x,z) = V(x,z) \leq V(y,z) = UV(y,z).$$

(ii) Associativity: We demonstrate that $UV(x,UV(y,z)) = UV(UV(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases.

1. Let $x \in [0,e]$.

1.1. $y \in [0,e]$,

1.1.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,UV(y,z)) = UV(x,U(y,z)) = U(x,U(y,z)) = U(U(x,y),z) = UV(U(x,y),z) = UV(UV(x,y),z)$$

1.1.2. $z \in [k,1]$,

$$UV(x,UV(y,z)) = UV(x,k) = k = UV(U(x,y),z) = UV(UV(x,y),z)$$

1.2. $y \in [e,k]$,

1.2.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,UV(y,z)) = UV(x,U(y,z)) = U(x,U(y,z)) = U(U(x,y),z) = UV(U(x,y),z) = UV(UV(x,y),z)$$

1.2.2. $z \in [k,1]$,

$$UV(x,UV(y,z)) = UV(x,k) = k = UV(U(x,y),z) = UV(UV(x,y),z)$$

1.3. $y \in [k,1]$,

1.3.1. $z \in [0,e]$,

$$UV(x,UV(y,z)) = UV(x,k) = k = U(k,z) = UV(k,z) = UV(UV(x,y),z)$$

1.3.2. $z \in [e,k]$,

$$UV(x,UV(y,z)) = UV(x,k) = k = UV(k,z) = UV(UV(x,y),z)$$

1.3.3. $z \in [k,1]$,

$$UV(x,UV(y,z)) = UV(x,V(y,z)) = k = V(k,z) = UV(k,z) = UV(UV(x,y),z)$$

2. Let $x \in [e,k]$.

2.1. $y \in [0,e]$,

2.1.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,UV(y,z)) = UV(x,U(y,z)) = U(x,U(y,z)) = U(U(x,y),z) = UV(U(x,y),z) = UV(UV(x,y),z)$$

2.1.2. $z \in [k,1]$,

$$UV(x,UV(y,z)) = UV(x,k) = U(x,k) = k = UV(U(x,y),z) = UV(UV(x,y),z)$$

2.2. $y \in [e,k]$,

2.2.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x,UV(y,z)) = UV(x,U(y,z)) = U(x,U(y,z)) = U(U(x,y),z) = UV(U(x,y),z) = UV(UV(x,y),z)$$

2.2.2. $z \in [k,1]$,

$$UV(x, UV(y,z)) = UV(x,k) = U(x,k) = k = UV(U(x,y),z) = UV(UV(x,y),z)$$

2.3. $y \in [k,1]$,
2.3.1 $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x, UV(y,z)) = UV(x,k) = U(x,k) = k = UV(k,z) = UV(UV(x,y),z)$$

2.3.2. $z \in [k,1]$,

$$UV(x, UV(y,z)) = UV(x, V(y,z)) = V(x, U(y,z)) = V(V(x,y),z) = UV(V(x,y),z) = UV(UV(x,y),z)$$

3. Let $x \in [k,1]$.
3.1. $y \in [0,e]$ or $y \in [e,k]$,
3.1.1. $z \in [0,e]$,

$$UV(x, UV(y,z)) = UV(x, U(y,z)) = k = U(k,z) = UV(k,z) = UV(UV(x,y),z)$$

3.1.2. $z \in [e,k]$,

$$UV(x, UV(y,z)) = UV(x, U(y,z)) = k = V(k,z) = UV(k,z) = UV(UV(x,y),z)$$

3.1.3. $z \in [k,1]$,

$$UV(x, UV(y,z)) = UV(x,k) = V(x,k) = k = V(k,z) = UV(k,z) = UV(UV(x,y),z)$$

3.2. $y \in [k,1]$,
3.2.1. $z \in [0,e]$ or $z \in [e,k]$,

$$UV(x, UV(y,z)) = UV(x,k) = V(x,k) = k = V(k,z) = UV(V(x,y),z) = UV(UV(x,y),z)$$

3.2.2. $z \in [k,1]$,

$$UV(x, UV(y,z)) = UV(x, V(y,z)) = V(x, V(y,z)) = V(V(x,y),z) = UV(V(x,y),z) = UV(UV(x,y),z).$$

It is trivial to see the commutativity and the fact that UV.

Let show that the class of all AMC operator obtained by (1) on L is represented by $\mathscr{U}\mathscr{V}(\mathscr{L})$.

