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An optimal multiple switching problem under weak assumptions

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Abstract: This work studies the problem of optimal multiple switching in finite horizon, when the switching costs functions are continous and belong to class $\mathbb{D}$. This problem is solved by means of the Snell envelope of processes.

Keywords: Real options, snell envelope, stopping time, optimal switching.

1 Introduction

In an optimal control problem there is a single control and a single criterion to be optimized while optimal multiple problem theory generalizes this to many controls and many criterias, one for each state. Let us begin with an example in order to introduce the problem we consider in this work. The optimal multi-modes switching problem in finite horizon, of real option type, can be described as follows:

Assume we have a power station which produces electricity and which has several modes of production (low, medium, intensive). It is well known that electricity is non-storable, once produced it should be consumed. Therefore, the station produces electricity only in its instantaneous most profitable mode known that when the station is in mode $i \in \mathscr{J}$, the yield per unit time is given by means of Ψ_i, and on the other hand, switching the station from the mode i to j is not free and generates costs given by g_{ij}. A management strategy for the power station is combination of two sequences.

(i) a non-decreasing sequence of stopping times $(\tau_n)_{n\geq 1}$, where at time τ_n, the manager decides to switch the production from its current mode to another one;

(ii) a sequence of indicators $(\xi_n)_{n\geq 1}$ taking values in $\{1,..,m\}$ of the state the production is switched to. At τ_n, the station switched from its current mode ξ_{n-1} to ξ_n. The value ξ_0 is deterministic and its the state of the station at time 0.

Using probabilistic tools like the snell envelope of processes, inspired by the work of Hamadène and Jeanblanc [7], Carmona and Ludkovski [1] solved this optimal switching problem. In order to tackle this problem, Djehiche et al. [5] provide existence and characterization of an optimal strategy of this multiple switching problem when the payoff rates Ψ_i, and the switching costs g_{ij} are adapted by a Brownian motion. The authors considered the case when the functions Ψ_i are $\mathbb{L}^p$-integrable, with $p > 1$ and the switching costs functions g_{ij} belong to the space $\mathscr{S}^p$ which is defined as the set of $\mathscr{P}$-measurable, continuous, $\mathbb{R}$-valued processes $(X_t)_{t\leq T}$ such that $\mathbb{E}[\sup_t |X_t|^p] < \infty$ with a fixed $p > 1$, then according to our knowledge, the case of $p = 1$ is still an open problem. Therefore the main objective of our work, and this is the originality of this paper, is to provide a solution to the optimal multiple switching problem when the functions g_{ij} are

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continuous and belong to class $\mathbb{D}$.

The outline of the paper is as follows. In Section 2, we formulate our problem. Section 3 and 4, we show our main result so we establish the Verification Theorem and provide an optimal strategy to our problem.

2 Problem formulation

Throughout this paper $(\Omega, \mathscr{F}, \mathbb{P})$ will be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathscr{F}_t^0 := \sigma\{B_s, s \leq t\})_{t\leq T}$. Let $(\mathscr{F}_t)_{t\leq T}$ is the completed filtration of $(\mathscr{F}_t^0)_{t\leq T}$ with the $\mathbb{P}$-null sets of $\mathscr{F}$, therefore $(\mathscr{F}_t)_{t\leq T}$ is right continuous and complete. We denote by $\mathscr{P}$ the σ-algebra of $\mathscr{F}_t$-progressively measurable sets on $[0,T]\times\Omega$. Furthermore, let
- $\mathscr{T}$ is the space of all $\mathscr{F}_t$-stopping times $\tau \in [0,T]$ and for any stopping time $\tau \in [0,T]$, $\mathscr{T}_\tau$ is the set of all stopping times θ such that $\tau \leq \theta \leq T$, $\mathbb{P}-a.s.$
Next for any stopping time $\tau \in [0,T]$, we denote by $\mathscr{F}_\tau$ the σ-algebra of events prior to τ.
- $\mathscr{M}^0$ is the set of $\mathscr{P}$-measurable processes $X := (X_t)_{t\leq T}$ with values in $\mathbb{R}$ such that $\int_0^T |X_s(\omega)|^2 ds < \infty$, $\mathbb{P}-a.s.$.
Let us now recall that a $\mathscr{P}$-measurable process $X := (X_t)_{t\in[0,T]}$ belongs to class $\mathbb{D}$ if the family of random variables $\{X_\tau,\, \tau \in \mathscr{T}\}$ is uniformly integrable. In ([3], pp.90) it is observed that the space of right continuous with left limits $\mathscr{F}_t$-adapted processes of $\mathbb{D}$ is complete under the norm

