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ON BOUNDS OF EXTREMAL EIGENVALUES OF MATRICES

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ABSTRACT

In this paper, bounds of dominant eigenvalues of a matrix are obtained in terms of absolute column sums and row sums. We received more exact inequalities on the bounds of eigenvalues. Several numerical examples are given to show that our formulas are effective, at least, for some classes of matrices.

Keywords: *maximal eigenvalues; iteration theorems.*

ÖZET

Bu makalede, matrisin dominant özdeğerleri satır elemanlarının mutlak değerleri toplamı ve sutun elemanlarının mutlak değerleri toplamı şeklinde bulunmuştur. Özdeğerler için daha doğru eşitsizlikler alınmıştır. Bulunmuş formüllerin en azından bir kaç matrisler sınıfı için daha efektif olduğunu göstermek için sayısal örnekler incelenmiştir.

Anahtar Kelimeler *: Maksimal özdeğerler; iterasyon teoremleri*

1. INTRODUCTION

In 1907, Perron [9] made the fundamental discovery that the dominant eigenvalue of a matrix with all positive entries is positive and also that there exists a corresponding eigenvector bwith all positive entries. This result was later extended to nonnegative matrices by Frobenius [3]. For a complex matrix

$C=(c_{ik}),\ i,k=1,2,...,n$ the well known Gershgorin theorem states that all eigenvalues λ of C must satisfy

$$|\lambda| \geq \min_i (|c_{ii}| - \sum_{k,k\neq i} |c_{ik}|)$$

[1]. The upper (Frobenius) bound for $|\lambda|$ easily can be found:

$$|\lambda| \leq \max_i (\sum_k |c_{ik}|).$$

Moreover, we have Brauer's theorem, Ostrowski's theorem and Brualdi's theorem etc, by which we can estimate the inclusions regions of eigenvalues of a matrix in terms of its entries (see [2], [4] and [7]). There are results on lower bound of eigenvalues (see, for example, [5] and [10]), results for stochastic matrices [6] and so on.

The goal of this paper is to give new bounds of $|\lambda|$.

2. GENERALIZED ITERATION THEOREMS AND UPPER BOUNDS FOR A DOMINANT EIGENVALUE.

We consider the system of linear equations

$$x = Cx + b, \tag{1}$$

where

$$x=(x_1,x_2,...,x_n)^{\mathrm{T}},\ b=(b_1,b_2,...,b_n)^{\mathrm{T}},...,C=(c_{ik}),\ i,k=1,2,...,n$$

are complex matrices. For the given set of real numbers $\{a_1,a_2,...,a_n\}$ we denote by $\overset{(2)}{\max}\{a_k\}$ the second maximum of the set $\{a_1,a_2,...,a_n\}$: $\overset{(2)}{\max}\{a_k\} = \max_{k\neq p} a_k$, where $a_p = \max_k\{a_k\}$. Similarly, we define $\overset{(i)}{\max_k}\{a_k\}$ and $\overset{(i)}{\min_k}\{a_k\}$. For example,

$$\overset{(3)}{\max}\{1,2,3,4,4,5\}=4,\ldots,\ \overset{(i)}{\min}\{1,2,3,4,4,5\}=3\,.$$

Now we denote by

$$\alpha_p \le \frac{\overset{(p)}{\max\limits_i}\sum\limits_{k=1}^{n}|c_{ki}|}{\max\limits_i\sum\limits_{k=1}^{n}|c_{ki}|},\ \ p=2,3,\ldots,n\ \text{ and }\ \alpha=\sqrt{\alpha_2}\,.$$

Theorem 1. Let $\max\limits_i\left\{\sum\limits_{k=1}^{n}|c_{ki}|\right\}=\sum\limits_{k=1}^{n}|c_{kq}|$ and

$$\sqrt{\alpha_2}\left|c_{1q}\right|+\ldots+\sqrt{\alpha_2}\left|c_{q-1,q}\right|+\left|c_{qq}\right|+\sqrt{\alpha_2}\left|c_{q+1,q}\right|+\ldots+\sqrt{\alpha_2}\left|c_{nq}\right|<1 \qquad (2)$$

then the system of equations (1) has a unique solution and the solution can be received by the iteration method.