Proposition 2. *Let $(L, \leq, 0, 1)$ be a chain, UV be an operator as (1) on L under constraints as given in Definition 6. Then, UV does not have unit element on L.*

Proof. Suppose that e^* is neutral element of UV on L. Then $UV(x, e^*) = x$ for all $x \in L$. If $e^* \in [0,k]$, $e^* = UV(e, e^*) = e$ since $e \in [0,k]$ is neutral element U on $[0,k]$ and e^* is neutral element of UV on L. In this case, $UV(x, e^*) = k \neq x$ for $x \in (k,1]$. It is contradiction. If $e^* \in [k,1]$, $UV(x, e^*) = k \neq x$ for $x \in [0,e]$. It is contradiction. Therefore, UV does not have unit element on L.

Proposition 3. *Let $(L, \leq, 0, 1)$ be a chain, UV be an operator as (1) on L under constraints as given in Definition 6. If U disjanktif uninorm on $[0,k]$, k is zero element of UV on L.*

Proof. Since U disjanktif uninorm, $U(0,k) = k$. Let show that k is zero element of UV on L. Let $x \in L$.

(i) $x \in [0,k]$. Then, $UV(x,k) = U(x,k) \geq U(0,k) = k$. Thus, $UV(x,k) = k$.
(ii) $x \notin [0,k]$. Then, it is obtained that $UV(x,k) = k$.

Therefore, if U disjanktif uninorm on $[0,k]$, k is zero element of UV on L.

Proposition 4. *Let $(L,\leq,0,1)$ be a chain, consider the AMC operator $U_{S_\vee}V_a{}^\vee$ by taking $U = U_{S_\vee}$ and $V = V_a{}^\vee$ in (1). Then, $U_{S_\vee}V_a{}^\vee(x,y) \leq UV(x,y)$ for all $UV \in \mathscr{U}\mathscr{V}(\mathscr{L})$ and $(x,y) \in L^2$, namely $U_{S_\vee}V_a{}^\vee \leq UV$.*

Proof. (i) $(x,y) \in [0,k]^2$. Then $U_{S_\vee}V_a{}^\vee = U_{S_\vee}$ and $UV = U$ for all $UV \in \mathscr{U}\mathscr{V}(\mathscr{L})$. Considering $U_{S_\vee}$ is the smallest uninorm, $U_{S_\vee} \leq U$. Therefore, $U_{S_\vee}V_a{}^\vee \leq UV$.
(ii) $(x,y) \in [e,1]^2$. Then $U_{S_\vee}V_a{}^\vee = V_a{}^\vee$ and $UV = V$ for all $UV \in \mathscr{U}\mathscr{V}(\mathscr{L})$. Considering $V_a{}^\vee$ is the smallest nullnorm, $V_a{}^\vee \leq V$. Therefore, $U_{S_\vee}V_a{}^\vee \leq UV$.
(iii) Otherwise, $U_{S_\vee}V_a{}^\vee = k = UV$.

Therefore, it is obtained that $U_{S_\vee}V_a{}^\vee \leq UV$.

4 Conclusion

Aggregation operators have been hot topic recent years. Researchers are studying the comparison of these operators in terms of some features and producing some new operators. UV operator is defined and studied on some properties such as associativity, monotonicity and commutativity via uninorms and nullnorms in this study. In addition, neutral and zero element are investigated. The smallest operator defined in this way was identified.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] G. Birkhoff, Lattice Theory, 3 rd edition, Providence, 1967.
[2] E. Czogoła, Associative monotonic operations, Fuzzy Sets and Systems, 12 (1984), 249-269.
[3] Ü. Ertuğrul, F. Karaçal, R. Mesiar, Modified ordinal sums of triangular norms and triangular conorms on bounded lattices, International Journal of Intelligent Systems, 30 (2015) 807-817.
[4] Ü. Ertuğrul, M. N. Kesicioğlu, F. Karaçal, Ordering based on uninorms, Information Sciences, 330 (2016), 315-327.
[5] Ü. Ertuğrul, M. N. Kesicioğlu, F. Karaçal, Ordering based 2-uninorms on bounded lattices, New Trends in Mathematical Sciences, in press.
[6] J. Fodor, R. Yager, and A. Rybalov, Structure of uninorms, Internata. J. Uncertain. Fuzziness Knowledge-Based Systems, 5 (1997), 411-427.
[7] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.
[8] F. Karaçal, M.A. İnce, R. Mesiar, Nullnorms on bounded lattices, Information Sciences, 325 (2015), 227-236.
[9] F. Karaçal, R. Mesiar, Uninorms on bounded lattices, Fuzzy Sets and Systems, 261 (2015), 33-43.
[10] R. R. Yager, A. Rybalov, Uninorm aggregation operators, Fuzzy Sets and Systems, 80 (1996), 111-120.
[11] R. R. Yager, Uninorms in fuzzy system modelling, Fuzzy Sets and Systems, 122 (2001), 167-175.