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Conjugate Tangent Vectors, Asymptotic Directions, Euler Theorem and Dupin Indicatrix For k-Kinematic Surfaces

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ABSTRACT

In this study, we define the k-kinematic surface M^g which is obtained from a surface M on Euclidean 3-space E^3 by applying rigid motion described by quaternions to points of M. Then we investigate and calculate for this surface some important concepts such as shape operator, asymptotic vectors, conjugate tangent vectors, Euler theorem and Dupin indicatrix which help to understand a surface differential geometrically well.

Keywords: Asymptotic direction, conjugate tangent vectors, Dupin indicatrix, Euler theorem.

1. INTRODUCTION

Surfaces have had application areas in many areas such as mathematics, kinematics, dynamics and engineering for many years and they have been in center of interest increasingly. Mathematicians have written many articles and books by investigating surfaces as Euclidean and non-Euclidean. For these studies, one can read [1-15] . Eisenhart defined parallel surfaces and their some properties in his book [3]. In [16], Ünlütürk and Özüsağlam investigated the parallel surfaces in Minkowski 3-space. In [17], Tarakçı and Hacısalihoğlu defined surfaces at a constant distance from edge of regression on a surface and gave some properties of such surfaces and then in [18-20] Sağlam and Kalkan investigated the other properties of this surface. Again Sağlam and Kalkan transported the surfaces at a constant distance from edge of regression on a surface to Minkowski 3-space and obtained their properties which they have in Euclidean space.
Quaternions have many application areas in both theoretical and applied mathematics. The quaternions described firstly by Hamilton applied mechanics in 3-dimensional space [21]. The quaternions as a set correspond to $\mathbb{R}^4$ 4-dimensional vector space on real numbers. The unit quaternions which are known as vensors provide a convenient mathematical description in rotations and directions in 3-dimension. They are simpler forming and numerically more stable and efficient than Euler angles and rotation matrices. The set of dual quaternions, invented by Clifford to describe space geometry in mathematics and mechanics, is a Clifford algebra which can be used for representation of rigid motions [22-24]. Motion of a point, line and objects has a great attraction in kinematics [25]. E. Study and Kotelnikov applied dual numbers and dual vectors

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to studies which they did in kinematics ([26], [27]). Homogeneous transformation is a point transformation. However, the line transformations in which transformed element is a line instead a point can be defined in 3-dimensional Cartesian space. Pottmann and Wallner studied on line transformations [28]. A screw is a 6-dimensional vector which is obtained from vectors such as power, torque, linear velocity and angular velocity emerged in rigid motion. When two lines are given it is easy to obtain one from another by screw motion [29]. Rigid motions include rotations, translations, reflections and combinations of these. Sometimes reflections are excluded in definition of rigid motion. The shape and dimension of any object remains same after the rigid motion. In kinematics, a suitable rigid motion represented by SE(3) is used for representing linear and angular changes. According to Charles theorem, every rigid motion can be expressed as a screw motion. A surface formed kinematically is a surface defined by a moving object envelope. This object can be a point, a line, a plane or any arbitrary figure. There are many applications of surfaces produced in many areas kinematically [30-35].
Selig and Husty took the dual quaternion which described a rigid motion and gave its effects on a point and a line in their study [36]. In [31-34] a computer-aided geometric design (CAGD) and surface design were combined. In these studies, they focused on the surfaces obtained by using point movements (substitution). The techniques for generating surfaces kinematically are more suitable in CAD/CAM, because these depend directly on the kinematic constraints of the bench and design requirements.