Proof. For simplicity let q=1, that is $\left|c_{11}\right|+\alpha\left|c_{21}\right|+\ldots+\alpha\left|c_{n1}\right|<1$ and

$$\sum_{k=1}^{n}|c_{k2}|=\overset{(2)}{\max_i}\sum_{k=1}^{n}|c_{ki}|=\alpha_2\sum_{k=1}^{n}|c_{k1}|. \qquad (3)$$

Consider the norm in $C^n=\{(x_1,x_2,\ldots,x_n)\}$ by

$$\|x\|=|x_1|+\alpha|x_2|+\ldots+\alpha|x_n|\,. \qquad (4)$$

And let $T:C^n\to C^n$ be a mapping $y=Tx=Cx+b$. If $y=Tx$ and $w=Tv$, we have

$$\|y-w\|=\left|\sum_{k=1}^{n}c_{1k}(x_k-v_k)\right|+\alpha\left|\sum_{k=1}^{n}c_{2k}(x_k-v_k)\right|+\ldots$$
$$+\alpha\left|\sum_{k=1}^{n}c_{nk}(x_k-v_k)\right|\le\left|c_{11}\right|\left|x_1-v_1\right|+\alpha\sum_{i=2}^{n}\left|c_{i1}\right|\left|x_1-v_1\right|+\ldots \qquad (5)$$

$$+|c_{1n}||x_n - v_n| + \alpha\sum_{i=2}^{n}|c_{in}||x_n - v_n| = |x_1 - v_1|\left(|c_{11}| + \sum_{i=2}^{n}\alpha|c_{i1}|\right)$$
$$+\ldots+|x_n - v_n|\left(|c_{1n}| + \sum_{i=2}^{n}\alpha|c_{in}|\right).$$

Then

$$\sum_{i=2}^{n}\alpha|c_{ip}| + |c_{1p}| = \alpha\sum_{i=1}^{n}|c_{ip}| + (1-\alpha)|c_{1p}|$$

$$=\alpha\alpha_p\left(|c_{11}|+|c_{21}|+\ldots+|c_{n1}|\right)+(1-\alpha)|c_{1p}| = \alpha\alpha_p|c_{11}|$$
$$+\left(\alpha\alpha_p-\alpha^2\right)\left(|c_{21}|+\ldots+|c_{n1}|\right)+\alpha^2\left(|c_{21}|+\ldots+|c_{n1}|\right) \qquad (6)$$

$$+(1-\alpha)|c_{1p}| = \alpha|c_{11}| + \alpha^2\left(|c_{21}|+\ldots+|c_{n1}|\right)+\left(\alpha^2-\alpha\right)|c_{11}|$$

$$+\left(\alpha\alpha_p-\alpha^2\right)\left(|c_{11}|+|c_{21}|+\ldots+|c_{n1}|\right)+(1-\alpha)|c_{1p}|.$$

Let us denote by $S = |c_{11}|+|c_{21}|\ldots+|c_{n1}|$ and show that

$$\left(\alpha^2-\alpha\right)|c_{11}|+\left(\alpha\alpha_p-\alpha^2\right)S+(1-\alpha)|c_{1p}| \le 0. \qquad (7)$$

Since $\left(\alpha^2-\alpha\right)|c_{11}| \le 0$ we need to show that

$$\left(\alpha\alpha_p-\alpha^2\right)S+(1-\alpha)|c_{1p}| \le 0 \quad \text{or}$$

$$\frac{1-\alpha}{\alpha(\alpha-\alpha_p)} \le \frac{S}{|c_{1p}|}. \qquad (8)$$

(If $c_{1p}=0,$ (7) trivially holds). Since $\dfrac{S}{|c_{1p}|} \ge \dfrac{1}{\alpha_p}$ we have that, (8) holds if

$$\frac{1-\alpha}{\alpha(\alpha-\alpha_p)} \leq \frac{1}{\alpha_p}.$$

Last one is equivalent to the true inequality $\alpha^2 \geq \alpha_p$ and that is (7) holds. Now (7), (6) and (5) together imply

$$\sum_{i=2}^{n} \alpha |c_{ip}| + |c_{1p}| \leq \beta(|c_{11}| + \alpha |c_{21}| + \ldots + \alpha |c_{n1}|), \quad p = 2,3,\ldots,n$$

and hence

$$\| y-w \| \leq \left(|c_{11}| + \alpha |c_{21}| + \ldots + \alpha |c_{n1}|\right) \| x-v \|.$$

Thus T is contraction mapping and has a unique fixed point. This fixed point is a solution of the equations (1).

Let $B(a,b) \equiv \{z : |z-a| \leq b\}$.

