



Does Algebra Curriculum Develop Cognitively Advanced Critical Minds? Comparative Analysis of Türkiye and the U.S.A Mathematics Textbooks

Cebir Müfredatı Bilişsel Olarak Gelişmiş Eleştirel Zihinler Geliştiriyor mu?
Türkiye ve ABD Matematik Ders Kitaplarının Karşılaştırmalı Analizi

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Abstract

Türkiye and the United States have the largest mathematics score gaps between bottom- and top-performing eighth-grade students compared to other countries in the 2019 Trends in International Mathematics and Science Study (TIMSS). Yet, the U.S.A. students, on average, have higher mathematics and algebra achievement. To explore this difference, this study aims to assess the alignment of Algebra content, the incorporation of higher cognitive demands and dimensions of mathematical critical thinking (namely, logical thinking, argumentation, deductive reasoning, critique, and critical judgment) in algebraic content in 6th to 8th grade mathematics textbooks in Türkiye and the United States. The analysis was conducted on 219 and 225 solved algebraic examples, respectively. Content analysis method was used to quantify content alignment, cognitive demands; and mathematical critical thinking. The analysis revealed a moderate level of alignment in solved algebraic examples across subtopics in Türkiye and the United States textbooks. Although both sets of textbooks had a high presence of examples with lower-level cognitive demands, the U.S.A. textbooks had a relatively more balanced distribution of cognitive demands across examples and subtopics. Except for logical thinking and deductive reasoning, critical thinking dimensions are underrepresented in both sets of textbooks. The implications of the notable findings are discussed.

Keywords: Algebra, Critical thinking, Cognitive demand, Opportunities to learn, Textbook Analysis

Öz

Türkiye ve Amerika Birleşik Devletleri, 2019 Uluslararası Matematik ve Fen Eğilimleri Araştırması'nda (TIMSS) diğer ülkelere kıyasla en düşük ve en yüksek performans gösteren sekizinci sınıf öğrencileri arasındaki matematik başarı farkının en yüksek olduğu iki ülkedir. Bununla birlikte, ABD'li öğrenciler ortalama olarak daha yüksek matematik ve cebir başarısına sahiptir. Bu farklılığı araştırmak için, bu çalışmada Türkiye ve Amerika Birleşik Devletleri'ndeki 6-8. sınıf matematik ders kitaplarında cebir içeriğinin uyumu, üst seviye bilişsel gereksinimlerin ve matematiksel eleştirel düşünme boyutlarının (yani mantıksal düşünme, argümantasyon, tümdengelimsel akıl yürütme, eleştiri ve eleştirel yargı) dahil edilme seviyesinin değerlendirilmesi amaçlanmaktadır. Analiz sırasıyla 219 ve 225 çözülmüş cebir örnekleri üzerinde gerçekleştirilmiştir. İçerik analizi yöntemi, içerik uyumunu, bilişsel gereksinimleri ve matematiksel eleştirel düşünmeyi nicelleştirerek kullanılmıştır. Analiz, Türkiye ve Amerika Birleşik Devletleri ders kitaplarındaki alt öğrenme alanları arasında çözülmüş cebir örneklerinde orta düzeyde bir uyum olduğunu ortaya koymuştur. Her iki ders kitabında da düşük seviyeli bilişsel gereksinimlere sahip örnekler yüksek oranda bulunmasına rağmen, ABD ders kitaplarında bilişsel gereksinimlerin örnekler ve alt öğrenme alanları arasında nispeten daha dengeli bir dağılımı vardır. Mantıksal düşünme ve tümdengelimsel akıl yürütme dışında, eleştirel düşünme boyutları her iki ders kitabı setinde de yeterince temsil edilmemektedir. Dikkate değer bulguların sonuçları tartışılmıştır.

Anahtar Kelimeler: Cebir, Eleştirel düşünme, Bilişsel gereksinim, Öğrenme fırsatları, Ders kitabı analizi

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Introduction

According to the 2019 Trends in International Mathematics and Science Study (TIMSS) results, the United States (U.S.A.) and Türkiye (formerly, Turkey) had the largest mathematics score gaps between bottom- and top-performing eighth-grade students (i.e., 10th and 90th percentiles, respectively) which is an indication of poor level of equity within these education systems (NCES, 2021). Although both countries have sporadically enacted curricular reforms, they were not ranked within top ten highest achieving countries. Yet, the U.S.A. students had higher proficiency in mathematics and algebra compared to their Turkish counterparts. Moreover, students in the United States had a higher proficiency in algebra (519.85) compared to their mathematics proficiency (515.44); however, students in Türkiye had a lower proficiency in algebra (493.81) compared to their proficiency in mathematics (495.63) (Martin et al., 2020).

Textbooks as a major conveyer of curriculum play a mediating role in the relationship between the intended curriculum (i.e., the aims and goals of the curriculum) and its implementation in the classroom (Schmidt et al., 1997; Tyson & Woodward, 1989). Other researchers also corroborated the primary medium role of textbooks for teachers to implement curriculum goals and standards in many disciplines including mathematics (Park & Leung, 2006; Rezat et al. 2021) and its instructional alignment to curricular goals was purported to serve as an indication of teaching quality (Polikoff & Porter, 2014). In other words, textbooks can be conceptualized as a *nexus of a curriculum*. For example, teachers in Singapore, which consistently ranked first in eighth-grade student mathematics achievement in TIMSS studies, exceedingly relies on textbooks on classroom instructions and homework assignments (Kaur et al., 2006). A study conducted in the United States revealed that teachers' choice of content was influenced by the assigned textbooks 50% of the time in mathematics lessons, and their instructional method was influenced 70% of the time (Weiss et al., 2003). Similarly, in Türkiye, where educational decision-making is excessively centralized with the exception of classroom-level flexibility in teaching methods and pupil assessment (Davutyan et al., 2010), the use of Ministry of National Education-approved mathematics textbooks that reflects the intended curriculum is mandatory. Therefore, to explain a portion of variance in disparity between mathematics and algebra achievement, there is a need to carefully scrutinize the quality of algebraic content in mathematics textbooks.

Empirical evidence regarding effectiveness of mathematics textbooks or their opportunities to learn is scarce. Some comparative analyses showed that the content of mathematics textbooks differ between high and lower achieving countries in terms of afforded opportunities for complex and in-depth learning (Chen, 2024), and problem-solving strategies and conceptual understanding (Kang, 2014). Student opportunity to learn offered by schools and socioeconomic status (SES), which includes equitable textbook content, has a significant association across most countries as well as within and between schools (Flores, 2007; Schmidt et al., 2015). The studies argued that differential content exposure across schools may exacerbate educational inequalities. For students who lack such contextual advantages in learning, well-designed textbooks may serve as a compensatory source of equitable opportunity to learn. Accordingly, to reduce inequity within education systems, the scope and quality of textbook content should be ensured to provide all students with equal opportunities for rigorous higher-level learning (Schmidt et al., 2015).

Algebra is one of the most critical mathematical domains that develop students' abstract thinking, generalization and symbolic operation skills from middle school onwards. Beyond arithmetic and geometry, it has a gatekeeping role for more advanced mathematics and STEM subjects (National Mathematics Advisory Panel, 2008). Students transition from arithmetic into abstract and more advanced mathematical reasoning through algebra (Susac et al., 2014). Through this transition, learners are afforded the opportunity to comprehend the intrinsic nature

of mathematical processes, concepts, and interrelations through a diverse array of mathematical tasks (Stein et al., 2000). While learning of arithmetic and geometry reinforce the student's procedural skills in familiar contexts, algebra tasks require higher levels of cognitive effort pertaining to symbolic processing and making conceptual connections, and additionally, verbal and visuospatial working memory resources are associated with success in algebra (Spiller et al., 2023). The exploration of algebraic concepts in textbooks, therefore, establishes a robust foundation for the potential for enhancement of students' advanced cognitive abilities, problem-solving competencies (The National Council of Teachers of Mathematics, 2000) and critical thinking (Namlı & Özcakir, 2024).

