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HOLDITCH'S THEOREM FOR CIRCLES IN 2-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT

The present study expresses and proves Holditch's Theorem for two different circles in two-dimensional Euclidean space through a new method.

Key Words: *Affine space, Euclidean space, Euclidean circle, Holditch's Theorem*

2-BOYUTLU ÖKLİD UZAYINDA ÇEMBER İÇİN HOLDITCH TEOREMİ

ÖZET

Bu çalışmada, 2-boyutlu Öklid uzayında farklı iki çember için Holditch Teoremi yeni bir metodla ifade ve ispat edilmiştir.

Anahtar Kelimeler: *Afin Uzay, Öklid uzayı, Öklid çemberi, Holditch Teoremi*

1. INTRODUCTION

In this section we give basic definitions and theorems used in this study.

Definition 1.1 Let A be a non-empty set and V be a vector space on a field F.

i. For $\forall P,Q,R \in A$, $f(P,Q)+f(Q,R)=f(P,R)$.
ii. For $\forall P \in A$ and $\forall \alpha \in V$, there is a unique point $Q \in A$ so that $f(P,Q)=\alpha$.

If there is a function $f: AxA \to V$, satisfying above propositions, then A is called an Affine space associated to V [1].

Definition 1.2 Let A be a real affine space and let V be a vector space associated to A. Using Euclidean inner product operation on V,

$$\langle , \rangle : VxV \to IR\, , \left\langle \vec{X}, \vec{Y} \right\rangle = \sum_{1}^{n} x_i y_i \, , \quad \vec{X} = (x_1,...,x_n), \vec{Y} = (y_1,...,y_n)$$

we can define the metric concepts such as distance and angle in A. Therefore, Affine space A is called a in Euclidean space and is denoted by $A = E^n$ [1].

Definition 1.3 The transformation defined by $\| \, \| : IR^n \to IR^+$

$$\left\| \vec{X} \right\| = \sqrt{\left\langle \vec{X}, \vec{X} \right\rangle}$$

is called the norm of the vector $\vec{X}$.

Definition 1.4 For $\forall \vec{X}, \vec{Y} \in IR^n$, the measure of the angle between $\vec{X}$ and $\vec{Y}$ is the real number θ derived from

$$\cos\theta = \frac{\left\langle \vec{X}, \vec{Y} \right\rangle}{\left\| \vec{X} \right\| \left\| \vec{Y} \right\|} .$$

Definition 1.5 (The Pythagoras Theorem) In a right triangle called $\overset{\Delta}{ABC}$, if $\left\| \vec{AB} \right\| = c$, $\left\| \vec{AC} \right\| = b$, $\left\| \vec{BC} \right\| = a$, then $a^2 = b^2 + c^2$ (Figure 1.1) [2].

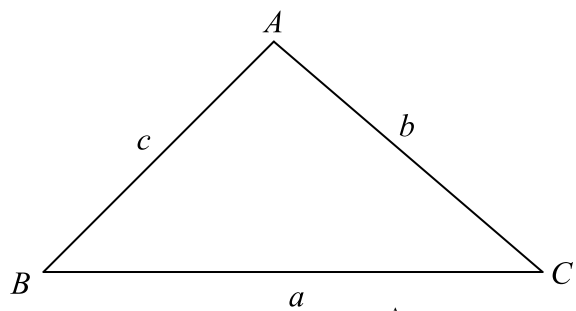

Figure 1.1 Right triangle $\overset{\Delta}{ABC}$

Definition 1.6 The set of the points in E^2 which are at a distance r from a point M is called a circle with center M and radial length r Euclid circle and is denoted by

$$C = \left\{ X : \left\| \vec{MX} \right\| = r, \; r = \text{constant} \right\} \quad [2].$$

Definition 1.7 The area of a circle with radius r and with point A on it is

$$A(C_A) = \pi r^2 \quad [2].$$

Theorem 1.1 (Holditch's Theorem) Let a chord with constant lengths of $a+b$ of a closed convex curve α be divided by a point P on it into two segments with a and b as lengths. Let us move the end-points of the chord so that they will entirely trace the curve. Then, the difference between the sizes of the area bounded by the closed curve drawn by point P and that bounded by the main convex curve α is πab [3].