Corollary 1. (Generalized Gershgorin Theorem). Let

$$\max_i \left\{ \sum_{k=1}^{n} |c_{ki}| \right\} = \sum_{k=1}^{n} |c_{kp}|,$$

then every eigenvalue of the matrix C lies on

$$\bigcup_{k=1}^{n} B(c_{kk}, R_k) \bigcap B(0,d),$$

where $R_k = \sum_{k=1, k\neq i}^{n} |c_{ki}|$ and

$$d = \sqrt{\alpha_2}\,|c_{1p}| + \ldots + \sqrt{\alpha_2}\,|c_{p-1,p}| + |c_{pp}| + \sqrt{\alpha_2}\,|c_{p+1,p}| + \ldots + \sqrt{\alpha_2}\,|c_{np}|.$$

The similar statement is true for the row sums. For the proof of Corollary 1 repeat the proof of the Theorem 1 for the matrix $\frac{1}{d+\varepsilon}C$ for any small enough positive ε. Since all eigenvalues of $\frac{1}{d+\varepsilon}C$ lies on $B(0,1)$, tending $\varepsilon \to 0$ we conclude that all eigenvalues of C lies on $B(0,d)$. It is clear that if $\alpha_2 \neq 1$, and $c_{mp} \neq 0$ for at least one $m \neq p$, then we have a strict inclusion

$$\bigcup_{k=1}^{n} B(c_{kk}, R_k) \bigcap B(0,d) \subset \bigcup_{k=1}^{n} B(c_{kk}, R_k).$$

Example 1. Consider the nonnegative matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{pmatrix}, \quad \rho(A) = 4.7321.$$

Here $\alpha_2 = 0.5$ and therefore $\rho(A) \leq \sqrt{0.5} \cdot (0+1+1) + 4 = 5.4142$.

Example 2 (Corollary). For the matrices with zeros in the main

$$\text{diagonal } A = \begin{pmatrix} 0 & c_{12} & c_{13} \cdots & c_{1n} \\ c_{21} & 0 & c_{23} \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} \cdots & 0 \end{pmatrix}$$

we have

$$\rho(A) \leq \left(\max_k \left\{ \sum_{i=1}^{n} | c_{ik} | \right\} \right)^{1/2} \left(\overset{(2)}{\max_k} \left\{ \sum_{i=1}^{n} | c_{ik} | \right\} \right)^{1/2}.$$

Now let us show that the conditions of the Theorem 1 can be weakened for diagonally dominant type matrices. Let us denote by $\beta = \sqrt{\dfrac{\alpha_2}{2-\alpha_2}}$.

Theorem 2. Let

$$\max_i \left\{ \sum_{k=1}^{n} |c_{ki}| \right\} = \sum_{k=1}^{n} |c_{kq}|, \quad |c_{ii}| \geq |c_{im}| \ \text{ and } \ |c_{ii}| \geq |c_{mi}|$$

for all $m, i = 1, 2, \ldots, n$ and

$$\beta |c_{1q}| + \ldots + \beta |c_{q-1,q}| + |c_{qq}| + \beta |c_{q+1,q}| + \beta |c_{nq}| < 1, \tag{9}$$

then the system of equations (1) has a unique solution and the solution can be received by the iteration method.

Proof. For simplicity let q=1. In the proof of Theorem 1 we take β instead of $\alpha = \sqrt{\alpha_2}$ and continue in like manner. The relationship (6) holds again, and let us prove (7) for β, that is show that

$$\left(\beta^2 - \beta\right)|c_{11}| + \left(\beta\alpha_p - \beta^2\right)S + (1-\beta)|c_{1p}| \leq 0. \tag{10}$$

This inequality is equivalent to

$$(1-\beta)|c_{1p}| \leq \left(\beta - \beta^2\right)|c_{11}| + \left(\beta^2 - \beta\alpha_p\right)S.$$

Since $|c_{11}| \geq |c_{1p}|$ and $\dfrac{S}{|c_{1p}|} \geq \dfrac{2}{\alpha_p}$ we need to show that

$$(1-\beta) \leq \beta - \beta^2 + \left(\beta^2 - \beta\alpha_p\right)\frac{2}{\alpha_p} \ \text{ or } \ \beta^2 \geq \frac{\alpha_p}{2-\alpha_p}.$$

Moreover, this inequality can easily derived from the fact that the function $\frac{x}{2-x}$ increases for $x \in (0,1)$. Then the mapping T is contraction and there exists a unique solution of the equation (1).