1. Objectives of the Study

1.1. Comparison in Algebraic Content and Cognitive Demand

The empirical evidence suggests that opportunities to learn are a significant predictor of academic achievement (Valverde et al., 2002) and educational equity (Stacey & Chick, 2004), and textbooks are a primary source of students' opportunity to learn (Zhang & Qi, 2019). Even though textbooks are a significant component of enacted curriculum, the content of textbooks may not align sufficiently with the educational goals of the curricula of the countries. Polikoff (2015) using the Survey of Enacted Curriculum (SEC; Center for Curriculum Analysis, 2007) framework found that cognitive demand levels of tasks in mathematics textbook in the United States does not align with the Common Core State Standards for Mathematics (CCSSM) due to an overemphasis on procedures and memorization rather than adhering to the development of meaningful learning emphasized in the standards. In the context of Türkiye, Iskenderoglu and Baki (2011) found that the textbook did not have enough rich content to improve students' higher-level cognitive skills in mathematics. Therefore, the quality of algebraic content in mathematics textbooks can partially explain the striking variation and distribution of algebra achievement across these diverse educational landscapes and provide insights into inherent shortcomings in textbooks that result in unequally distributed opportunities to learn. Thus, the first objective of this study is to examine the alignment of algebra learning domains in the curricula of Türkiye and the United States and the coverage of intended cognitive skill levels within textbooks by utilizing the SEC framework that reflects the hierarchical structure of cognitive demand levels on a continuum.

1.2. Critical Thinking in Algebraic Content

Developing critical thinking is a ubiquitous goal in many countries as a form of higher-order cognitive skills (Aizikovitsh & Amith, 2010). One goal of cultivating critical thinking skills in students is to help them develop cognitive structures, or mental schemas, that allow for objective processing and organization of information based on evidence and associated analysis, rather than "emotion, prejudice, or dogma" (Abrami et al., 2008, p. 1103), and to use these mental schemas in adulthood to benefit society as critical thinking citizens. Although critical thinking significantly predicts mathematics achievement (Firdaus et al., 2015; Üredi & Kösece, 2020), there is little empirical evidence about educational curricula adequately incorporating critical thinking skills in mathematics education (Polikoff, 2015). Likewise, to the author's best knowledge, there is not a research study that specifically analyzed the coverage of critical thinking within algebra content in the major databases (i.e., EBSCO, JSTOR, Web of Science) and the subdomains of mathematical critical thinking in algebra content are rarely examined in these databases. To account for these shortcomings, the second objective of this study is to analyze the potential of these countries' algebraic content to develop mathematical critical thinking skills.

1.3. Distribution of Cognitive Demands and Critical Thinking Across Learning Domains

According to Kieran (2018), achieving proficiency in algebra necessitates significant mindful attention, and cognitive effort ought to raise awareness among teachers and researchers regarding the cognitive demands involved in doing algebra. To acquire higher-order thinking abilities, including critical thinking skills, students must learn mathematics through a balance of lower-order and higher-order cognitive tasks in a scaffolded and balanced manner. For example, a comparative study by Xin (2007) on U.S.A. and Chinese students with learning difficulty showed that unbalanced distribution of word problems within their textbooks was a probable cause for U.S.A. students' lower performances. As instructional content can drastically impact learning outcomes (Benson-O'Connor et al., 2019) along with the design of the textbooks, another purpose of this study is to examine the potential educational effectiveness of the U.S.A. and Türkiye mathematics textbooks by analyzing the distributional pattern of algebraic content with respect to varying levels of cognitive demand to solve algebraic problems. Additionally, the study investigates the distributional pattern of incorporation of critical thinking domains in algebraic content. By determining embedded connections between these facets, this research is intended to provide empirical evidence to mathematics education researchers, curriculum and textbook developers, and policy makers regarding effective curricular decisions in the teaching and learning of algebra that may support equal opportunities to learn through designing balanced and rich algebraic content within mathematics textbooks.

2. Conceptual Framework

Textbooks are instructional materials that promote a distinct pedagogical model for enacting curricular intentions (Valverde et al., 2002). According to psychological and neuroscientific research, cognitive tasks within textbooks that demand a higher level of cognitive processing skills or executive functions, such as flexibility in novel problem-solving, inhibitory control (or executive attention), and utilization of working memory (Blair, 2006) are associated with higher levels of mathematical proficiency, self-regulation and mathematical reasoning (e.g., Espy et al., 2004; Silver & Stein, 1996). Therefore, textbook content systematically embedding higher level of cognitive tasks that offer opportunities that engage students in these tasks can help achieve the aims and goals of implemented curriculum through its significant impact on their reasoning abilities, knowledge acquisition, learning methodologies, and cognitive development (Hadar, 2017). Wakhata et al. (2023) also found that instructions with tasks with varying levels of cognitive demand improved students' average grades in linear programming course. Therefore, instructional materials in algebra should be delivered through a balance of lower-order and higher-order cognitive tasks in a scaffolded manner.

According to Piaget's concepts of assimilation and accommodation, higher-order cognitive problems that require the use of different forms of critical thinking can create perturbation that are balanced by the accommodation of new knowledge through the creation of new mental schemas (von Glasersfeld, 1989). Furthermore, Rabardel's theory of instrument (Laisney & Chatoney, 2018) elucidates the process of instrumental genesis (the transition of a mere artefact—in this context, a mathematics textbook—into a genuine instrument facilitating subject's cognitive activity) through the interrelated mechanisms of instrumentation (where the affordances and constraints of the artefact influence and reorganize the action schemes of students) and instrumentalization (in which students modify and customize the functionalities of the artefact to accommodate their individual learning requirements). Utilizing this theoretical lens, Rezat (2013) discerned that the design components of textbooks—such as the perceptual prominence of worked examples, the arrangement of problem sets, and the integration of

metacognitive question prompts—significantly influenced the manner in which students select and engage with content for self-regulated practice. Provided that textbooks are mostly viewed and used as primary instructional instruments to meet the curricular goals and objectives, as well as self-regulating sources of learning for students, these theoretical perspectives and findings corroborate the proposition that algebraic content in textbooks and its presentation is associated not only with algebra learning and achievement but also with critical thinking.

Critical thinking is a non-linear and repetitive higher-order cognitive process of performing actions through purposeful and evaluative judgments that are contingent on the situation, and forming beliefs based on evaluating the fairness and consequences of practical decisions (Facione et al., 1995; Paul & Elder, 2001). Being a critical thinker requires not only the use of these skills but also a disposition to use these skills, and curriculum content can reinforce subject-specific critical thinking dispositions (Ennis, 2018). Increasing level of critical thinking skills in students is a predictor of higher mathematics achievement at the secondary school level (Firdaus et al., 2015; Üredi & Kösece, 2020) and academic performance at the university level (Aizikovitsh & Amit, 2010; Ren et al., 2020). The curriculum from the Ministry of National Education (MoNE, 2018) in Türkiye emphasizes the development of critical thinking in the curriculum. In contrast, lack of explicit discussion of the critical thinking in the Common Core State Standards for Mathematics (CCSSM; Common Core State Standard Initiative, 2010) may stem from the mathematics education community's perception of critical thinking as an inherent outcome of mathematical learning. Nevertheless, it is erroneous to presume that critical thinking will unfailingly manifest as a mere by-product of implemented curriculum.