In this study, we define the kinematic surface by applying rigid motion expressed by dual quaternions as in [36] to points of a surface M in 3-dimensional Euclidean space E^3 and obtain a k-kinematic surface M^g by taking the rotation axis specially as the unit vector k. The k-kinematic surface M^g is a more general case of surfaces at a constant distance from edge of regression from a point on a surface and the parallel surfaces on which many studies have been done by mathematicians and differential geometers until now. In special cases one can obtain surfaces at a constant distance from edge of regression from a point on a surface and the parallel surfaces from the k-kinematic surfaces. Then, we calculate shape operator, asymptotic vectors, conjugate tangent vectors, Euler theorem and Dupin indicatrix, which are well-known concepts in differential geometry, of the k-kinematic surface M^g and investigate the changes in these concepts under the rigid motion.

2. PRELIMINARIES

Let M be a surface of E^3 with the metric tensor $\langle , \rangle$. Let D be the Riemannian connection on E^3 and N be a unit normal C^∞ vector field on M. Then, for every $p \in M$ and $X \in T_p(M)$ we have $\langle N_p, N_p \rangle = 1$ and $\langle N_p, X \rangle = 0$. Let $S: T_p(M) \to T_p(M)$ be the shape operator defined by $S(X) = D_X N$. The Gaussian curvature $K(p)$ and mean curvature $H(p)$ of M at p are the determinant and the trace of S at $p \in M$, respectively. The eigenvalues of S are called the principal curvatures of M. If tangent of a curve is a principal vector at each of its points then this curve is a curvature line in M.

Definition 1. Let M and M^r be two surfaces in Euclidean space. Let $\overrightarrow{N}$ be the unit normal vector field of M and $r \in \mathbb{R}$ be a constant. If there is a function

$$f: M \to M^r$$
$$p \to f(p) = p + r\overrightarrow{N}_p$$

between the surfaces M and M^r then M^r is called parallel surface of M and the function f is called the parallelization function between the surfaces M and M^r [37].

Definition 2. Let M be an Euclidean surface in E^3 and S be the shape operator of M. For $X_p \in T_p(M)$ if

$$\langle S(X_p), X_p \rangle = 0$$

then X_p is called an asymptotic direction of M at $p \in M$ [38].

Definition 3. Let M be an Euclidean surface in E^3 and S be the shape operator of M. For $X_p, Y_p \in T_p(M)$ if

$$\langle S(X_p), Y_p \rangle = 0$$

then X_p and Y_p are called conjugate tangent vectors of M at $p \in M$ [38].

Definition 4. Let M be an Euclidean surface in E^3 and S be the shape operator of M. For an umbilic point $p \in M$ the function

$$k_n : T_p(M) \to R$$
$$k_n(X_p) = \frac{1}{\|X_p\|^2}\langle S(X_p), X_p\rangle$$

is called the normal curvature function of M at p [37].

Definition 5. Let M be an Euclidean surface in E^3 and S be the shape operator of M. Then the Dupin indicatrix of $p \in M$ is

$$\mathrm{D}_p = \{X_p \mid \langle S(X_p), X_p\rangle = \pm 1,$$
$X_p \in T_p(M)\}$ [37].

Definition 6. Let M and M^f be two surfaces in E^3 and N_p be a unit normal vector of M at a point $p \in M$. Let $T_p(M)$ be the tangent space at $p \in M$ and $\{X_p, Y_p\}$ be an orthonormal basis of $T_p(M)$. Let $Z_p = d_1 X_p + d_2 Y_p + d_3 N_p$ be a unit vector where $d_1, d_2, d_3 \in \mathbb{R}$ are constant numbers such that $d_1^2 + d_2^2 + d_3^2 = 1$. If a function with the condition

$$f : M \to M^f, f(p) = p + rZ_p, r \text{ constant},$$

M^f is called as the surface at a constant distance from edge of regression on M [17].

2.1. Quaternions

Let us firstly begin with Hamilton's quaternions and their connection with rotations. A rotation of angle θ, about a unit vector $v = (v_x, v_y, v_z)^T$ is represented by the quaternion,

$$r = cos\frac{\theta}{2} + sin\frac{\theta}{2}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

The conjugation

$$p' = rp\overline{r}$$

gives the action of such a quaternion on a point $p = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space, where

$$\overline{r} = cos\frac{\theta}{2} - sin\frac{\theta}{2}(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}).$$

The quaternions representing rotations satisfy $r\overline{r} = 1$ and also r and $\overline{r}$ represent the same rotation. The set of unit quaternions, those satisfying $r\overline{r} = 1$, comprise the group Spin(3), which is the double cover of the group of rotations $SO(3)$.