Corollary 2. Let C be a matrix with $|c_{ii}| \geq |c_{im}|$ and $|c_{ii}| \geq |c_{mi}|$, $i,m = 1,2,\ldots,n$, then every eigenvalue of the matrix C lies on $\bigcup_{k=1}^{n} B(c_{kk}, R_k) \cap B(0,d)$,

where $d = \sqrt{\frac{\alpha_2}{2-\alpha_2}}|c_{1p}| + \ldots + \sqrt{\frac{\alpha_2}{2-\alpha_2}}|c_{p-1,p}| + |c_{pp}| + \sqrt{\frac{\alpha_2}{2-\alpha_2}}|c_{p+1,p}|$

$$+ \ldots + \sqrt{\frac{\alpha_2}{2-\alpha_2}}|c_{np}|.$$

Example 3. For the matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0.95 \\ 1 & 1 & 1 & 0.95 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0.95 & 1 & 1 \\ 1 & 0,95 & 1 & 1 & 1 \end{pmatrix}$$

we have

$\alpha_2 = 0.99$, $\alpha = \sqrt{\alpha_2} = 0.99498..$ and $\beta = \sqrt{\frac{0.99}{2-0.99}} = 0.9900495...$

We easily obtain, by applying Corollary 2 $\rho(C) \leq 4.96019...$ This is very good upper bound since $\rho(C) = 4.95998...$

Corollary 3. Let C be a weak diagonally dominant matrix, that is

$$|c_{ii}| \geq \sum_{m\neq i, m=1}^{n} |c_{mi}|, \quad i = 1, 2, ..., n,$$

then

$$\rho(C) \leq \max\{|c_{ii}|\}\left(1 + \sqrt{\frac{\alpha_2}{2-\alpha_2}}\right).$$

Theorems 1 and 2 can be improved in terms of third, fourth, etc. maxima of summations. First let us denote by

$$\alpha_r \leq \frac{\overset{(r)}{\max_i} \sum_{k=1}^{n} |c_{ki}|}{\max_i \sum_{k=1}^{n} |c_{ki}|}, \quad \gamma_r = \max\{\sqrt{\alpha_r}, \alpha_2\} \text{ and let}$$

$$\max_i \left\{\sum_{k=1}^{n} |c_{ki}|\right\} = \sum_{k=1}^{n} |c_{kq}|, \quad \overset{(m)}{\max_i} \left\{\sum_{k=1}^{n} |c_{ki}|\right\} = \sum_{k=1}^{n} |c_{ki_m}|.$$

Theorem 3. Let $P = \{i_r, i_{r+1}, ..., i_n\}$, $r \geq 2$ and

$$\gamma_r \sum_{m\in P} |c_{mq}| + \sum_{m\in P'} |c_{mq}| < 1. \tag{11}$$

Then the system of equations (1) has a unique solution and the solution can be received by the iteration method.

Proof. For simplicity let $q = 1$, $r = 3$ and $\sum_{k=1}^{n} |c_{ks}| = \overset{(s)}{\max_i} \left\{\sum_{k=1}^{n} |c_{ki}|\right\}$. That is $i_m = m$ and

$$|c_{11}| + |c_{21}| + \gamma_3 \left(|c_{31}| + ... + |c_{n1}|\right) < 1.$$

Consider the norm in $C^n = \{(x_1, x_2, ..., x_n)\}$ by

$$\| x \| = | x_1 | + | x_2 | + \gamma_3 (| x_3 | + \ldots + \alpha | x_n |)$$

and the mapping $T : C^n \to C^n$ by $Tx = Cx + b$. For y = Tx and w = Tv, we have

$$\| y - w \| = \left| \sum_{k=1}^{n} c_{1k} (x_k - v_k) \right| + \left| \sum_{k=1}^{n} c_{2k} (x_k - v_k) \right| + \gamma_3 \left| \sum_{k=1}^{n} c_{3k} (x_k - v_k) \right|$$

$$+ \gamma_3 \left| \sum_{k=1}^{n} c_{nk} (x_k - v_k) \right| \le |x_1 - v_1| \left(| c_{11} | + | c_{21} | + \gamma_3 \sum_{i=3}^{n} | c_{i1} | \right) \qquad (12)$$

$$+ \ldots + |x_n - v_n| \left(| c_{1n} | + | c_{2n} | + \gamma_3 \sum_{i=3}^{n} | c_{in} | \right).$$