Algebra can be conceptualized as the generalization and formulation of relations, the study of structures, the manipulation of symbol arrays representing unknowns, the study of functions, relations, and covariances, and the language of modeling (Bell, 1996; Kaput, 1995). It is often considered a gateway to abstract thought, and the level of abstract thinking required is generally higher than in other areas of mathematics (Mason, 1996). While one of the primary objectives of the mathematics curriculum is to develop abstract algebraic thinking, immersing critical thinking in mathematical topics is an international effort to prepare students with this 21st century skills (e.g., MoNE, 2018; NCTM, 2000). Conceptually, algebraic thinking (or reasoning) and critical thinking are carried out by students with fundamentally different cognitive processes. While algebraic thinking involves *innate* cognitive processes such of pattern recognition, generalization and the use of symbolic representation (Kieran, 2004), critical thinking distinctively involves *reflective* metacognitive strategies including analysis of information and evaluation of arguments (Aizikovitsh & Amit, 2010; Alpindo et al., 2024). Neurocognitively, critical thinking relies heavily on executive functions that are mainly responsible for decision-making and logical reasoning, which are mostly happens in prefrontal cortex (Ku & Ho, 2010) while mathematical cognition occurs in a network of regions, mainly in parietal cortex, and relies heavily on working memory (Gruber et al., 2001). Additionally, mathematical critical thinking, as a specialized form of critical thinking within the discipline of mathematics, amalgamates domain-specific knowledge with cognitive processes that are distinctly adapted to meet the exigencies of mathematical inquiry. For instance, the proficiency in domain-specific skills such as the manipulation of abstract symbols, the comprehension of constructs and concepts along with their interrelations (NCTM, 2000), as well as the systematic approach to problem formulation and solution within the stringent confines of mathematical theories, serves to distinguish mathematical critical thinking skills from generalized and context-neutral critical thinking competencies. Thus, it should be evaluated differently.

2.1 Evaluation of Critical Thinking

In *Encyclopedia of Mathematics Education*, Jablonka (2020) proposes that mathematical critical thinking involves logical thinking, argumentation, deductive reasoning, critique, and critical judgment processes in solving mathematical problems. Although *logic* denotes a structured reasoning form, including formal logic as its most rigid variant (Hanna, 2020), *logical thinking* encompasses the application of concepts, judgments, and conclusions to comprehend reality. It necessitates abstraction and generalization from specific cases to reach overarching principles (Wille, 2009). In the literature, it is mostly addressed indirectly through discussions of critical thinking, as both demand a profound understanding of reality and the capacity for rational predictions and solution discovery through coherent problem identification, strategy formulation, and the application of logical rules. *Argumentation* involves deriving conclusions through reasoning (Umland & Sriraman, 2020) that includes diverse methods of logical debate and various levels of proof and rationale aimed at persuading the validity of one's reasoning to achieve optimal conclusions—sometimes despite limited knowledge (Hanna, 2020). *Deductive reasoning* constitutes a method wherein new knowledge is extracted from an assemblage of premises through a sequence of deductive inferences, culminating in a conclusion that is inferred from the premises, which are posited to have rational justification that the premises necessitate the conclusion (Harel & Weber, 2020). In general, *critique* embodies reflective and questioning thought, necessitating skepticism towards various claims and practices including the examination and questioning of one's own reasoning and practices (Antonacopoulou, 2008). *Critical judgment* refers to the cognitive decision-making abilities to analyze problems and suggested solutions within a broader context, which demands the integration of established facts with novel information pertinent to a given problem (Cheung et al., 2017).

Table 1 provides definitions of the dimensions of *mathematical critical thinking* and related concepts studied in the mathematics education literature, which are adaptations of the definitions in the cited literature into the context of mathematics, and fairly correspond with the California Critical Thinking Scale (CCTS) skills. The common feature of these proposed constructs is that they address different aspects of mathematical critical thinking. Notably, logical thinking is the foundation of other dimensions of critical thinking. Although the dimensions of mathematical critical thinking can be distinguished from each other with slight overlaps, other concepts play distinct and equally important roles in the process of critical thinking. The dimensions are organized sequentially to some extent. While it is desirable to emphasize some of these dimensions in the curriculum, the process of mathematical reasoning that enculturates students into a single type of critical thinking can be perceived as disempowering (Jablonka, 2020). Therefore, the development of critical thinking skills in students requires their development in these fairly comprehensive dimensions of mathematical critical thinking.

Table 1. Definition of mathematical critical thinking dimensions in algebra

Dimensions (Conceptual cues)	Corresponding CCTS skills	Definition
Logical Thinking [LT] (<i>problem analysis, logical reasoning</i>)	Overall reasoning skills	The process of systematically evaluating mathematical problem-solving steps for clarity, coherence, and correctness to correctly derive conclusions
Argumentation [AR] (<i>justification, rationale</i>)	Analysis	The process of justifying the selection of problem-solving strategies and solution steps through a chain of reasoning
Deductive Reasoning [DR] (<i>deductive inference, proof, generalization</i>)	Induction and Deduction	The process of deriving new information and evaluating its significance based on given principles and premises leading to a conclusion
Critique [CR] (<i>analytical thinking, error identification, open-mindedness</i>)	Evaluation	The process of analyzing the effectiveness (good and bad qualities) of solutions, problem-solving strategies and obtained generalizations/rules
Critical Judgment [CJ] (<i>reflection, metacognition, decision-making</i>)	Inference	The process of decision-making for and reflection on solutions to problems in their totality, considering their reasonableness, validity, and transferability to other contexts

2.2 Aims and Goals

The aim of this paper is to conduct a comparative analysis of the algebra content of middle school mathematics textbooks in Türkiye and the United States. By identifying the cognitive demands embedded in solved algebraic examples and the potential of these worked examples to promote mathematical critical thinking, this study aims to investigate the alignment of certain aspects of algebraic content, which is of utmost importance in accounting for some of the variations in differential algebra achievement between Türkiye and the United States. The reason for selecting solved examples, and not the end of unit questions, is its observable role in shaping students' cognitive processes and yielding the evaluation of subdomains of mathematical critical thinking. The scope of the research is limited to the potential opportunities to learn in mathematics textbooks in terms of cognitive and critical thinking skills. However, instructional methods of utilizing textbooks or student outcomes as a result of using different textbooks were not investigated. The research questions to be answered are as follows:

1. To what extent do the algebra learning domains in the curricula of Türkiye and the United States align?
2. To what extent does the opportunity to learn middle school algebra vary in terms of cognitive demand and critical thinking skills embedded in solved algebraic examples?
3. Are there discernible patterns in the distribution of cognitive demand levels and critical thinking dimensions in solved examples across algebra subtopics?

3. Method

This study analyzed solved examples in the algebra learning domain of commonly used middle school mathematics textbooks in Türkiye and the United States. The content was analyzed using the content analysis method to discover content-related differences in the curricula. Additionally, a novel index was used to examine the frequency and distribution of cognitive demand and critical thinking embedded in solved algebra examples. It was deemed ethically appropriate with the letter dated 11.09.2023 and numbered 365035 of Gaziantep University Rectorate Ethics Committee.

3.1 Selection of Textbook and Content

Three mathematics textbooks from each country were examined, including the Free Easy Access Student Edition of 'Big Ideas Math: Modeling Real Life' (BIM) for 6th-8th grades (Larson & Boswell, 2019), which is in alignment with the Common Core State Standards for Mathematics (CCSSM), which have been adopted by 41 states and the District of Columbia. The mathematics textbooks for grades 6-8 in Türkiye that are distributed to schools by MoNE were approved by the Board of Education and Discipline. Although middle school in Türkiye involve 5th grade, for comparative purposes this grade level was not included. Schools in Türkiye are permitted to choose some other publications from private publishers that are also approved by the same board. However, based on personal communication with educators, the use of mathematics textbooks published by MoNE is more common. Both sets of textbooks were prepared by publishers in accordance with the curriculum and standards of the country, including CCSSM and the 2018 curriculum by the Ministry of National Education. Thus, their data should provide representative content in this comparative analysis. Analyses were performed after obtaining ethical approval from the author's institution.

