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# On $\psi$-Hilfer fractional differential equation with complex order

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**Abstract**

The objectives of this paper is to investigate some adequate results for the existence of solution to a $\psi$-Hilfer fractional derivatives (HFDEs) involving complex order. Appropriate conditions for the existence of at least one solution are developed by using Schauder fixed point theorem (SFPT) to the consider problem. Moreover, we also investigate the Ulam-Hyers stability for the proposed problem.

## 1. Introduction

Fractional calculus deals with the study of fractional order integral and derivative operators over real or complex domains and some of their applications are in the area of fluid flow, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, viscoelasticity, electrochemistry of corrosion, dynamical processes in self-similar and porous structures, optics and rheology etc. There has been significant development in fractional differential equations in recent year (see [1]-[6])

The generalization of Riemann-Liouville and Caputo fractional derivatives was introduced by R. Hilfer [1] in 1999. A significant development and interest has been shown by many researchers. Vanterler et al. interpolated HFD and $\psi$-fractional derivative is called as $\psi$-HFD [7]. This fractional derivative is different from the other classical fractional derivative because the kernel is in terms of function. The study on $\psi$-HFD with classical properties and interpolation of many fractional derivatives.

Alternatively, the stability problem of functional equations initiated form a question of Ulam, created in 1940, relating to the stability of group homomorphism. In the next year, Hyers gave a partial affirmative respond to the question of Ulam in the background of Banach spaces that was the opening momentous breakthrough and a step towards more solutions in this area. In view of the fact that a large number of papers have been published in connections with various generalizations appeared devoted to the data dependence in the theory of fractional differential equations [8]-[11].

Inspired by the above discussion, we introduce complex order to $\psi$-HFD and we establish the existence, uniqueness and stability of solutions. Consider the differential equations with $\psi$-HFD with complex order of the form

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t)),\ t \in J := (a,b], \tag{1.1}$$

$$\mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \theta = \theta_1 + \theta_2 - \theta_1\theta_2, \tag{1.2}$$

where $D^{\theta_1,\theta_2;\psi}(\theta_1,\theta_2 \in \mathbb{C})$ is $\psi$-HFD of order $\theta_1 = \alpha + i\beta$ and type $\theta_2 = \gamma + i\eta$. Here, $0 < \Re(\theta_1) < 1$ and $0 \le \Re(\theta_2) \le 1$, with $\alpha,\beta,\gamma$ and $\eta$ are constants. Consider a Banach space $R$ and $\mathfrak{g} : J \times R \to R$ be a continuous function.

The paper is organised as follows. In Section 2, we give some basic definitions and results concerning with the $\psi$-HFD. In Section 3, we present existence results based on SFPT and further stability result is also discussed. Finally, an example is included to check the theoretical results.

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## 2. Preliminaries

For the ease of the readers, we discuss some basic definitions and lemmas. The ideas are adopted from [12, 13]. Next, consider the following spaces, let $C(J)$ a space of continuous functions from $J$ into $R$ with the norm

$$\|x\|_C = \max\{|x(t)| : t \in J\}.$$

The weighted space $C_{1-\xi,\psi}(J)$ of functions $\mathfrak{g}$ on $J$ is defined by

$$C_{1-\xi,\psi}(J) = \left\{\mathfrak{g} : J \to R : (\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t) \in C(J)\right\}, 0 \leq \xi (= \Re(\theta)) < 1,$$

with the norm

$$\|\mathfrak{g}\|_{C_{1-\xi,\psi}} = \left\|(\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t)\right\|_{C[a,b]} = \max_{t\in J}\left|(\psi(t)-\psi(a))^{1-\xi}\,\mathfrak{g}(t)\right|.$$

**Definition 2.1.** *The $\psi$-Riemann Liouville (RL) fractional integral of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function $\mathfrak{g}$ is defined by,*

$$\mathfrak{I}^{\theta;\psi}\mathfrak{g}(t) = \frac{1}{\Gamma(\theta)}\int_0^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{\theta-1}\,\mathfrak{g}(s)ds, \quad t \geq 0.$$