Let us estimate the summations $| c_{1k} | + | c_{2k} | + \gamma_3 \sum_{i=3}^{n} | c_{ik} |,$

k = 2,3,…,n. For k = 2 from the definition of α_2 we have

$$| c_{12} | + | c_{22} | + \gamma_3 \sum_{i=3}^{n} | c_{i2} | = | c_{11} | + | c_{21} | + \gamma_3 | c_{31} | + \ldots + \gamma_3 | c_{n1} |$$

$$+ (\alpha_2 - 1) \gamma_3 S + (\gamma_3 - 1) \big(| c_{11} | + | c_{21} | \big) + (1 - \gamma_3) \big(| c_{12} | + | c_{22} | \big), \qquad (13)$$

where $S = | c_{11} | + | c_{21} | + | c_{31} | + \ldots + | c_{n1} |$. Let's show that

$$(\alpha_2 - 1) \gamma_3 S + (\gamma_3 - 1) \big(| c_{11} | + | c_{21} | \big) + (1 - \gamma_3) \big(| c_{12} | + | c_{22} | \big) \le 0. \qquad (14)$$

Since all expressions here except last one are not positive and $\frac{S}{| c_{12} | + | c_{22} |} \ge \frac{1}{\alpha_2}$, to obtain (14) we need to establish the inequality

$$1 - \gamma_3 \le (1 - \alpha_2) \gamma_3 \frac{1}{\alpha_2},$$

which is equivalent to the true inequality $\alpha_2 \le \gamma_3$. That is we have

$$|c_{12}|+|c_{22}|+\gamma_3\sum_{i=3}^{n}|c_{i2}|=|c_{11}|+|c_{21}|+\gamma_3|c_{31}|+\ldots+\gamma_3|c_{n1}|. \quad (15)$$

Now for $k>2$ by using the definition of α_k we have

$$|c_{1k}|+|c_{2k}|+\gamma_3\sum_{i=3}^{n}|c_{ik}|=\gamma_3\left(|c_{11}|+|c_{21}|+\gamma_3|c_{31}|+\ldots+\gamma_3|c_{n1}|\right)$$
$$+(\alpha_k-\gamma_3)\gamma_3 S+(\gamma_3^2-\gamma_3)\left(|c_{11}|+|c_{21}|\right)+(1-\gamma_3)\left(|c_{1k}|+|c_{2k}|\right).$$

Let us show that

$$(\alpha_k-\gamma_3)\gamma_3 S+(\gamma_3^2-\gamma_3)\left(|c_{11}|+|c_{21}|\right)+(1-\gamma_3)\left(|c_{1k}|+|c_{2k}|\right)\le 0. \quad (16)$$

Again, all terms here except last one are not positive and $\frac{S}{|c_{1k}|+|c_{2k}|}\ge\frac{1}{\alpha_k}$, hence we need to establish the inequality $1-\gamma_3\le(\gamma_3-\alpha_k)\gamma_3\frac{1}{\alpha_k}$, which is equivalent to the true inequality $\alpha_k\le\gamma_3^2$, for k=3,4,…,n. That is we obtain

$$|c_{1k}|+|c_{2k}|+\gamma_3\sum_{i=3}^{n}|c_{ik}|=\gamma_3\left(|c_{11}|+|c_{21}|+\gamma_r|c_{31}|+\ldots+\gamma_r|c_{n1}|\right). \quad (17)$$

The inequalities (17), (15), (13) and (12) together imply

$$\|y-w\|\le\left(|c_{11}|+|c_{21}|+\gamma_3|c_{31}|\ldots+\gamma_3|c_{n1}|\right)\|x-v\|$$

and therefore T is contraction mapping and has a unique fixed point.

Definition. The matrix C is called a maxi-dominant if

$$\sum_{i\in Q}|c_{ki}|\ge\sum_{i\in Q}|c_{mi}|$$

for any subset Q of $\{1,2,\ldots,n\}$ with $k\in Q$, where

$$\sum_{i=1}^{n} | c_{ki} | = \max_{m} \sum_{i=1}^{n} | c_{mi} |.$$

The conditions of the Theorem 3 can be weakened for the maxi-dominant matrices:

Theorem 4. Let C be a maxi-dominant matrix and $P = \{i_r, i_{r+1}, \ldots, i_n\}$, $r \geq 2$, and

$$\sqrt{\alpha_r} \sum_{p \in P} | c_{pq} | + \sum_{p \in P'} | c_{pq} | < 1. \qquad (18)$$

Then the system of equations (1) has a unique solution and the solution can be received by the iteration method.