All chapters in each book were examined, and only those related to algebra learning domains were selected. The analysis included a total of 219 solved examples from MoNE textbooks (for sixth, seventh and eighth grades 12, 30 and 177 examples, respectively) and 225 solved examples from BIM textbooks (for sixth, seventh and eighth grades 53, 38 and 134 examples, respectively).

3.2 Analytical Framework

To answer the research questions, the SEC framework (Center for Curriculum Analysis, 2007) was utilized. The SEC assessment framework is widely recognized in the literature, particularly in studies on curriculum alignment (Porter et al., 2011). To compare the alignment of the learning domains of *Basic Algebra* between textbooks from different countries, we used subtopics of Basic Algebra and their corresponding 3-digit codes, as suggested by SEC K-12 Mathematics Taxonomy (Center for Curriculum Analysis, 2007). Each example in the textbook is matched with its corresponding subtopic. A two-dimensional framework is used to represent the intersection of topics and their related cognitive demand categories. Porter et al. (2011) identified five categories of cognitive demand levels: memorization, perform procedures, demonstrate understanding, conjecture/analyze, and solve non-routine problems and make connections.

To measure mathematical critical thinking embedded in algebraic tasks, '*Assessment Framework for the Dimensions of Mathematical Critical Thinking*' (refer to Table 2) was developed based on the definition of mathematical critical thinking dimensions in algebra (See Table 1), which include the multidimensional components of mathematical critical thinking. Each concept was operationalized by 3 operational definitions. To enhance the content validity, this assessment framework was evaluated by three experts—two of them specialized in mathematics education and one specialized in critical thinking. The validity of operational definitions and their compatibility with pre-defined concepts were evaluated by the experts and finalized taking the expert recommendations and opinions into account. Representation of cognitive demand levels and critical thinking dimensions in solved examples were rated on a scale of 0 to 3 (0 = not representative at all and 3 = highly representative). The scoring guidelines were tested on a sample set of examples with an expert in mathematics education in order to refine, adjust and finalize scoring criteria. Given thick descriptions and evaluation of solved examples based on an acclaimed SEC framework provided evidence for credibility; detailed descriptions of assessment framework for coding and scoring criteria as well as expert

opinions provided evidence for dependability; and delineation of descriptions regarding the context and constructs offers readers evidence for transferability.

Table 2. Assessment Framework for the Dimensions of Mathematical Critical Thinking

Dimensions	Operational Definitions
Logical Thinking	<ul style="list-style-type: none"> - Problem-solving stages are given in a clear and coherent step-by-step manner with a precise and unambiguous language - Correctness and coherence of solution steps are evaluated - Cause-and-effect relationships in problem solving procedures are discussed
Argumentation	<ul style="list-style-type: none"> - Explanatory evidence or justification are given for chosen algebraic claims within solutions - Reasons for the choice of problem-solving strategy(ies) are given - The validity of the used algebraic operations is discussed by comparing with alternative concepts and methods
Deductive Reasoning	<ul style="list-style-type: none"> - Estimating and finding new inferences using assumptions and known facts, proofs or formulas - Logical conclusions to complex problems are drawn on the basis of given algebraic principles and premises - Providing additional algebraic examples and explanations to assess the sensibility of obtained algebraic statements or proofs
Critique	<ul style="list-style-type: none"> - Multiple solutions to an algebraic problem are given and feedback on their results is provided - The effectiveness of used problem-solving techniques is discussed with respect to its broader applicability - In-depth analysis of algebraic reasoning and proof are given, including addressing potential errors to avoid mistakes
Critical Judgment	<ul style="list-style-type: none"> - Challenging algebraic problems that demonstrate informed decision-making are presented - The reasonableness and validity of assumptions and solutions to algebraic problems (e.g., communicating precision of solutions, proper notation, and clarification of confusing steps or statements) are evaluated - Reflections on the implications of algebraic results (such as applicability of the model or procedure to real-life situation, relevancy of given information, factual or value judgments, and bias) are given

Using this assessment framework in solved examples, the study utilized a two-dimensional scoring framework to map the intersection of sub-topics defined in the SEC framework and critical thinking concepts. The overall proportions for each cognitive demand level and the dimensions of critical thinking were calculated using these values. A procedure, similar to that of Porter et al. (2011), was followed to calculate the alignment index.

Although the number of solved algebra examples in the U.S. and Türkiye textbooks is similar, their distribution among subtopics differs significantly (explore the frequencies in Table 5). To visually examine the coverage of each subtopic, as well as the cognitive demand levels and critical thinking dimensions embedded within its examples, a weighted proportion formula called the 'Normalized Proportion Index' (NPI) was developed:

$$NPI_j = \frac{\sum_{i=1}^n x_{ij}}{3 * N}$$

where j represents a particular subtopic, i is the i -th item within a subtopic, x_{ij} is the rating of the i -th item in subtopic j , n is the number of items within the subtopic j , and N is the total number of rated examples within a textbook. The denominator is multiplied by 3 to find the maximum possible sum for the total rating of cognitive demand levels (since each item is rated

once for cognitive demand levels, there is no need to multiply by 15). To compare and illustrate the distribution of cognitive demand levels and critical thinking dimensions across algebra subtopics, this formula is used instead of calculating raw proportions. The formula divides the sum of ratings within each subtopic by the maximum possible sum of ratings. This provides a normalized proportion that allows for the comparison of the cells at the intersection of subtopics and cognitive demand levels for each textbook. The index values were manually calculated by determining the frequency of cognitive demand levels and critical thinking dimensions for each subtopic using the Stata software.

3.2.1 Sample Coding Decisions

In Table 3, sample coding decisions are given for measuring levels of perform procedure and logical thinking using similar examples. These levels of cognitive demand and critical thinking were selected because, as is reported in the Results section, they had the highest frequency within the algebra content of the textbooks.

Table 3. Sample coding decisions for perform procedure and logical thinking levels

	Perform Procedure - Level 1 & Logical Thinking - Level 1	Perform Procedure - Level 2 & Logical Thinking - Level 2	Perform Procedure - Level 3 & Logical Thinking - Level 3
Examples	Solve the equation $-4x = -20$. <u>Answer:</u> $\frac{-4x}{-4} = \frac{-20}{-4}$ $x = 5$	Solve the equation $-4x+8 = -12$. <u>Answer:</u> Write the equation: $-4x + 8 = -12$ Subtraction property of equality: $-4x+8 - 8 = -12-8$ $-4x = -20$ Division property of equality: $\frac{-4x}{-4} = \frac{-20}{-4}$ We find that $x=5$	Solve the equation $-4x+8 = -12$. <u>Answer:</u> Write the equation: $-4x + 8 = -12$ Subtraction property of equality: $-4x+8 - 8 = -12-8$ $-4x = -20$ Division property of equality: $\frac{-4x}{-4} = \frac{-20}{-4}$ We find that $x=5$ So, the solution is $x=5$. Check: $-4*(5)+8 = -20+8 = -12$. The solution is correct for $x=5$
Subtopics	One-step equations (504)	Multi-step equations (507)	Multi-step equations (507)

In the first example, because only a one-step solution was used in the solution, this sample example is coded as ‘1’ under the perform procedure level. In the second example, the application of a multiple-step solution (i.e., subtraction property, and then, division) is coded as ‘2’, indicating a higher level of cognitive demand within the perform procedure level. In the third example, both the application of the multiple-step solution and the checking of the solution (i.e., justifying the correctness of the equation for $x=5$) resulted in a coding of ‘3’, representing an example solution that embeds the highest cognitive demand within this level. Regarding the mathematical critical thinking dimension of logical thinking, the first example is coded ‘1’ because among the coding decision options—i.e., i) step-by-step solution, ii) algebraic operations used in each step are given with clear reasons and terms/properties, and iii) cause-and-effect relationship or checking solutions for consistency is given, only step-by-step solution is present in the solution. Therefore, this example receives a score of 1 for the levels of logical thinking. In the second example, the solution includes the first two options because, in addition to the step-by-step solution, the names of the applied algebraic properties are given to support clearer and more coherent logical thinking. The third example is scored ‘3’ because not only

does it meet the first two criteria, but also the verification of the solution contributes to a higher level of critical thinking in terms of its logical thinking dimension.