Let ε be the dual unit which satisfies the relation $\varepsilon^2 = 0$ and commutes with the quaternion units $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. For ordinary quaternions q_0 and q_1,

$$h = q_0 + \varepsilon q_1$$

indicates a general dual quaternion. A rigid transformation is represented by a dual quaternion

$$g = r + \frac{1}{2}\varepsilon tr,$$

where r is a quaternion representing a rotation as above and $t = t_x\mathbf{i} + t_y\mathbf{j} + t_z\mathbf{k}$ is a pure quaternion representing the translational part of the transformation [36].

Points in space are represented by dual quaternions of the form,

$$\hat{p} = 1 + \varepsilon p,$$

where p is a pure quaternion as above. The action of a rigid transformation on a point is given by,

$$\begin{aligned}\hat{p}' &= (r + \frac{1}{2}\varepsilon tr)\hat{p}(r + \frac{1}{2}\varepsilon\overline{rt}) \\ &= (r + \frac{1}{2}\varepsilon tr)(1 + \varepsilon p)(r + \frac{1}{2}\varepsilon\overline{rt}) \\ &= 1 + \varepsilon(rp\overline{r} + t).\end{aligned}$$

Note that, as with the pure rotations, g and $-g$ represent the same rigid transformation [36].

3. KINEMATIC SURFACES AND k-KINEMATIC SURFACES

Firstly, let us give the definition of the kinematic surface:

Definition 7. Let M and M^g be two surfaces in E^3 and $p \in M$. Let $r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(v_x i + v_y j + v_z k)$ be a rotation by an angle of θ radian about the unit vector $\vec{v} = (v_x, v_y, v_z)$ and $\vec{t}$ be the translational vector. If there is a function defined as

$$f : M \to M^g$$
$$p \to f(p) = rp\overline{r} + t$$

then the surface M^g is called a kinematic surface of the surface M.

Let the rotation axis be the unit vector k and the translational vector be any unit vector Z_p at a point $p \in M$. Then, we can obtain a new kinematic surface, let us call this surface as "k-kinematic surface".

Definition 8. Let M and M^g be two surfaces in E^3 and $p \in M$. Let $r = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}k$ be a rotation by an angle of θ radian about the unit vector k and $\overrightarrow{Z_p}$ be the translational vector. If there is a function defined as

$$f(p) = \cos\theta p + \sin\theta \vec{k} \wedge p$$
$$+(1-\cos(\theta)\langle \vec{k}, p\rangle \vec{k} + \lambda \overrightarrow{Z_p}$$

then M^g is called a k-kinematic surface of the surface M.

As an example, let us consider the half cylinder

$$M = \{\phi(u,v) \mid \phi(u,v) = (\cos u, \sin u, v),$$
$$0 \le u \le \pi/2, 0 \le v \le 2\}.$$

Let the rotation angle be $\pi/2$ and translational vector be $\vec{Z} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$. Rotating every point of M by $\pi/2$ angle over the $\vec{k}$ and translating 6 unit along $\vec{Z}$ gives the k-kinematic surface

$$M^g = \{\psi(u,v) \mid \psi(u,v) = (-\sin u + 2\sqrt{3}, \cos u + 2\sqrt{3}, v + 2\sqrt{3})$$
$$0 \le u \le \pi/2, 0 \le v \le 2\}.$$

Image of a point $P = (0,1,2) \in M$ will be $p' = (-1+2\sqrt{3}, 2\sqrt{3}, 2+2\sqrt{3}) \in M^g$ (Figure 1).

