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# THE HIGHER DIMENSIONAL WHEELER-DEWITT EQUATION AND WAVE FUNCTION OF THE UNIVERSE

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**ABSTRACT**

The higher dimensional Arnowitt-Deser-Misner (ADM) formalism is briefly explained. The higher dimensional Wheeler-DeWitt equation is obtained for the ten dimensional effective string theory ground state manifold which has the $M_4 \times C_6$ form, where $M_4$ denotes the actual 4-dimensional spacetime and $C_6$ denotes the 6-dimensional compact internal space. The $M_4$ spacetime manifold is the four dimensional sphere in the Euclidean region or in the quantum gravity region, whereas, the four dimensional hypersphere in the Lorentzian region or in the very early universe region. The $C_6$ space manifold is the six dimensional Euclidean sphere in every region. The Wheeler-deWitt equation is solved by using of suitable transformations of the relevant metrics scale factors and the standart methods of the second quantized field theory methods, i.e., using of normal ordering rules of the field operators. The solution wave function of the universe has the exponantial varying character in the Euclidean region, whereas, the oscillatory character in the Lorentzian region. Furthermore, the wave function has the harmonic oscillator ground state wave function form for the small values of the relevant metrics scale factors.

**Key words:** *The higher dimensional ADM formalism, the Euclidean and Lorentzian regions, the higher dimensional Wheeler-deWitt equation.*

# YÜKSEK BOYUTLU WHEELER-DEWITT DENKLEMİ VE EVRENİN DALGA FONKSİYONU

**ÖZET**

Yüksek boyutlu Arnowitt-Deser-Misner (ADM) formalizmi kısaca anlatılır. Yüksek boyutlu Wheeler-deWitt denklemi, $M_4 \times C_6$ formuna sahip olan on boyutlu etkin sicim alan kuramı taban durum manifoldu için elde edilir, burada $M_4$ gerçek 4 boyutlu uzayzamanı ve $C_6$ 6 boyutlu kompakt iç uzayı gösterir. $M_4$ uzayzaman manifoldu Euclidyen bölgede veya kuantum kütleçekimi bölgesinde dört boyutlu küre, oysa, Lorentzyen veya erken evren bölgesinde dört boyutlu hiperküredir. $C_6$ uzay manifoldu her bölgede altı boyutlu Euclidyen küredir. Wheeler-deWitt denklemi ilgili metriklerin ölçek çarpanlarının uygun dönüşümlerini kullanarak ve ikinci kuantumlanmış alan kuramının standart yöntemlerini yani alan operatörlerinin normal sıralama kurallarını kullanarak çözülür. Evrenin çözüm dalga fonksiyonu, Euclidyen bölgede üstel değişen karaktere, Lorentzian bölgede ise salınımlı karaktere sahiptir. Ayrıca, dalga fonksiyonu ilgili metrik ölçek çarpanlarının küçük değerleri için harmonik salınıcı taban durum dalga fonksiyonu formuna sahiptir.

**Anahtar Kelimeler:** *Yüksek boyutlu ADM formalizmi, Euclidyen ve Lorentzyen bölgeler, yüksek boyutlu Wheeler-deWitt denklemi.*

## 1. INTRODUCTION

The four dimensional Wheeler-deWitt equation is the fundamental equation of the four dimensional quantum gravity. But, the four dimensional quantum gravity theories have many problems, for instance, the huge cosmological constant, the renormalization difficulties, the ambiguties which arises from the statistical symmetry (Supersymmetry) lacking. These problems can be solved in the higher dimensional quantum gravity theories framework, for example Superstring Theory. In this paper, the higher dimensional Wheeler-deWitt equation is considered and solved [1,3,4].