4. Results

The study analyzed 225 solved algebra examples in the BIM's 6th-8th grade textbooks and 219 solved algebra examples in the MoNE's 6th-8th grade mathematics textbooks. The SEC framework was used to classify each selected algebraic example into an algebraic subtopic using the content analysis method, and then a two-dimensional framework was used to quantify the levels of cognitive demand and dimensions of critical thinking embedded in each of the 'Basic Algebra' subtopics.

4.1 Congruity of Algebra Learning Domains in Türkiye and the U.S.A. Textbooks

The algebra examples in each country's textbook were matched with their corresponding subtopics in the SEC framework. The alignment between the algebraic contents of Türkiye and the U.S. was computed using Porter's (2002) 'alignment index' (i.e., the sum of the difference between each country's proportions of each subtopics is divided by 2, and this value was subtracted from 1). According to Porter (2002), the alignment index is between 0 and 1 and perfect alignment occurs when the index is 1, and the lower values of this index indicate divergence from alignment. The alignment index for algebra learning domains in the U.S. and Turkish textbooks was 0.66. Therefore, the algebra learning domains in Turkish and U.S. mathematics textbooks are moderately aligned.

4.2 Cognitive Demand Levels in Algebra Content

First, Table 4 presents the frequencies of cognitive demand levels for solved examples, categorized by their ratings as well as the total frequencies. In codifying the cognitive demand levels, the highest level of cognitive demand of the examples was coded. For instance, if an example demonstrated both 'memorization' and 'perform procedure' levels, it was coded for 'perform procedure' only.

Table 4. Comparative frequencies of highest cognitive demand levels

Cognitive Demand Levels	US-BIM					TR-MoNE				
	<u>M</u>	<u>P</u>	<u>D</u>	<u>C</u>	<u>N</u>	<u>M</u>	<u>P</u>	<u>D</u>	<u>C</u>	<u>N</u>
1	4.4%	5.3%	2.2%	6.7%	4.9%	4.6%	41.1%	4.1%	3.2%	8.2%
2	6.2%	16.0%	11.6%	6.2%	17.8%	3.2%	12.3%	14.2%	0.9%	3.7%
3	0.9%	8.4%	0.9%	1.3%	7.1%	0%	0.9%	0.9%	0%	3.2%
TOTAL	11.5%	29.7%	14.7%	14.2%	29.4%	7.8%	54.3%	19.2%	4.1%	15.1%

Note. M: Memorize; P: Perform procedures; D: Demonstrate/communicate understanding; C: Conjecture and analyze; N: Novel problems and connections

According to the statistics, the MoNE textbooks have a higher representation of the cognitive demand levels 'perform procedure' and 'demonstrate/communicate understanding'. Personal observation during the analyses showed that most of solved examples at 'perform procedure' level repeatedly used similar problem-solving strategies. Upon examination of the ratings, it was evident that a significant majority (54.3%) of the solved examples in the MoNE textbooks were designed to facilitate student cognition in algebra at the 'perform procedure' level, with most of them being rated '1'. Additionally, both textbooks had a low frequency of rating values for their respective cognitive demand levels.

A block of solved examples (perform procedures level) are given in Figure 1.

Birlikte Yapalım 9
Aşağıdaki üslü ifadelerin çarpımına ilişkin işlemleri yapalım.

$$2^3 \cdot 2^5 = 2^{3+5} = 2^8$$
$$12^{75} \cdot 12^{50} = 12^{75+50} = 12^{125}$$
$$5^7 \cdot 5^6 \cdot 5^5 \cdot 5^4 = 5^{7+6+5+4} = 5^{22}$$
$$3^0 \cdot 3^3 = 3^{0+3} = 3^3$$

Figure 1. A sample of examples from the 8th grade MoNE textbook (pp. 24). [Reproduced for educational research purposes under Article 34 of Turkish Copyright Law (5846 Fikir ve Sanat Eserleri Kanunu)]

Figure 2 shows that the BIM textbooks had a higher normalized proportion of cognitive demand levels—except for ‘perform procedure’ and ‘demonstrate/communicate understanding’. Additionally, the distribution of cognitive demands embedded in solved examples in the BIM textbooks is more balanced than in the MoNE textbooks. The latter mostly embedded solved examples with lower levels of cognitive demand, which also received lower ratings even in those levels.

However, the BIM textbooks offer proportionally more tasks that are designed to promote higher level cognitive learning (such as the use of multiple representations and alternative methods/checking the solution, and the inclusion of modeling problems in the subsection of each chapter; See Figure 3). An ideal textbook design should have an effective scaffolding strategy that shows a gradual increase in NPIs from ‘memorize’ to ‘novel problems/connections.’ The majority of solved examples is expected to cluster around higher levels of cognitive demand. Although the BIM textbooks fit better to this scenario, both textbooks have a large number of solved examples at the ‘perform procedure’ level and a relatively low proportion at the ‘conjecture/analyze’ level.

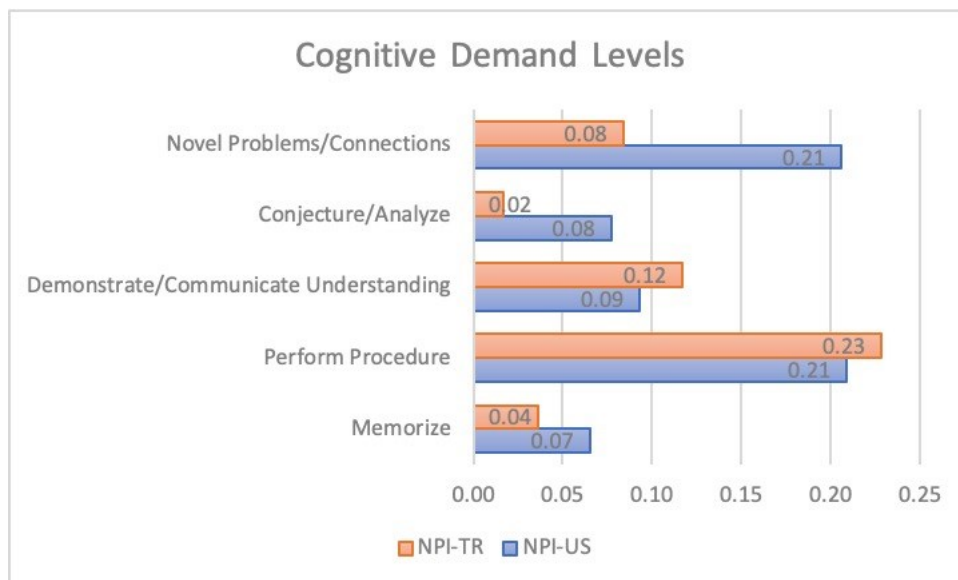


Figure 2. NPI values for cognitive demand levels in algebra content in the U.S. (BIM) and Türkiye (MoNE)

EXAMPLE 3 Modeling Real Life

Skateboard kits cost d dollars and you have a coupon for \$2 off each one you buy. After assembly, you sell each skateboard for $(2d - 4)$ dollars. Find and interpret your profit on each skateboard sold.