**Definition 2.2.** *The $\psi$-RL fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of a function $\mathfrak{g}$ is defined by,*

$$\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t) = \frac{1}{\Gamma(n-\theta)}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \int_0^t \psi^{'}(s)\,(\psi(t)-\psi(s))^{n-\theta-1}\,\mathfrak{g}(s)ds, \quad t \geq 0,$$

*where $n = [\Re(\theta)]+1$.*

**Definition 2.3.** *The $\psi$-Caputo fractional derivative of order $\theta \in \mathbb{C}, (\Re(\theta) > 0)$ of function $\mathfrak{g}$ is defined by,*

$$\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t) = \mathfrak{I}^{n-\theta;\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)^n \mathfrak{g}(t) \quad t \geq 0,$$

*where $n = [\Re(\theta)]+1$.*

**Definition 2.4.** *The $\psi$-HFD of order $0 < \theta_1 < 1$ and $0 \leq \theta_2 \leq 1$ of function $\mathfrak{g}(t)$ is defined by*

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\theta_2(1-\theta_1);\psi}\left(\frac{1}{\psi^{'}(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\theta_2)(1-\theta_1);\psi}\mathfrak{g}(t). \tag{2.1}$$

*The $\psi$-HFD as above defined, can be written in the following*

$$\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{g}(t) = \mathfrak{I}^{\theta-\theta_1;\psi}\mathfrak{D}^{\theta;\psi}\mathfrak{g}(t).$$

**Remark 2.5.** *(a) If $\theta_2 = 0(\gamma = 0,\ \eta = 0)$, then $\mathfrak{D}^{\theta_1,\theta_2;\psi} = \mathfrak{D}^{\theta_1,0;\psi}$ is called the RL fractional derivative of complex order.*
*(b) If $\theta_2 = 1(\gamma = 1,\ \eta = 0)$, then $\mathfrak{D}^{\theta_1,\theta_2;\psi} = \mathfrak{I}^{1-\theta_1;\psi}\mathfrak{D}^{1;\psi}$ is called the Caputo fractional derivative of complex order.*

**Definition 2.6.** *The Stirling asymptotic formula of gamma function for $z \in \mathbb{C}$ is following*

$$\Gamma(z) = (2\pi)^{1/2} z^{z-\frac{1}{2}} e^{-z}\left[1+O\left(\frac{1}{z}\right)\right] \quad (|\arg(z)| < \pi; |z| \to \infty),$$

*and its result for $|\Gamma(a+ib)|$, $(a,b \in R)$ is*

$$|\Gamma(a+ib)| = (2\pi)^{1/2}\,|b|^{a-\frac{1}{2}}\,e^{-a-\frac{\pi|b|}{2}}\left[1+O\left(\frac{1}{z}\right)\right] \quad (b \to \infty).$$

Here, we shall give the definitions of Ulam-Hyers(U-H) stability and Ulam-Hyers-Rassias(U-H-R) stability for $\psi$-HFDEs of complex order. Let $\varepsilon > 0$ be a positive real number and $\varphi : J \to R^+$ be a continuous function. We consider the following inequalities:

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varepsilon, \quad t \in J, \tag{2.2}$$

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varepsilon\varphi(t), \quad t \in J, \tag{2.3}$$

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t) - \mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varphi(t), \quad t \in J. \tag{2.4}$$

**Definition 2.7.** *Eq. (1.1) is U-H stable if there exists a real number $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ of the inequality (2.2) there exists a solution $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ of Eq. (1.1) with*

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \leq C_f\,\varepsilon, \quad t \in J.$$

**Definition 2.8.** *Eq.* (1.1) *is generalized U-H stable if there exist* $\varphi \in C_{1-\xi,\psi}(J)$, $\varphi_f(0)=0$ *such that for each solution* $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ *of the inequality* (2.2) *there exists a solution* $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ *of Eq.* (1.1) *with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq \varphi_f\, \varepsilon, \quad t \in J.$$