Proof. We need to repeat the proof of the theorem 3, by taking $\sqrt{\alpha_r}$ instead of γ_r and taking into account that now (14) trivially holds. Note that the inequality (18) can be weakened as (by using (14))

$$\frac{-\alpha_r + \sqrt{4\alpha_r + 3\alpha_r^2}}{2(1-\alpha_r)} \sum_{p \in P} | c_{pq} | + \sum_{p \in P'} | c_{pq} | < 1. \qquad (19)$$

and easily seen that $\dfrac{-\alpha_r + \sqrt{4\alpha_r + 3\alpha_r^2}}{2(1-\alpha_r)} < \sqrt{\alpha_r}$.

Let us compare the result of the Theorem 4 with the Brauer's result [6]: If C is positive matrix, then

$$\rho(A) \leq S - m\left(1 - \frac{2(s-m)}{S - 2m + \sqrt{S^2 4m(S-s)}}\right),$$

where

$$S = \max_q \sum_{p=1}^{n} c_{pq}, \quad s = \min_q \sum_{p=1}^{n} c_{pq}, \quad \min_{i,j} a_{ij}.$$

Example 4. Consider the matrix [8]

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \quad \rho(A) = 17.207\,.$$

We have S = 21 and s = 6 and easily obtain, by applying Brauer's formula $\rho(A) \le 20.\ 260$. Now, since $\alpha_3 = \frac{18}{21}$ we have

$$\rho(A) \le 6 + 5 + (1 + 2 + 3 + 4) \cdot \sqrt{\frac{18}{21}} = 20.258.$$

We can obtain more interesting result for $\rho(A)$, by using formula (19), for example if P={3,4,5,6} we have

$$\rho(A) \le 6 + 5 + (1 + 2 + 3 + 4) \cdot \frac{-\frac{18}{21} + \sqrt{4 \cdot \frac{18}{21} - 3 \cdot \left(\frac{18}{21}\right)^2}}{2 \cdot \left(1 - \frac{18}{21}\right)} = 19.730$$

or by taking P={4,5,6} we obtain

$$\rho(A) \le 6+5+4+(1+2+3)\cdot\frac{-\frac{15}{21}+\sqrt{4\cdot\frac{15}{21}-3\cdot\left(\frac{15}{21}\right)^2}}{2\cdot\left(1-\frac{15}{21}\right)} = 19.593...$$

Remark. Note that when C is positive matrix , Brauer's formula and other many formulae (e.g., Ledermann's and Ostrowski's formulae, see [6]) for the bounds of C are closely relative to the least entry of C. So our results in general, become more effective even in case of positive matrices if the least entry is small enough.

Example 5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \rho(A) = 10.066.$$

We have $S = 21$ and $s = 6$ and easily obtain, by applying Brauer's formula $\rho(A) \le 20.\ 260$. That is, in spite of serious differences in spectral radii of the matrices in the examples 4 and 5 the Brauer's formula gives the same result. But we obtain, by applying (18) for $r = 4$

$$\rho(A) \le 1+2+3+15\cdot\sqrt{\frac{9}{21}} = 15.820.$$

REFERENCES

[1] Bellman, R.: Introduction to matrix analysis, McGraw-Hill, New York. MR 23 #A153 (1960)

[2] Brauer, A. and Gentry, I.C.: Bounds for the greatest characteristic root of an irreducible non-negative matrix. Linear Algebra Appl. 8 105-107 (1974)

[3] Frobenius, G.: Uber matrizen aus nicht negativen elementen, S.--B. Preuss Acad. Wiss., Berlin (1912).

[4] Horn, R. A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, New York (1985)

[5] Huang, T.Z., Zhang W., Shen, S.Q.: Regions containing eigenvalues of a matrix, ELA, v.15, 215-224 (2006)

[6] Marcus, M., Minc, H.: A Survey of Matrix Theory and Matrix Inequalities. Boston (1992)

[7] Minc, H.: Nonnegative Matrices. John Wiley & Sons, New York (1988)

[8] Neumann, M.: Inverse of Perron complements of inverse M-matrices. Linear Algebra Appl. 313, 163-171 (2000)

[9] Perron, O.: Zur theorie der matrizen, Math Ann. 64 248-263 (1907)

[10] Wo, C.W.: On bounds of extremal eigenvalues of irreducible and m-reducible matrices, Linear Algebra Appl. 402 29-45 (2005)