Understand the problem. You are given information about purchasing skateboard kits and selling the assembled skateboards. You are asked to find and interpret the profit made on each skateboard sold.

Make a plan. Find the difference of the expressions representing the selling price and the purchase price. Then simplify and interpret the expression.

Solve and check. You receive \$2 off of d dollars, so you pay $(d - 2)$ dollars for each kit.

Profit (dollars)	=	Selling price (dollars)	-	Purchase price (dollars)	
		$= (2d - 4)$		$- (d - 2)$	Write the difference.
		$= (2d - 4) + (-d + 2)$			Add the opposite.
		$= 2d - d - 4 + 2$			Group like terms.
		$= d - 2$			Combine like terms.

▶ Your profit on each skateboard sold is $(d - 2)$ dollars. You pay $(d - 2)$ dollars for each kit, so you are doubling your money.

Look Back Assume each kit is \$40. Verify that you double your money.
 When $d = 40$: You pay $d - 2 = 40 - 2 = \$38$.
 You sell it for $2d - 4 = 2(40) - 4 = 80 - 4 = \76 .
 Because $\$38 \cdot 2 = \76 , you double your money. ✓




Figure 3. A sample modeling example from the 7th grade BIM textbook (pp. 100). [Reproduced for scholarly purposes under the Fair Use provision of U.S. Copyright Law (17 U.S. Code § 107)]

A deeper level of scrutiny is provided in Table 5 to corroborate the previous findings. In this table, NPIs for cognitive demand levels for each algebra subtopics is given with color-coded visualization. While the distribution of NPIs for cognitive demand levels of solved examples in the BIM is more dispersed both horizontally and vertically (by cognitive demand levels and subtopics, respectively), these values are clustered around a few subtopics and lower cognitive demand levels in the MoNE textbooks. In the BIM textbooks, highest level of cognitive demand (i.e., novel problems/connections) is mostly represented in subtopics of ‘linear and nonlinear relations’, ‘rate of change/slope/ratio’, ‘one-step equations’ and ‘exponentials’, while in MoNE textbooks, this level is mostly represented only in the subtopic of ‘rate of change/slope/ratio’.

Finally, to determine which subtopics were designed to require higher level cognitive skills, their NPIs were summed for both countries' textbooks and a weighted composite score was calculated by dividing the sum of the domain NPIs by the number of solved examples. The BIM textbooks had the highest weighted composite NPIs in ‘one-step equations,’ ‘multi-step equations,’ and ‘factoring,’ and the lowest values in ‘absolute value,’ ‘coordinate planes,’ and ‘square roots and radicals’ (There was no chapter devoted to the subtopics of ‘patterns’ and ‘rational expressions’). The MoNE textbooks had the highest weighted composite NPIs in ‘patterns,’ ‘linear and nonlinear relations,’ and ‘rate of change/slope/line’ and the lowest values in ‘operations on radicals,’ ‘rational expressions,’ ‘exponents,’ and ‘factoring’ (there was no chapter section focused on ‘absolute value’).

Table 5. Distribution of NPIs for cognitive demand levels among algebra subtopics

	f	BIM (the U.S.)					MoNE (TR)					f
		M	P	D	C	N	M	P	D	C	N	
501	3	0.003	0.000	0.000	0.000	0.003	0.000	0.000	0.000	0.000	0.000	0
502	8	0.016	0.000	0.003	0.000	0.003	0.015	0.003	0.000	0.000	0.011	15
503	17	0.000	0.016	0.000	0.018	0.012	0.000	0.002	0.000	0.000	0.006	3
504	20	0.000	0.040	0.000	0.010	0.024	0.000	0.018	0.003	0.000	0.008	12
505	22	0.003	0.013	0.024	0.000	0.013	0.006	0.000	0.023	0.000	0.002	11
506	0	0.000	0.000	0.000	0.000	0.000	0.000	0.006	0.000	0.006	0.008	6
507	13	0.000	0.031	0.000	0.003	0.010	0.000	0.017	0.000	0.000	0.002	8
508	21	0.003	0.012	0.028	0.000	0.013	0.009	0.005	0.021	0.000	0.003	19
509	39	0.018	0.007	0.019	0.028	0.041	0.000	0.003	0.012	0.000	0.000	5
510	19	0.000	0.004	0.007	0.013	0.031	0.002	0.012	0.006	0.000	0.032	16
511	17	0.000	0.027	0.000	0.003	0.019	0.000	0.027	0.033	0.000	0.006	26
512	7	0.000	0.009	0.006	0.000	0.009	0.000	0.030	0.008	0.006	0.000	25
513	8	0.010	0.003	0.006	0.000	0.000	0.003	0.035	0.009	0.005	0.002	21
514	5	0.000	0.006	0.000	0.001	0.006	0.000	0.027	0.000	0.000	0.000	18
515	0	0.000	0.000	0.000	0.000	0.000	0.000	0.003	0.000	0.000	0.000	2
590	26	0.012	0.040	0.000	0.000	0.021	0.002	0.040	0.002	0.000	0.006	32

Note. 501-Absolute value; 502-Use of variables; 503-Evaluation of formulas, expressions, and equations; 504-One-step equations; 505-Coordinate planes; 506-Patterns; 507-Multi-step equations; 508-Inequalities; 509-Linear and non-linear relations; 510-Rate of change/slope/line; 511-Operations on polynomials; 512-Factoring; 513-Square roots and radicals; 514-Operations on Radicals; 515-Rational Expressions; 590-Others (Exponents)

4.3 Critical Thinking Dimensions in Algebra Content

The frequencies of cognitive demand levels presented in Table 4 sum up to 100 for each textbook. However, since multiple critical thinking dimensions were evaluated for solved examples (namely, for each solved example, the dimensions were rated multiple times when necessary), each dimension will be analyzed individually. Table 6 shows that the MoNE textbooks have lower frequencies as well as lower ratings in each critical thinking dimensions. Moreover, both textbooks have lower frequencies in ‘critique’ and ‘critical judgment’ dimensions while they have higher frequencies in ‘logical thinking’ and ‘deductive reasoning’ dimensions. Nevertheless, both of them have strikingly low degree of high ratings in each dimension.

Table 6. Comparative frequencies of critical thinking dimensions

Critical Thinking	US-BIM					TR-MoNE				
	LT	A	DR	Cr	CJ	LT	A	DR	Cr	CJ
1	26.7%	42.2%	48.4%	34.7%	32.0%	61.2%	28.8%	45.6%	17.8%	22.9%
2	46.2%	13.8%	19.1%	6.7%	4.9%	28.3%	4.1%	12.8%	0.9%	0%
3	20.9%	0.4%	8.9%	0%	0%	5.5%	0%	0.5%	0%	0%

Note. LT: Logical thinking; A: Argumentation; DR: Deductive reasoning; Cr: Critique; CJ: Critical judgment

In Figure 4, NPI values for critical thinking dimensions in algebra content of the BIM and the MoNE textbooks are given. There is no strict hierarchical order among these conceptual dimensions, and there is slight overlap between these concepts. However, it can be conveniently conceptualized that ‘logical thinking’ represents an overall systematic evaluation of solution

steps in terms of clarity, coherence, and step-by-step solutions that provides the foundation for other dimensions; ‘argumentation’ and ‘deductive reasoning’ focus on problem-solving strategies, rationales for selecting operations and strategies, and their relationships to given premises; and ‘critique’ and ‘critical judgment’ focus on the end-products (i.e., alternative solutions or methods, checking solutions or retrospective error diagnosis, and general conclusion or transferability of the solution method to other problem contexts). For interpretation purposes, it can be suggested that these categories roughly correspond to the *integral logic, reasoned action, and judgment* aspects of mathematical critical thinking, respectively. Using this practical hierarchical ordering, Figure 4 shows that while solved examples contain a sufficient level of ‘integral logic’, a moderate-to-low degree of ‘reasoned action’, and a very low degree of ‘judgment’ aspects of critical thinking in both sets of textbooks (the MoNE textbooks have considerably lower values in these categories). In other words, solved examples do not fully incorporate higher-level critical thinking skills that are necessary for students to apply the skill in real-life situations.