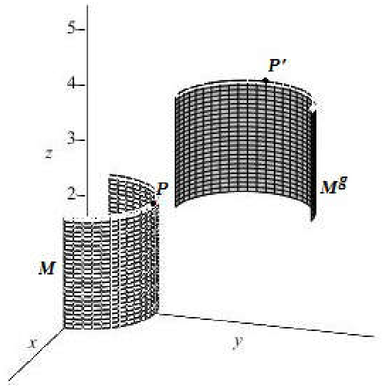

Figure 1. k-kinematic surface of an half cylinder

One can easily see that for $\forall X_p \in T_p M$

$$f_*\left(X_p\right) = \cos\theta X_p + \sin\theta \vec{k} \wedge X_p$$
$$+(1-\cos\theta)\langle \vec{k}, X_p\rangle \vec{k} + \lambda \overrightarrow{Z_p},$$

so the tangent vectors on M can be transferred to the surface M^g by the transformation f_*.

Let (ϕ, U) be a parametrization of the surface M. Then, one can write that

$$\phi : U \subset E^3 \to M$$
$$(u,v) \to p = \phi(u,v).$$

It follows that $\{\phi_u, \phi_v\}_p$ is a basis of $T_p(M)$. Let N_p be a unit normal vector at $p \in M$ and $d_1, d_2, d_3 \in \mathbb{R}$ be constant real numbers. Then we can write that $\overrightarrow{Z_p} = d_1\phi_u|_p + d_2\phi_v|_p + d_3 N_p$. Since

$$M^g = \{f(p) \mid f(p) = \cos\theta p + \sin\theta \vec{k} \wedge p$$
$$+(1-\cos\theta)\langle \vec{k}, p\rangle \vec{k} + \lambda \overrightarrow{Z_p}\},$$

a parametric representation of the surface M^g is

$$\psi(u,v) = \cos\theta\phi(u,v) + \sin\theta \vec{k} \wedge \phi(u,v)$$
$$+(1-\cos\theta)\langle \vec{k}, \phi(u,v)\rangle \vec{k} + \lambda \overrightarrow{Z_p}$$

and

$$M^g = \{\psi(u,v) \mid \psi(u,v) = \cos\theta\phi(u,v) + \sin\theta \vec{k} \wedge \phi(u,v)$$
$$+(1-\cos\theta)\langle \vec{k}, \phi(u,v)\rangle \vec{k} + \lambda\left(d_1\phi_u + d_2\phi_v + d_3 N(u,v)\right),$$
$$d_1, d_2, d_3, \lambda \text{ are constants}\}$$

or

$$M^g = \{\psi(u,v) \mid \psi(u,v) = \cos\theta\phi(u,v) + \sin\theta \vec{k} \wedge \phi(u,v)$$
$$+(1-\cos\theta)\langle \vec{k}, \phi(u,v)\rangle \vec{k} + \lambda_1\phi_u + \lambda_2\phi_v + \lambda_3 N(u,v),$$
$$\lambda_1, \lambda_2, \lambda_3 \text{ are constants}\}.$$

where $\lambda_1 = \lambda d_1$, $\lambda_2 = \lambda d_2$ and $\lambda_3 = \lambda d_3$. Let us take ϕ_u and ϕ_v as the principal directions of the surface M. Let k_1 and k_2 be the associated principal curvatures, respectively. Then, we get

$$\psi_u = (\cos\theta + \lambda_3 k_1)\phi_u + \sin\theta\langle k, N\rangle\phi_v$$
$$-(\lambda_1 k_1 + \sin\theta\langle k, \phi_v\rangle)N + (1-\cos\theta)\langle k, \phi_u\rangle k$$

and

$$\psi_v = -\sin\theta\langle k, N\rangle\phi_u + (\cos\theta + \lambda_3 k_2)\phi_v$$
$$-(\lambda_2 k_2 - \sin\theta\langle k, \phi_u\rangle)N + (1-\cos\theta)\langle k, \phi_v\rangle k.$$

Therefore, the unit normal vector field of the surface M^g can be calculated as

$$N^g = \frac{\psi_u \wedge \psi_v}{A}$$

where $A = \|\psi_u \wedge \psi_v\|$.