2. THE CLASSICAL HOLDITCH THEOREM

Theorem 2.1 Let an AB chord with a constant length of $a+b$ on a circle (C) with a radius r in Euclidean plane E^2 be divided by a point D into two segments with lengths of a and b, respectively. When the end-points A and B of the chord draw the circle in full, then geometric location of D forms an inner circle (Figure 2.1).

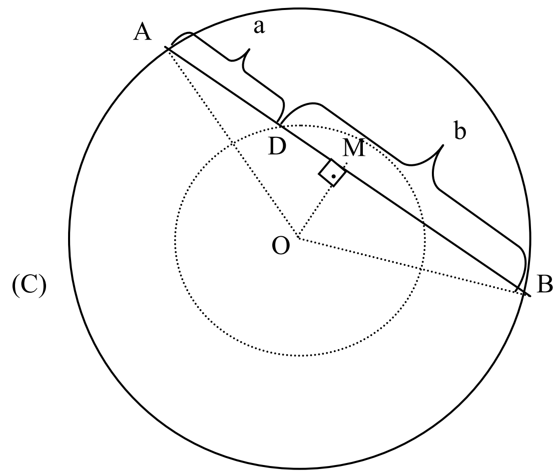

Figure 2.1 Geometric location of D on the chord

Proof. $\|AB\| = a+b = 2l = \text{constant}$. Let M be the midpoint of AB. Then,

$$\|AD\| = a$$
$$\|BD\| = b$$
$$\|MA\| = \|MB\| = \frac{a+b}{2} = l.$$

From the right triangle $\overset{\Delta}{OMA}$

$$\|OA\|^2 = \|OM\|^2 + \|MA\|^2$$

or

$$\|OM\|^2 = \|OA\|^2 - \|MA\|^2 = r^2 - \left(\frac{a+b}{2}\right)^2$$

$$\|MD\| = b - l = b - \frac{a+b}{2} = \frac{b-a}{2}.$$

Similarly, from the right triangle $\overset{\Delta}{OMD}$

$$\|OD\|^2 = \|OM\|^2 + \|MD\|^2$$
$$= r^2 - \left(\frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2$$
$$= r^2 - ab.$$

Since a, b and r are constant, $\|OD\|$ is also constant. Therefore, the geometric location of D is a circle with a centre O and a radial length of $\sqrt{r^2 - ab}$.

Theorem 2.2 Let a chord AB with a constant length of $a+b$ on a circle (C) with a radius R in Euclidean plane E^2 be divided by a point D into two segments with lengths of *a* and *b*, respectively. When the end-points A and B of the chord draw the circle in full, the size of the ring-shaped region between the orbit of D (inner circle) and circle (C) is independent from the radial length of circle (C).

Proof. For the radial length $z = \sqrt{r^2 - ab}$ of the inner circle is and its two chords AB and EF intersecting at point D of circle (C), let the chord EF be the diameter of (C). The chords AB and EF are divided by D into line segments whose lengths are a, b and r+z, r-z, respectively (Figure 2.2).

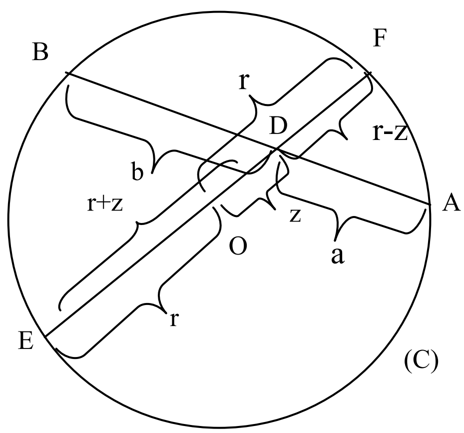

Figure 2.2 Two chords intersecting in circle (C)

Since the triangles $\overset{\Delta}{BDE}$ and $\overset{\Delta}{FDA}$ in the interior of circle (C) are similar triangles,

$$\frac{\|BD\|}{\|FD\|} = \frac{\|DE\|}{\|DA\|}$$
$$\frac{b}{r-z} = \frac{r+z}{a}$$
$$ab = r^2 - z^2$$

Since the area of circle (C) is $A(C) = \pi r^2$ and the area of the inner circle is πz^2, the area of the ring-shaped region between these two circles is found as

$$\pi r^2 - \pi z^2 = \pi\left(r^2 - z^2\right)$$
$$= \pi ab$$

Therefore, the size of the ring-shaped region that falls between the orbit of D and circle (C) is independent from the radial length of circle (C).