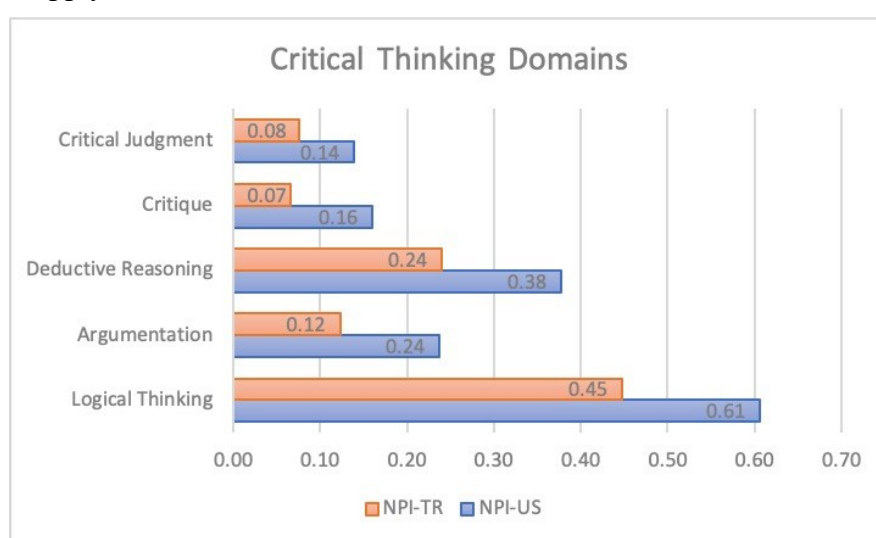


Figure 4. NPI values for critical thinking dimensions in algebra content in the U.S and Türkiye

Table 7 presents the distribution of critical thinking dimensions in solved algebraic examples. The visualization is color-coded and supports the previous inferences. The distribution of NPIs for critical thinking dimensions in the BIM textbooks is more spread vertically, but their horizontal distribution indicates a fairly similar pattern, although to a lesser degree. NPI values are mostly clustered in logical thinking and deductive reasoning dimensions. To further examine which subtopics had the potential to sufficiently improve critical thinking as a whole, a weighted composite score was calculated as was done for the cognitive domains. The BIM textbooks had the highest weighted composite NPIs in ‘multi-step equations,’ ‘rate of change/slope/line,’ and ‘operations on radicals’, and the lowest values in ‘use of variables,’ ‘coordinate planes,’ and ‘square roots and radicals’ (there was no chapter section focused on the subtopics of ‘patterns’ and ‘rational expressions’). The MoNE textbooks had the highest weighted composite NPIs in ‘linear and nonlinear relations’, but the lowest values in ‘use of variables’, ‘evaluations of formulas, expressions, and equations’, ‘inequalities’, ‘operations on radicals’, ‘rational expressions’, and ‘exponents’. There was no chapter section that focused on ‘absolute value’.

Table 7. *Distribution of critical thinking dimensions among algebra subtopics*

	f	US_BIM					TR_MoNE					f
		LT	A	DR	Cr	CJ	LT	A	DR	Cr	CJ	
501	3	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
502	8	0.007	0.001	0.003	0.001	0.003	0.018	0.002	0.006	0.000	0.006	15
503	17	0.050	0.025	0.033	0.012	0.012	0.005	0.000	0.000	0.000	0.002	3
504	20	0.056	0.012	0.041	0.019	0.006	0.032	0.012	0.023	0.009	0.002	12
505	22	0.041	0.012	0.010	0.015	0.010	0.021	0.005	0.015	0.005	0.003	11
506	0	0.000	0.000	0.000	0.000	0.000	0.018	0.005	0.015	0.000	0.003	6
507	13	0.046	0.019	0.033	0.009	0.010	0.023	0.005	0.015	0.003	0.005	8
508	21	0.055	0.004	0.027	0.013	0.010	0.029	0.002	0.023	0.005	0.006	19
509	39	0.107	0.059	0.087	0.039	0.025	0.015	0.006	0.011	0.009	0.003	5
510	19	0.061	0.028	0.036	0.021	0.021	0.049	0.020	0.024	0.005	0.009	16
511	17	0.047	0.009	0.034	0.012	0.016	0.059	0.021	0.032	0.009	0.011	26
512	7	0.019	0.010	0.019	0.000	0.006	0.038	0.012	0.038	0.011	0.005	25
513	8	0.013	0.003	0.006	0.004	0.004	0.050	0.027	0.011	0.005	0.005	21
514	5	0.018	0.009	0.010	0.004	0.003	0.037	0.006	0.008	0.005	0.006	18
515	0	0.000	0.000	0.000	0.000	0.000	0.003	0.000	0.003	0.000	0.000	2
590	26	0.081	0.044	0.039	0.010	0.012	0.050	0.002	0.017	0.002	0.012	32

Note. For subtopic codes (e.g., 501), refer to the descriptions in Table 5

5. Discussion

The first objective of this study was to determine the alignment between the emphasis given to algebra learning domains in middle school mathematics textbooks of Türkiye and the United States. The results of the study revealed a moderate level of alignment in the solved algebra examples between the textbooks of Türkiye and the United States. International comparisons suggest that mathematics curricula are moving toward general harmonization in terms of the number of topics covered (Johansson & Hansen, 2019; Valverde et al., 2002), influenced by a growing awareness of international educational practices (Romberg, 1997). Therefore, the observed level of alignment (0.66) is indeed indicative of an inadequate level of overlap between the textbooks. In addition to the moderate variation in the number of topics across countries, the emphasis on topics in high school mathematics textbooks has been found to differ between the United States and several countries (Wang & Lu, 2018). In a competitive and globalized world, it is important to minimize not only quantitative but also qualitative differences in the algebraic content of national curricula and textbooks, which this study reports.

The second objective of this study was to identify levels of cognitive demand and predetermined critical thinking dimensions embedded in solved algebraic examples. In terms of cognitive demand, the textbooks of both countries did not have a sufficient presence of solved examples requiring a higher level of cognitive demand, highlighting the need for further efforts to advance learning opportunities for low-achieving students in order to promote equity. The MoNE textbooks had not only a significantly higher proportion of solved examples with lower levels of cognitive demand (specifically, at the ‘perform procedure’ level), but also a higher frequency of lower ratings at these levels. Previous studies found that low-demand tasks were prevalent in eighth-grade Turkish mathematics textbooks (Polat & Dede, 2023) and middle school teachers generally prefer to teach mathematics tasks that have low cognitive demands (Ubuz & Sarpkaya, 2020). Similar complaints can be found in the U.S. context. Polikoff (2015),

for example, reports that U.S. textbooks are not aligned with the goals of the CCSSM and place more emphasis on procedures and memorization. Research on U.S. curricula and textbooks suggests that these resources overemphasize procedures and algorithms at the expense of conceptual learning (e.g., Jacob, 2001; Porter et al., 2011). Ideally, mathematics textbooks should hierarchically incorporate a higher proportion of scaffolded examples that have the potential to develop higher levels of cognitive demand.