Theorem 1. Let the pair (M, M^g) be given in E^3. Let $\{\phi_u, \phi_v\}$ be orthonormal and principle vector fields on M and k_1, k_2 be principle curvatures of M. Then the matrix of the shape operator S^g of M^g is

$$S^g = \frac{1}{A^2}\begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}$$

where

$$\mu_1=\left[\langle N^g,\psi_{uv}\rangle\langle\psi_u,\psi_v\rangle-\langle N^g,\psi_{uu}\rangle\langle\psi_v,\psi_v\rangle\right],$$
$$\mu_2=\left[\langle N^g,\psi_{uu}\rangle\langle\psi_u,\psi_v\rangle-\langle N^g,\psi_{uv}\rangle\langle\psi_u,\psi_u\rangle\right],$$
$$\mu_3=\left[\langle N^g,\psi_{vv}\rangle\langle\psi_u,\psi_v\rangle-\langle N^g,\psi_{uv}\rangle\langle\psi_v,\psi_v\rangle\right],$$
$$\mu_4=\left[\langle N^g,\psi_{uv}\rangle\langle\psi_u,\psi_v\rangle-\langle N^g,\psi_{uu}\rangle\langle\psi_u,\psi_u\rangle\right].$$

4. ASYMPTOTIC DIRECTIONS AND CONJUGATE TANGENT VECTORS FOR k-KINEMATIC SURFACES

Theorem 2. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u,\phi_v\}$ be orthonormal and principle vector fields on M and k_1, k_2 be principle curvatures of M. Let $X_p\in T_p(M)$. Then $f_*(X_p)\in T_{f(p)}(M^g)$ is an asymptotic direction of M^g if and only if

$$\mu_1^* x_1^2+\mu_2^* x_1x_2+\mu_3^* x_2^2=0 \tag{1}$$

where

$x_1=\langle X_p,\phi_u\rangle,\qquad x_2=\langle X_p,\phi_v\rangle,$

$\mu_1^*=\mu_1\langle\psi_u,\psi_u\rangle+\mu_2\langle\psi_u,\psi_v\rangle,$

$\mu_2^*=\mu_1\langle\psi_u,\psi_v\rangle+\mu_2\langle\psi_v,\psi_v\rangle+\mu_3\langle\psi_u,\psi_u\rangle+\mu_4\langle\psi_u,\psi_v\rangle,$

$\mu_3^*=\mu_3\langle\psi_u,\psi_v\rangle+\mu_4\langle\psi_v,\psi_v\rangle.$

Proof. Let $X_p\in T_p(M)$. Then, we can write that $X_p=x_1\phi_u+x_2\phi_v$, where $x_1=\langle X_p,\phi_u\rangle$ and $x_2=\langle X_p,\phi_v\rangle$. Besides, one can write that

$$\begin{aligned} f_*(X_p)&=x_1f_*(\phi_u)+x_2f_*(\phi_v)\\ &=x_1\psi_u+x_2\psi_v. \end{aligned}\tag{2}$$

On the other hand, calculating $S^g(f_*(X_p))$ gives

$$\begin{aligned} S^g(f_*(X_p))&=x_1S^g(f_*(\phi_u))+x_2S^g(f_*(\phi_v))\\ &=(\mu_1x_1+\mu_3x_2)\psi_u+(\mu_2x_1+\mu_4x_2)\psi_v \end{aligned}\tag{3}$$

Calculating inner product of (2) and (3) gives the result.