$$\|X\|_1 = \sup_{\tau\in\mathscr{T}} \mathbb{E}[|X_\tau|].$$

Let $\mathscr{J} := \{1,...,m\}$ be the set of all possible activity modes of the production of the commodity. A management strategy of the project consists, on the one hand, of the choice of a sequence of non-decreasing stopping times $(\tau_n)_{n\geq 1}$ (i.e. $\tau_n \leq \tau_{n+1}$ and $\tau_0 = 0$) where the manager decides to switch the activity from its current mode, say i, to another one from the set $\mathscr{J}^{-i} \subseteq \{1,...,i-1,i+1,...,m\}$. On the other hand, it consists of the choice of the mode ξ_n, a random variable $\mathscr{F}_{\tau_n}$-measurable with values in $\mathscr{J}$, to which the production is switched at τ_n from its current mode i. Therefore a strategy for our multiple switching problem will be denoted by δ. A strategy $\delta := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ will be called admissible if it satisfies

$$\lim_{n\to\infty} \tau_n = T \quad \mathbb{P}-a.s.$$

and the set of these strategies is denoted by $\mathscr{D}$.

Assume that the production activity is in mode 1 at the initial time $t = 0$, we denote by $(\delta_t)_{t\leq T}$ the indicator of the production activity's mode at time $t \in [0,T]$:

$$\delta_t = \mathbf{1}_{[0,\tau_1]}(t) + \sum_{n\geq 1} \xi_n \mathbf{1}_{(\tau_n,\tau_{n+1}]}(t).$$

Now for $i \in \mathscr{J}$, let $\Psi_i := (\Psi_i(t))_{0\leq t\leq T}$ be, the instantaneous profit when the system is in state i, a stochastic process which belongs to $\mathscr{M}^0$. On the other hand, for $i \in \mathscr{J}$ and $j \in \mathscr{J}^{-i}$, let $g_{ij} := (g_{ij}(t))_{0\leq t\leq T}$ be, the switching cost of the production at time t from current mode i to another mode j, a continuous process of class $\mathbb{D}$. When a startegy $\delta := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ is implemented the optimal yield is given by

$$J(\delta) = \mathbb{E}[\int_0^T \Psi_{\delta_s}(s) ds - \sum_{n\geq 1} g_{\delta_{\tau_{n-1}}\delta_{\tau_n}}(\tau_n)\mathbf{1}_{[\tau_n<T]}].$$

Therefore the problem we are interested in is to find an optimal startegy, i.e. a strategy δ^* such that $J(\delta^*) \geq J(\delta)$ for any $\delta \in \mathscr{D}$.

3 Verification Theorem

The problem described previously is reduced to the existence of m continuous processes of class $\mathbb{D}$, $(Y^1,..,Y^m)$ expressed via Snell envelopes (see [8]). The verification Theorem for the m-states optimal switching problem is the following.

Theorem 1. *Assume there exist m continuous processes of class* $\mathbb{D}$ $(Y^i := (Y^i_t)_{0\leq t\leq T}, i = 1,...,m)$ *that satisfy for all stopping time* $\tau \leq T$,

$$Y^i_\tau = ess\sup_{\tau\geq t} \mathbb{E}[\int_t^\tau \Psi_i(s)ds + max_{i\neq j}(-g_{ij}(\tau) + Y^j_\tau)\mathbf{1}_{[\tau<T]}|\mathscr{F}_\tau], \quad (1)$$

Then $(Y^1,..,Y^m)$ *are unique. Furthermore.*

(i) $Y^1_0 = \sup_{\delta\in\mathscr{D}} J(\delta)$.
(ii) The sequence of $\mathscr{F}_t$*-stopping times* $(\tau_n)_{n\geq 1}$ *defined as follows*

$$\tau_1 = \inf\{s \geq 0, Y^1_s = max_{j\in\mathscr{J}^{-1}}(-g_{1j}(s) + Y^j_s)\} \wedge T, \quad (2)$$

and for $n \geq 2$,

$$\tau_n = \inf\{s \geq \tau_{n-1}, Y^{\delta_{\tau_{n-1}}}_s = max_{k\in\mathscr{J}^{-\tau_{n-1}}}(-g_{\tau_{n-1}k}(s) + Y^k_s)\} \wedge T, \quad (3)$$