**Definition 2.9.** *Eq.* (1.1) *is U-H-R stable with respect to* $\varphi \in C_{1-\xi,\psi}(J)$ *if there exists a real number* $C_{f,\varphi}>0$ *such that for each* $\varepsilon>0$ *and for each solution* $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ *of the inequality* (2.3) *there exists a solution* $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ *of Eq.* (1.1) *with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq C_{f,\varphi}\, \varepsilon\varphi(t), \quad t \in J.$$

**Definition 2.10.** *Eq.* (1.1) *is generalized U-H-R stable with respect to* $\varphi \in C_{1-\xi,\psi}(J)$ *if there exists a real number* $C_{f,\varphi}>0$ *such that for each solution* $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ *of the inequality* (2.4) *there exists a solution* $\mathfrak{h} \in C_{1-\xi,\psi}(J)$ *of Eq.* (1.1) *with*

$$|\mathfrak{v}(t)-\mathfrak{h}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in J.$$

**Remark 2.11.** *A function* $\mathfrak{v} \in C_{1-\xi,\psi}(J)$ *is a solution of the inequality*

$$\left|\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t)-\mathfrak{g}(t,\mathfrak{v}(t))\right| \leq \varepsilon, \;\; t \in J,$$

*iff there exist a function* $g \in C_{1-\xi,\psi}(J)$ *such that*

*(i)* $|g(t)| \leq \varepsilon, t \in J.$
*(ii)* $\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{v}(t)=\mathfrak{g}(t,\mathfrak{v}(t))+g(t), t\in J.$
*(iii) If* $\mathfrak{v}$ *is solution of the inequality* (2.2)*, then* $z$ *is a solution of the following integral inequality*

$$\left|\mathfrak{v}(t)-\frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1}-\frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \leq \frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\theta_1)|}\varepsilon.$$

**Lemma 2.12.** *Suppose* $\alpha(=\Re(\theta))>0$, $a(t)$ *is a nonnegative function locally integrable on* $a \leq t < b$ *(some* $b \leq \infty$*), and let* $g(t)$ *be a nonnegative, nondecreasing continuous function defined on* $a \leq t < b$*, such that* $g(t) \leq K$ *for some constant* $K$*. Further let* $\mathfrak{h}(t)$ *be a nonnegative locally integrable on* $a \leq t < b$ *function with*

$$|\mathfrak{h}(t)| \leq a(t)+g(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\mathfrak{h}(s)ds, \;\; t \in J$$

*with some* $\alpha>0$*. Then*

$$|\mathfrak{h}(t)| \leq a(t)+\int_a^t\left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\psi'(s)(\psi(t)-\psi(s))^{n\alpha-1}\right]a(s)ds, \;\; a \leq t < b.$$

**Theorem 2.13.** *(SFPT) Let* $E$ *be a Banach space and* $Q$ *be a nonempty bounded convex and closed subset of* $E$ *and* $N: Q \to Q$ *is compact, and continuous map. Then* $N$ *has at least one fixed point in* $Q$*.*

**Lemma 2.14.** *A function* $\mathfrak{h}$ *is the solution of*

$$\begin{cases}\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t)=\mathfrak{g}(t), \;\; t \in J,\\ \mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a}=\mathfrak{h}_a, \;\; \theta=\theta_1+\theta_2-\theta_1\theta_2,\end{cases} \tag{2.5}$$

*equivalent to the solution of integral equation:*

$$\mathfrak{h}(t)=\frac{\mathfrak{h}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1}+\frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s)ds. \tag{2.6}$$

## 3. Main results

Consider the following assumptions in order to solve the problem (1.1)-(1.2).