**2. THE HIGHER DIMENSIONAL ADM FORMALISM**

The higher dimensional ADM formalism is a procedure for reducing $I = \int R\,(-G)^{1/2}\, d^{d+1} x$ Einstein-Hilbert action, to canonical form, where R is scalar curvature, G is determinant of the spacetime metric tensor $G_{AB}$ [8]. In this work, italic and capital Latin letters in the indices such as *(A,B,C,…)* refers to spacetime coordinates and capital Latin letters in the indices such as (A,B,C,…) refers to d-space coordinates. The index o denotes time coordinate. The d+1 dimensional spacetime is splitted into the time and space constituents by means of the $n_A$ vector satisfying the $n_A n^A = 1$ in the Euclidean region. The covariant and the contravariant components of $n_A$ are given by $\{n_o = N,\ n_A = 0,\ n^o = 1/N,\ n^A = N^A / N\}$, where N is the Lapse function and $N^A$ is the shift vector [2,7].

The higher dimensional spacetime metric in the Euclidean region has the form of $ds^2 = d\tau^2 + G_{AB}\, dx^A\, dx^B$. From the field redefineable freedom of the theory, the substitutions of $d\tau$ with $N\, d\tau$ and $dx^A$ with $dx^A + N^A d\tau$ can be applied. As a result, the final metric has the form of $ds^2 = N^2 d\tau^2 + G_{AB}\,(dx^A + N^A d\tau)\,(dx^B + N^B d\tau)$. Hence the covariant and contravariant components of the $G_{AB}$ metric tensor are $\{ G_{oo} = N^2 + N^A N_A,\ G_{oA} = N_A,\ G_{AB} = g_{AB},\ G^{oo} = 1/N^2,\ G^{oA} = N^A/N^2,\ G^{AB} = g^{AB} + N^A N^B / N^2\}$ where $g_{AB}$ is the d-space metric tensor and $g^{AB}$ is the inverse of $g_{AB}$. The extrinsic curvature tensor or the second fundamental form of the d-space geometry $K_{AB}$ is defined by $K_{AB} = [N_{A|B} + N_{B|A} - (\partial g_{AB} / \partial \tau)] / 2N$, where | denotes the d-space covariant derivative [4,5,6,7].

On the other hand, the extrinsic curvature tensor $K_{AB}$ can also be expressed as $K_{AB} = [\mathcal{L}_N g - (\partial g_{AB} / \partial \tau)] / 2N$, where $\mathcal{L}_N g = N_{A|B} + N_{B|A}$ is the Lie derivative of the $g_{AB}$ d-space metric tensor with respect to the $N_A$ vector field. When the $N_A$ vectors are thought as the generators of the isometry group of the d-space, the $\mathcal{L}_N g$ Lie derivative vanishes. This means that the isometry group symmetries of the d-space give rise to the observed symmetries of the fundamental interactions of physics. For instance, from the antisymmetric $N_{A|B}$ quantities, the totally antisymmetric tensor $H_{ABC} = N_{A|B|C} + N_{B|C|A} + N_{C|A|B}$ can be constrıcted which has 84 degrees of freedom of the relevant string theory ( d! / ( 3! (d-3)! = 84 for d=9 ) . Hence, the internal symmetries of the fundamental interactions of physics in our universe are spacetime symmetries in higher dimensional universe [3].

According to the ADM formalism, the zeroth order Einstein-Hilbert action is $I = \int [\pi^{AB} (\partial g_{AB} / \partial \tau) - N C^o - N_A C^A]\, d^{d+1} x$, where a total divergence term has been discarded. The Hilbert-Palatini variation is taken with respect to the $\pi^{AB}$, $g_{AB}$, N and $N_A$ quantities, separately, where, $\pi^{AB} = g^{1/2} (g^{AB}\, \mathrm{Tr}\, K - K^{AB})$, $\mathrm{Tr}\, K = g_{AB} K^{AB}$, $C^o = g^{-1/2} [\mathrm{Tr}\, \pi^2 - (\mathrm{Tr}\, \pi)^2 /(d-1)] - g^{1/2}\, {}^{d}R$, $\mathrm{Tr}\, \pi^2 = \pi_{AB} \pi^{AB}$, $\mathrm{Tr}\, \pi = \pi_A{}^A = g_{AB} \pi^{AB}$, $C^A = -2\, \pi^{AB}{}_{|B}$, g is the determinant of the d-space metric $g_{AB}$, and ${}^{d}R$ is the d-space curvature scalar which is constructed by d-space metric tensor $g_{AB}$ [7].