where,
- $\delta_{\tau_1} = \sum_{j\in\mathscr{J}} j\mathbf{1}_{\{max_{k\in\mathscr{J}^{-1}}(-g_{1k}(\tau_1)+Y^k_{\tau_1}) = -g_{1j}(\tau_1)+Y^j_{\tau_1}\}}$
- *for any* $n \geq 1$ *and* $t \geq \tau_n$, $Y^{\delta_{\tau_n}}_t = \sum_{j\in\mathscr{J}} \mathbf{1}_{[\delta_{\tau_n}=j]} Y^j_t$
- *for any* $n \geq 2$, $\delta_{\tau_n} = l$ *on the set* $\{max_{k\in\mathscr{J}^{-\tau_{n-1}}}(-g_{\delta_{\tau_{n-1}}k}(\tau_n) + Y^k_{\tau_n}) = -g_{\delta_{\tau_{n-1}}l}(\tau_n) + Y^l_{\tau_n}$, *with* $g_{\delta_{\tau_{n-1}}k}(\tau_n) = \sum_{j\in\mathscr{J}} \mathbf{1}_{[\tau_{n-1}=j]} g_{jk}(\tau_n)$ *and* $\mathscr{J}^{\delta_{\tau_{n-1}}} = \sum_{j\in\mathscr{J}} \mathbf{1}_{[\tau_{n-1}=j]} \mathscr{J}^{-j}$.

Then the strategy $\delta := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ *is optimal i.e.* $J(\delta) \geq J(u)$ *for any* $u \in \mathscr{D}$.

Proof. The proof is obtained similarly as in the proof of theorem 1 in [5]. □

4 Existence of the processes $(Y^1,...,Y^m)$

The existence of the processes $(Y^1,...,Y^m)$ which satisfy (1) is also adressed in [8]. These processes will be obtained as a limit of a sequence of processes $(Y^{1,n},...,Y^{m,n})_{n\geq 0}$ defined recursively as follows: for $i \in \mathscr{J}$ we set,

$$Y^{i,0}_\tau = ess\sup_{\tau\geq t} \mathbb{E}[\int_t^\tau \Psi_i(s)ds|\mathscr{F}_\tau], \quad (4)$$

and for $n \geq 1$,

$$Y^{i,n}_\tau = ess\sup_{\tau\geq t} \mathbb{E}[\int_t^\tau \Psi_i(s)ds + max_{i\neq k}(-g_{ik}(\tau) + Y^{k,n-1}_\tau)\mathbf{1}_{[\tau<T]}|\mathscr{F}_\tau]. \quad (5)$$

Then the sequence of processes $(Y^{1,n},...,Y^{m,n})_{n\geq 0}$ have the following properties.

Proposition 1.

(i) For each $n \geq 0$ *and any* $i \in \mathscr{J}$, *the processes* $Y^{1,n},...,Y^{m,n}$ *are continous and belong to class* $\mathbb{D}$ *and verify*

$$\forall\tau \leq T, Y^{i,n}_\tau \leq Y^{i,n+1}_\tau \leq \mathbb{E}[\int_\tau^T \{max_{i=1,m}|\Psi_i(s)|\}ds|\mathscr{F}_\tau]; \quad (6)$$

(ii) there exist m processes $Y^1,\ldots,Y^m$ of class $\mathbb{D}$ such that for any $i\in\mathscr{J}$:

(a) $\forall t\leq T, Y_t^i=\lim_{n\to\infty}\nearrow Y_t^{i,n}$.
(b) these limit processes Y^i satisfy the Verification Theorem 1,

$$Y_t^i=ess\sup_{\tau\geq t}\mathbb{E}[\int_t^\tau \Psi_i(s)ds+max_{i\neq j}(-g_{ij}(\tau)+Y_\tau^j)\mathbf{1}_{[\tau<T]}|\mathscr{F}_t],\tag{7}$$

(c) For any $i\in\mathscr{J}$,

$$\sup_{\tau\in\mathscr{T}}\mathbb{E}[|Y_\tau^{i,n}-Y_\tau^i|]\to 0 \ as \ n\to\infty.$$

Proof. (i) To show the first point, we use recurrence. For $n=0$, $Y^{i,0}$ is the sum of a continuous process. Therefore $Y^{i,0}$ is continuous and since the process Ψ_i belongs to $\mathscr{M}^0$, we obtain from 4 that $Y^{i,0}$ is of class $\mathbb{D}$ by using Snell envelope notion (see Appendix). Suppose now that the property is true for some n and prove it for the term $n+1$. $Y^{i,n+1}$ is the Snell envelope of the process $(\int_0^t\Psi_i(s)ds+max_{i\neq k}(-g_{ik}(t)+Y_t^{k,n})\mathbf{1}_{[t<T]})_{0\leq t\leq T}$ and verifies $Y_T^{i,n+1}=0$. Since $max_{i\neq k}(-g_{ik}(t)+Y_t^{k,n})<0$ on $t=T$, this process is continuous on $[0,T)$ and have a positive jump at T. Therefore $Y^{i,n+1}$ is continuous and belongs to class $\mathbb{D}$ (see Appendix). So we conclude for every $i\in\mathscr{J}$, $Y^{i,n}$ is continuous and of class $\mathbb{D}$ for any $n\geq 0$.