(H1) Let $\mathfrak{g}: J \times R \to R$ be continuous. For $\mathfrak{h},\mathfrak{v} \in R$, there exists a positive constant $L>0$ such that

$$|\mathfrak{g}(t,\mathfrak{h})-\mathfrak{g}(t,\mathfrak{v})| \leq L|\mathfrak{h}-\mathfrak{v}|, \quad t \in J.$$

(H2) The constant

$$\rho=\frac{L}{|\Gamma(\theta_1)|}(\psi(b)-\psi(a))^{\alpha}B(\xi,\alpha)<1.$$

(H3) Let $\mathfrak{g}: J \times R \to R$ be continuous. For $\mathfrak{h} \in R$, there exists $M \geq 0$ and $N>0$ such that

$$|\mathfrak{g}(t,\mathfrak{h})| \leq M|\mathfrak{h}|+N.$$

(H4) Suppose that there exists $\lambda_\varphi>0$ such that

$$\mathfrak{I}^{\theta_1;\psi}\varphi(t) \leq \lambda_\varphi\varphi(t).$$

**Theorem 3.1.** *If assumptions (H1) and (H2) are satisfied. Then, the Eq.* (1.1)-(1.2) *has a unique solution.*

*Proof.* Consider the operator $N : C_{1-\xi;\psi}(J) \to C_{1-\xi;\psi}(J)$ given by

$$(N\mathfrak{h})(t) = \frac{\mathfrak{h}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds. \tag{3.1}$$

Define a ball $B_r = \left\{\mathfrak{h} \in C_{1-\xi;\psi}(J) : \|\mathfrak{h}\| \le r\right\}$. First, we show $N(B_r) \subset B_r$, for $\mathfrak{h} \in B_r$

$$\begin{aligned}
|(N\mathfrak{h})(t)| &= \left|\frac{\mathfrak{h}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left|(\psi(t)-\psi(a))^{\theta-1}\right| + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)\left|(\psi(t)-\psi(s))^{\theta_1-1}\right||\mathfrak{g}(s,\mathfrak{h}(s))|ds \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}(\psi(t)-\psi(a))^{\xi-1} + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{h}(s))-\mathfrak{g}(s,0)|ds \\
&\qquad + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,0)|ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(N\mathfrak{h})(t)\|_{C_{1-\xi;\psi}} &\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}L|\mathfrak{h}(s)|ds \\
&\qquad + \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|\tilde{\mathfrak{g}}(s)|ds \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\theta_1)|}B(\xi,\alpha)\left(L\|\mathfrak{h}\|_{C_{1-\xi;\psi}} + \|\tilde{\mathfrak{g}}\|_{C_{1-\xi;\psi}}\right) \\
&:= r.
\end{aligned}$$

Let $\mathfrak{h},\mathfrak{v} \in C_{1-\xi;\psi}(J)$ and for $t \in J$, we have

$$\begin{aligned}
&\left|((N\mathfrak{h})(t)-(N\mathfrak{v})(t))(\psi(t)-\psi(a))^{1-\xi}\right| \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}L|\mathfrak{h}(s)-\mathfrak{v}(s)|ds \\
&\le \frac{L(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}(\psi(t)-\psi(a))^{\alpha+\xi-1}B(\xi,\alpha)\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}} \\
&\le \frac{L}{|\Gamma(\theta_1)|}(\psi(b)-\psi(a))^{\alpha}B(\xi,\alpha)\|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}} \\
&\le \|\mathfrak{h}-\mathfrak{v}\|_{C_{1-\xi;\psi}}.
\end{aligned}$$

□

**Theorem 3.2.** *Assume that [H3] is satisfied. Then, Eq.(1.1)-(1.2) has at least one solution.*

*Proof.* Consider the operator $N$, we check $N(B_r) \subset B_r$. For $\mathfrak{h} \in C_{1-\xi;\psi}(J)$ and $\|\mathfrak{h}\|_{C_{1-\xi;\psi}} < r'$. By using assumption [H3], we can obtain

$$\begin{aligned}
|(N\mathfrak{h})(t)| &= \left|\frac{\mathfrak{h}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}\left|(\psi(t)-\psi(a))^{\theta-1}\right| + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)\left|(\psi(t)-\psi(s))^{\theta_1-1}\right||\mathfrak{g}(s,\mathfrak{h}(s))|ds \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|}(\psi(t)-\psi(a))^{\xi-1} + \frac{1}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(M|\mathfrak{h}|+N)ds
\end{aligned}$$

$$\begin{aligned}
\|(N\mathfrak{h})(t)\|_{C_{1-\xi;\psi}} &\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}M|\mathfrak{h}|ds \\
&\qquad + \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}Nds \\
&\le \frac{|\mathfrak{h}_a|}{|\Gamma(\theta)|} + M\frac{(\psi(b)-\psi(a))^{\alpha}}{|\Gamma(\theta_1)|}B(\xi,\alpha)\|\mathfrak{h}\|_{C_{1-\xi;\psi}} + N\frac{(\psi(b)-\psi(a))^{\alpha-\xi+1}}{\alpha|\Gamma(\theta_1)|} \\
&:= r'.
\end{aligned}$$