The $K_{AB}$ is expressed in terms of the conjugate momentum $\pi_{AB}$ reading the $K_{AB} = - g^{-1/2} (\pi_{AB} - g_{AB}\, \mathrm{Tr}\, \pi / (d-1))$, which is one of the necessary conditions for an extremum of I [7].

**3. THE WHEELER-DEWITT EQUATION**

The Wheeler-DeWitt equation is a infinite dimensional nonlinear equation. It is not well-known to solve this infinite dimensional equation. Hence this equation is partly solved in a finite dimensional submanifold named as minisuperspace by Wheeler in the literature.

This submanifold is the $M_4 \times C_6$ manifold which has the dimension of $N = d + 1 = 3 + 6 + 1 = 10$. In this paper, the Gaussian normal coordinates were used for all calculations. The submanifold metric in the Gaussian normal coordinates has the form $ds^2 = dn^2 + R(n)^2\, d\Omega_3{}^2 + S(n)^2 d\Omega_6{}^2$, where $dn = N\, d\tau$, $d\Omega_3{}^2$ is the three dimensional sphere, $d\Omega_6{}^2$ is the six dimensional sphere, R(n) is the actual space scale factor and S(n) is the internal compact space scale factor.

The Wheeler-deWitt equation has the form of $i \hbar \, \partial\psi / \partial t = H \psi = 0$ where H is the ADM Hamiltonian. For the above submanifold, the ADM Hamiltonian, in terms of the geometrized units, i.e., the Planck constant ħ , the velocity of light c and the gravitational constant G are unit, has the form of $H = R(\tau)^3 S(\tau)^6 N \, [ ( \partial R(\tau) / R(\tau) \, N \, \partial \tau)^2 + 6 \, ( \partial R(\tau) / R(\tau) \, N \, \partial \tau) ( \partial S(\tau) / S(\tau) \, N \, \partial \tau) + 5 ( \partial S(\tau) / S(\tau) \, N \, \partial \tau )^2 - (1 / R(n) )^2 - (1 / S(n) )^2 + \lambda]$ where $\lambda$ is a positive cosmological constant. By choosing $N = R(n)$ for Lapse function N, where $d n = N d \tau$, and by using of $e = R(n) S(n)$ and $g = R(n) S(n)^5$ transformations, the Hamiltonian H can be expressed as $H = \dot{e} \dot{g} - e g - e^4 - \lambda e^{(7/2)} g^{(1/2)}$ , where an overdot denotes the derivative with respect to n . Under these conditons, the Wheeler-deWitt equation becomes as $H \psi = \partial^2 \psi / \partial e \, \partial g - e g \, \psi - e^4 \psi + \lambda e^{(7/2)} g^{(1/2)} \psi = 0$. By making of $\psi = \exp ( Ø )$ substitution, the Wheeler-deWitt equation takes the form of $\chi \partial^2 Ø / \partial e \, \partial g + (\partial Ø / \partial e) (\partial Ø / \partial g) = e g + e^4 - \lambda e^{(7/2)} g^{(1/2)}$ where $\chi$ reflects the normal ordering ambiguity for the second quantized operators. By choosing $\chi = Ø$, the last equation can be easily integrated and by returning to R(n) and S(n) variables , wave function of the universe is obtained as $\psi = \exp \{ [ (R(n)^4 S(n)^{12} /2 + 2 (R(n)^6 S(n)^{10} / 5 - 8 \lambda (R(n)^6 S(n)^{12} / 27 ]^{1/2} \}$.

In fact, for the small R(n) and S(n), the wave function has the form of $\psi = \exp [ (R(n)^2 S(n)^6 ]$.

## 4. CONCLUSION

The higher dimensional ADM formalism has been briefly mentioned. The higher dimensional Wheeler-deWitt equation has been solved for the $M_4 \times C_6$ minisuperspace and the wave function of the universe were found. The solution wave function of the universe has the exponantial varying character in the Euclidean region, whereas, the oscillatory character in the Lorentzian region. Furthermore, the wave function has the harmonic oscillator ground state wave function form for the small values of the relevant metrics scale factors.

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