To prove (ii-a) let us set $\mathscr{D}_\theta^{i,n}=\{\delta\in\mathscr{D}$, such that $\delta_0=i,\tau_1\geq\theta$ and $\tau_{n+1}=T$. Similaraly as in the Verification Theorem the following property yields.

$$Y_\theta^{i,n}=ess\sup_{\delta\in\mathscr{D}_\theta^{i,n}}\mathbb{E}[\int_\theta^T\Psi_{\delta_s}(s)ds-\sum_{j=1}^n g_{\delta_{\tau_{j-1}}\delta_{\tau_j}}(\tau_j)\mathbf{1}_{[\tau_j<T]}|\mathscr{F}_\theta].\tag{8}$$

Since $\mathscr{D}_\theta^{i,n}\subset\mathscr{D}_\theta^{i,n+1}$, we have $Y_\theta^{i,n}\leq Y_\theta^{i,n+1}$ for any stopping time θ thanks to the continuity of $Y^{i,n}$.

Since $g_{ij}>0$, we obtain for each $i\in\mathscr{J}$,

$$\forall\theta, Y_\theta^{i,n}\leq\mathbb{E}[\int_\theta^T max_{i=1,..,m}|\Psi_i(s)|ds|\mathscr{F}_\theta].\tag{9}$$

Therefore, this sequence converges to another process $Y_t^i=\lim_{n\to\infty}Y_t^{i,n}$ and this limit process verifies

$$Y_\theta^{i,0}\leq Y_\theta^i\leq\mathbb{E}[\int_\theta^T max_{i=1,..,m}|\Psi_i(s)|ds|\mathscr{F}_\theta].\tag{10}$$

Now using the same arguments as in the paper of Djehiche et al. [5] theorem 2, we obtain that the processes $Y^1,..,Y^m$ are continuous and by uniqueness these limit processes satisfy the Verification Theorem. It remais to show the final point of convergence $(ii-c)$ and this is obvious since the continuity of the processes $Y^{i,n}$ and the limit Y^i.

5 Appendix

The Snell envelope notion Let $U=(U_t)_{t\leq T}$ be an $\mathscr{F}_t$-adapted, $\mathbb{R}$-valued càdlàg process without negative jumps and which belongs to the class $\mathbb{D}$, i.e., the set of random variables $\{U_\tau,\tau\in\mathscr{T}_0\}$ is uniformly integrable. Then there exists a unique $\mathscr{F}_t$-adapted $\mathbb{R}$-valued continuous process $Z:=(Z_t)_{t\leq T}$ (see e.g. [2,6]), called the Snell envelope of U, such that. Z is the smallest super-martingale which dominates U, i.e, if $(\bar{Z}_t)_{t\leq T}$ is another càdlàg super-martingale of class $\mathbb{D}$ such that $\forall t\leq T,\bar{Z}_t\geq U_t$ then $\bar{Z}_t\geq Z_t$ for any $t\leq T$. The following properties of the process Z hold true.

(i) Z can be expressed as : for any $\mathscr{F}_t$-stopping time γ,

$$Z_\gamma = esssup_{\tau\in\mathscr{T}_\tau}\mathbb{E}[U_\tau|\mathscr{F}_\gamma] \text{ (and then } Z_T = U_T).$$

(ii) let γ be an $\mathscr{F}_t$-stopping time and $\tau^*_\gamma = \inf\{s \geq \gamma, Z_s = U_s\} \wedge T$ then τ^*_γ is optimal after γ, i.e.,

$$Z_\gamma = \mathbb{E}[Z_{\tau^*_\gamma}|\mathscr{F}_\gamma] = \mathbb{E}[U_{\tau^*_\gamma}|\mathscr{F}_\gamma] = esssup_{\tau\geq\gamma}\mathbb{E}[U_\tau|\mathscr{F}_\gamma].$$

(iii) if $U_n, n \geq 0$, and U are càdlàg and of class $\mathbb{D}$ such that the sequence $(U_n)_{n\geq 0}$ converges increasingly and pointwisely to U then $(Z^{U_n})_{n\geq 0}$ converges increasingly and pointwisely to Z^U ; Z^{U_n} and Z^U are the Snell envelopes of respectively U_n and U. If U belongs to class $\mathbb{D}$ then Z^U belongs to class $\mathbb{D}$.

The proof of (iii) is given in the appendix of [2]. For more details on the Snell envelope notion, one can refer to [6].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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