Now we show that $N : B_r \to B_r$ is continuous. Let $\mathfrak{h}_n$ be a sequence such that $\mathfrak{h}_n \to \mathfrak{h}$ in $B_r$. Then for each $t \in J$, we have

$$\begin{aligned}
&\left|(N\mathfrak{h}_n(t) - N\mathfrak{h}(t))(\psi(t)-\psi(a))^{1-\xi}\right| \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_0^t \psi^{'}(s)\left|(\psi(t)-\psi(s))^{\theta_1-1}\right| |\mathfrak{g}(t,\mathfrak{h}_n(t)) - \mathfrak{g}(t,\mathfrak{h}(t))|\,ds \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1} |\mathfrak{g}(t,\mathfrak{h}_n(t)) - \mathfrak{g}(t,\mathfrak{h}(t))|\,ds \\
&\le \frac{(\psi(t)-\psi(a))^{1-\xi}}{|\Gamma(\theta_1)|}(\psi(t)-\psi(a))^{\alpha+\xi-1} B(\xi,\alpha)\,\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot))\|_{C_{1-\xi,\psi}} \\
&\le \frac{1}{|\Gamma(\theta_1)|}(\psi(b)-\psi(a))^{\alpha} B(\xi,\alpha)\,\|\mathfrak{g}(\cdot,\mathfrak{h}_n(\cdot)) - \mathfrak{g}(\cdot,\mathfrak{h}(\cdot))\|_{C_{1-\xi,\psi}}.
\end{aligned}$$

Since $\mathfrak{g}$ is continuous, then by the Lebesgue dominated convergence theorem which implies

$$\|(N\mathfrak{h}_n)(t) - (N\mathfrak{h})(t)\|_{C_{1-\xi,\psi}} \to 0 \ \text{ as } \ n \to \infty.$$

Thus $N(B_r)$ is uniformly bounded. It is clear that $N(B_r) \subset B_r$ is bounded. Next we show that $N(B_r)$ is equicontinuous. Let $t_1, t_2 \in J$, such that $t_1 < t_2$, we get

$$\begin{aligned}
&\left|(\psi(t_2)-\psi(a))^{1-\xi}(N\mathfrak{h})(t_2) - (\psi(t_1)-\psi(a))^{1-\xi}(N\mathfrak{h})(t_1)\right| \\
&= \left|\frac{(\psi(t_2)-\psi(a))^{1-\xi}}{\Gamma(\theta_1)}\int_a^{t_2} \psi^{'}(s)(\psi(t_2)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right. \\
&\qquad \left. + \frac{(\psi(t_1)-\psi(a))^{1-\xi}}{\Gamma(\theta_1)}\int_a^{t_1} \psi^{'}(s)(\psi(t_1)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \frac{\|\mathfrak{g}\|_{C_{1-\xi,\psi}}}{|\Gamma(\theta_1)|}B(\xi,\alpha)\left|(\psi(t_2)-\psi(a))^{\alpha} + (\psi(t_1)-\psi(a))^{\alpha}\right|.
\end{aligned}$$

Thus from Steps 1 to 3 with the Arzelӓ-Ascoli theorem, the operator $N$ is continuous and compact. From Theorem 2.13 the operator $N$ has a fixed point $\mathfrak{h}$ which is a solution of the problem Eq.(2.5). □