Corollary 2.1 The size of the ring-shaped region that falls between the orbit of D and circle (C) is dependent on the selection of point D on the chord; that is, on the segments with lengths of *a* and *b*.

3. HOLDITCH'S THEOREM FOR TWO DIFFERENT CIRCLES IN A 2-DIMENSIONAL EUCLIDEAN SPACE

Theorem 3.1 For a circle C with a radial length of $R+a+b$ and a circle C' with radius $R<\left(R+a+b\right)$, let an AB rod with a constant length of $a+b=\text{constant}$, with end B attached to circle C, and the other end A attached to circle C' by a joint, be divided by point X on it into two segments with lengths of *a* and *b*, respectively. When the rod with the constant length of $a+b=\text{constant}$ draws the circles *C* and C' with its end-points within these circles *C* and C', the geometric location of X forms another inner circle (Figure 3.1). During this motion, the relation between the regions bounded by circles is independent from the selection of circles *C* and C'.

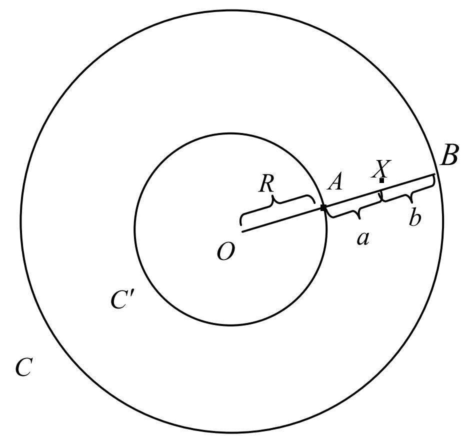

Figure 3.1 Circles in E^2

Proof.

$$\left\|\vec{AX}\right\|=a\,,\ \left\|\vec{XB}\right\|=b\,,\ \left\|\vec{OX}\right\|=R+a$$

The area of circle C' with a radial length R and with a point A on it is

$$A(C_A)=\pi R^2$$

The area of circle *C* with a radial length $R+a+b$ and with a point B on it is

$$A(C_B)=\pi\left(R+a+b\right)^2$$

Therefore, the area of the circle, which is the geometric location of point X is

$$A(C_X) = \pi (R+a)^2$$

Thus,

$$\begin{aligned} A(C_B) - A(C_X) &= \pi\left[(R+a+b)^2 - (R+a)^2\right] \\ &= \pi b\left[2(R+a)+b\right] \\ A(C_A) - A(C_X) &= \pi\left[R^2 - (R+a)^2\right] \\ &= -\pi a(2R+a) \end{aligned}$$

and by adding these two equations side by side, we get

$$\begin{aligned} A(C_A) + A(C_B) - 2A(C_X) &= \pi\left[2(R+a)b + b^2 + R^2 - (R+a)^2\right] \\ &= \pi\left[2Rb + 2ab + b^2 + R^2 - R^2 - 2aR - a^2\right] \\ &= \pi\left[2R(b-a) + 2ab + b^2 - a^2\right] \\ &= \pi\left[2ab + (b-a)(b+a+2R)\right] \end{aligned}$$

$$A(C_A) + A(C_B) - 2A(C_X) = \pi\left[2ab + (b-a)\left(b+a+2\sqrt{\frac{A(C_A)}{\pi}}\right)\right]$$

Corollary 3.1 The relation between the regions bounded by circles with radial lengths of R, $R+a$ and $R+a+b$ is *a* and *b*; in other words, it is independent from the rod's motion on the circles.

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