Although the critical thinking dimensions were generally applied to a reasonable number of solved examples, most of them had the lowest score (i.e., 1 out of 3) in terms of sophistication. The logical thinking and deductive reasoning dimensions were fairly well represented in the examples. By their definitions, the fact that they both inherently involve some properties of mathematics may partly account for this. However, argumentation, critique, and critical judgment were used in a small proportion of the solved examples. Moreover, the examples in which these skills were embedded had a very low rating of 2, and their rating of 3 was very rare. In other words, most examples in MoNE and BIM textbooks have a superficial presence of higher level of these critical thinking dimensions in the solved algebra examples. Not only can the incorporation of tasks that require higher cognitive effort through novel non-routine problems facilitate critical thinking skills in a perturbation and equilibrium mechanism (von Glasersfeld, 1989), the incorporation of these dimensions into algebraic tasks may be more effective in helping students critically apply algebraic knowledge to real-world scenarios. Although critical thinking skills have been enthusiastically promoted internationally and are considered one of the foundations of 21st century skills (MoNE, 2018; NCTM, 2000), selected mathematics textbooks have been found to fall short of this goal by underrepresenting critical thinking dimensions in algebraic content.

Erbağcı and Kaf (2020) found a weak coverage of the application of critical thinking in Turkish elementary mathematics textbooks. Similarly, Bieda et al. (2014) reported a very low presence of mathematical argumentation in upper elementary school textbooks. Similar findings have been reported, such as the inadequate coverage of argumentation in Croatian middle school mathematics textbooks (Glasnovic Gracin, 2018) or the shallow incorporation of deductive reasoning in Australian eighth grade mathematics textbooks (Vincent & Stacey, 2008). To improve critical thinking, several approaches have been proposed that can be applied to algebraic content in mathematics textbooks. For example, positive attitudes toward science, technology, engineering, arts, and mathematics (STEAM) have been found to enhance students' critical thinking and dispositions, further improving their decision-making skills and academic performance (e.g., Aydın-Gürler & Kaplan, 2023; Ren et al., 2020; Üredi & Kösece, 2020). Greater adaptation of STEAM-related content, in solved examples can help students develop critical thinking through algebraic content. The BIM textbooks had a small section in each chapter devoted to making connections to STEAM disciplines; however, the MoNE textbooks did not have any distinct section of solved examples on STEAM disciplines. In addition, the discussion of incorrectly solved problems was found to develop students' critical thinking skills (Star et al., 2015). This method corresponds to the 'critique' dimension in this study. Because both sets of textbooks lacked sufficient levels of higher levels of critical thinking domains (specifically, critique and critical judgment), incorporation of these domains into mathematics textbooks should be given more attention.

The third objective of this study was to examine discernible patterns in the distribution of cognitive demand levels and critical thinking dimensions across algebra subtopics. Based on Piaget's constructivist perspective, "cognitive change and learning take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, leads to accommodation that establishes a new equilibrium" (von Glasersfeld, 1989, p. 128). In other words, for novel learnings to occur, algebraic tasks with higher cognitive demand can facilitate

critical thinking through perturbation-and-equilibrium mechanism. However, the visual map of the distribution of cognitive demands across subtopics showed that the majority of the solved algebra examples in the MoNE textbooks were at the ‘perform procedure’ level, especially on the subtopics ‘square root and radicals’ and ‘exponents’. The emphasis on procedures together with the very limited coverage of real-world connections and multiple representations in algebraic tasks should naturally force students to memorize these learning domains in these subtopics. A similar conclusion can be drawn for the BIM textbooks in the learning domains of ‘single-step equations’, ‘multi-step equations’, and ‘exponents’. In conclusion, the lack of use of a variety of problem-solving strategies and methods deprives students of conceptual learning through accommodation.

Although each of the critical thinking dimensions has a higher proportional representation in the BIM textbooks, their distribution is quite similar to that in the MoNE textbooks. In both sets of textbooks, logical thinking and deductive reasoning are embedded in many of the subtopics; however, other critical thinking dimensions are less embedded. Considering that logical thinking and deductive reasoning are closely related to mathematical thinking, the results are not surprising. The BIM textbooks compensate for the presence of a higher frequency of solved examples with a low level of cognitive demand in the subtopics ‘exponents’ and ‘operations on polynomials’ by embedding a higher proportion of logical thinking and deductive reasoning in them. However, the same conclusion cannot be drawn for ‘one-step equations’. Similarly, the MoNE textbooks compensate for the presence of a higher frequency of solved examples that offer a low level of cognitive demand in the subtopics ‘operations on polynomials’ and ‘exponents’, but the same shortcoming is not compensated for the subtopics ‘factoring’, ‘square root and radicals’ and ‘operations on radicals’. It should be reiterated that the inclusion of argumentation, critique, and critical judgment in solved algebraic examples is inadequate. Thus, ongoing reform efforts aimed at preparing students with the necessary 21st century skills should pay more attention to the quality of algebra content in curricula and take the necessary steps to develop more effective mathematics textbooks that include a higher proportion of high-level cognitive demand and critical thinking dimensions.

5.1 Limitations

In this study, the textbooks were analyzed using the content analysis method, which involves qualitative coding accompanied by quantification procedures. The scoring and interpretation of the data involves a degree of subjective interpretation of the author. However, evidence for credibility, dependability and transferability were provided. In addition, although the operational definitions given in Table 2 may not fully capture all aspects of the corresponding mathematical critical thinking dimensions, they were found to be sufficient in determining to match solved examples to their corresponding dimensions.

6. Concluding Remarks

This study primarily aimed to operationalize mathematical critical thinking within algebra content as mathematics education community still lacks a consensus thereof. As one of the most pronounced 21st century skills, establishing the theorization of critical thinking in mathematical strands is essential. More in-depth research is needed to uncover the theoretical underpinnings of argumentation, critique, and critical judgment skills and the effectiveness on the transferability of these skills to real-world contexts. Cross-cultural findings indicated that both countries lack sufficient proportions of higher levels cognitive demand and critical thinking dimensions within solved algebra examples. However, a relatively better coverage of cognitive demands and mathematical critical thinking dimensions in the U.S.A. textbooks offer some explanation for the differences in algebra achievement. Because classroom instructions are a function of textbook quality, textbook developers should include more opportunities to

learn to foster equity within education systems. In conclusion, curriculum developers should be encouraged to make data- and theory-driven decisions about the design of algebraic content in mathematics textbooks in order to prepare cognitively advanced critical thinkers by avoiding superficial changes in textbooks.

Değerlendirme/Evaluation:

Çift Taraflı Kör Hakemlik Sistemi
Double Blind Refereeing System

Etik Kurul İzni/ Ethics Committee Permission:

Gaziantep Üniversitesi Rektörlüğü Etik Kurulu'nun 11.09.2023 tarih ve 365035 sayılı yazısı ile etik açıdan uygun görülmüştür.

It was deemed ethically appropriate with the letter dated 11.09.2023 and numbered 365035 of Gaziantep University Rectorate Ethics Committee.

Etik Beyanı/ Ethical Statement:

Bu çalışmanın hazırlanma sürecinde bilimsel ve etik ilkelere uyulduğu ve yararlanılan tüm çalışmaların kaynakçada belirtildiği beyan olunur.

It is hereby declared that scientific and ethical principles were adhered to during the preparation of this study and that all studies used as references are listed in the bibliography.

Etik Bildirim/Ethics Notification: info@esosder.org

Çıkar Çatışması Beyanı/ Declaration of conflict of interest:

Bu çalışmada herhangi bir potansiyel çıkar çatışması bulunmamaktadır.

There is no potential conflict of interest in this study.

Yapay Zekâ Kullanımı /Use of Artificial Intelligence:

Bu çalışmanın hazırlanma sürecinde yapay zekâ tabanlı herhangi bir araç veya uygulama kullanılmamıştır. Çalışmanın tüm içeriği, yazar(lar) tarafından bilimsel araştırma yöntemleri ve akademik etik ilkelere uygun şekilde üretilmiştir.

No artificial intelligence-based tools or applications were used in the preparation of this study. All content of the study was produced by the author(s) in accordance with scientific research methods and academic ethical principles.

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