**Theorem 3.3.** *The assumptions [H1] and [H4] hold. Then Eq.(1.1)-(1.2) is generalised U-H-R stable.*

*Proof.* Let $\mathfrak{v}$ be solution of 2.4 and by Theorem 3.1 there $\mathfrak{h}$ is unique solution of the problem

$$\begin{aligned}
&\mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \mathfrak{g}(t,\mathfrak{h}(t)), \\
&\mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{I}^{1-\theta;\psi}\mathfrak{v}(t)|_{t=a} = \mathfrak{h}_a.
\end{aligned}$$

Then we have

$$\mathfrak{h}(t) = \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} + \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds.$$

By differentiating inequality (2.4), we have

$$\begin{aligned}
&\left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \\
&\le \left|\frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\theta_1-1}\varphi(s)ds\right| \\
&\le \lambda_\varphi \varphi(t).
\end{aligned}$$

Hence it follows that,

$$\begin{aligned}
&|\mathfrak{v}(t) - \mathfrak{h}(t)| \\
&\le \left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{h}(s))ds\right| \\
&\le \left|\mathfrak{v}(t) - \frac{\mathfrak{v}_a}{\Gamma(\theta)}(\psi(t)-\psi(a))^{\theta-1} - \frac{1}{\Gamma(\theta_1)}\int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\theta_1-1}\mathfrak{g}(s,\mathfrak{v}(s))ds\right| \\
&+ \int_a^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|\mathfrak{g}(s,\mathfrak{v}(s)) - \mathfrak{g}(s,\mathfrak{h}(s))|\,ds \\
&\le \lambda_\varphi\varphi(t) + \frac{L(\psi(b)-\psi(a))^{\alpha}}{\alpha|\Gamma(\theta_1)|}|\mathfrak{v}(t) - \mathfrak{h}(t)|
\end{aligned}$$

By Lemma 2.12, there exists a constant $K^* > 0$ independent of $\lambda_\varphi \varphi(t)$ such that

$$|\mathfrak{v}(t) - \mathfrak{h}(t)| \le K^*\varphi(t) := C_{f,\varphi}\varphi(t).$$

Thus, Eq.(1.1)-(1.2) is generalized U-H-R stable. □

## 4. An example

In this section, here we consider the following Cauchy problem in order to verify our results.

$$\begin{cases} \mathfrak{D}^{\theta_1,\theta_2;\psi}\mathfrak{h}(t) = \frac{1}{20}(\psi(t)-\psi(a))\cos(t)\mathfrak{h}(t),\ t\in J:=(a,b], \\ \mathfrak{I}^{1-\theta;\psi}\mathfrak{h}(t)|_{t=a} = \mathfrak{h}_a, \theta = \theta_1+\theta_2-\theta_1\theta_2, \end{cases} \tag{4.1}$$

By taking $\psi(t)=\ln t,\ a=1,\ b=e,\ \theta_1=\frac{1}{2}+\frac{1}{3}i,\ \theta_2=\frac{1}{3}+\frac{1}{2}i$, then we get a particular case of the proposed problem (4.1) using the Hadamard fractional derivative.

$$\mathfrak{D}^{\theta_1,\theta_2;\ln t}\mathfrak{h}(t) = \frac{1}{20}\ln t^{1/2}\cos(t)\mathfrak{h}(t),\ t\in(1,e], \tag{4.2}$$

$$\mathfrak{I}^{1-\theta;\ln t}\mathfrak{h}(1) = 1. \tag{4.3}$$

Here the function $\mathfrak{g}$ is continuous. Then, for all $\mathfrak{h},\mathfrak{v}\in R$, and $t\in(1,e]$, we have

$$|\mathfrak{g}(t,\mathfrak{h})-\mathfrak{g}(t,\mathfrak{v})| \leq \frac{1}{20}|\mathfrak{h}-\mathfrak{v}|$$

Thus condition (H2) is satisfied with $L=\frac{1}{20}$. Then, for $\lambda_\varphi = \frac{2}{\sqrt{\pi}}\varphi(t) = \ln t^{1/2}$, condition (H4) is satisfied. Hence, by Theorem 3.1 and Theorem 3.3, the problem has a unique solution and it is U-H-R stability.

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