Theorem 3. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u,\phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M and k_1, k_2 be principle curvatures of M. Let θ_1 and θ_2 be the angles between the unit vector X_p and ϕ_u and ϕ_v, respectively. Then $f_*(X_p)\in T_{f(p)}(M^g)$ is an asymptotic direction of M^g if and only if

$$\mu_1^*\cos^2\theta_1+\mu_2^*\cos\theta_1\cos\theta_2+\mu_3^*\cos^2\theta_2=0. \tag{4}$$

Proof. Let θ_1 be the angle between X_p and ϕ_u and θ_2 be the angle between X_p and ϕ_v. Then we have

$$\cos\theta_1=\langle X_p,\phi_u\rangle=x_1. \tag{5}$$

Similarly, we can obtain

$$\cos\theta_2=\langle X_p,\phi_v\rangle=x_2. \tag{6}$$

Substituting (5) and (6) into (1) completes the proof.

Theorem 4. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u,\phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M. Then for $X_p,Y_p\in T_p(M)$, $f_*(X_p)\in T_{f(p)}(M^g)$ and $f_*(Y_p)\in T_{f(p)}(M^g)$ are conjugate tangent vectors if and only if

$$\mu_1^*x_1y_1+\mu_2^*x_1y_2+\mu_3^*x_2y_1+\mu_4^*x_2y_2=0, \tag{7}$$

where

$$x_1=\langle X_p,\phi_u\rangle,\qquad x_2=\langle X_p,\phi_v\rangle,$$
$$y_1=\langle Y_p,\phi_u\rangle,\qquad y_2=\langle Y_p,\phi_v\rangle,$$
$$\mu_1^*=\mu_1\langle\psi_u,\psi_u\rangle+\mu_2\langle\psi_u,\psi_v\rangle,$$
$$\mu_2^*=\mu_1\langle\psi_u,\psi_v\rangle+\mu_2\langle\psi_v,\psi_v\rangle,$$
$$\mu_3^*=\mu_3\langle\psi_u,\psi_u\rangle+\mu_4\langle\psi_u,\psi_v\rangle,$$
$$\mu_4^*=\mu_3\langle\psi_u,\psi_v\rangle+\mu_4\langle\psi_v,\psi_v\rangle.$$

Proof. Let $X_p,Y_p\in T_p(M)$. Then, since $\{\phi_u,\phi_v\}$ is an orthonormal basis on $T_p(M)$ we have $X_p=x_1\phi_u+x_2\phi_v$ and $Y_p=y_1\phi_u+y_2\phi_v$, where $x_1=\langle X_p,\phi_u\rangle$, $x_2=\langle X_p,\phi_v\rangle$, $y_1=\langle Y_p,\phi_u\rangle$ and $y_2=\langle Y_p,\phi_v\rangle$. It follows that

$$\begin{aligned} f_*(X_p)&=x_1f_*(\phi_u)+x_2f_*(\phi_v)\\ &=x_1\psi_u+x_2\psi_v \end{aligned}$$

and

$$\begin{aligned} f_*(Y_p)&=y_1f_*(\phi_u)+y_2f_*(\phi_v)\\ &=y_1\psi_u+y_2\psi_v. \end{aligned}\tag{8}$$

On the other hand, one can obtain that

$$\begin{aligned} S^g(f_*(X_p))&=x_1S^g(f_*(\phi_u))+x_2S^g(f_*(\phi_v))\\ &=(\mu_1x_1+\mu_3x_2)\psi_u+(\mu_2x_1+\mu_4x_2)\psi_v \end{aligned}\tag{9}$$

Inner product of (8) and (9) gives

$$\begin{aligned} \langle S^g(f_*(X_p)),f_*(Y_p)\rangle&=\mu_1^*x_1y_1+\mu_2^*x_1y_2\\ &\quad+\mu_3^*x_2y_1+\mu_4^*x_2y_2, \end{aligned}$$

where

$$x_1=\langle X_p,\phi_u\rangle,\qquad x_2=\langle X_p,\phi_v\rangle,$$
$$y_1=\langle Y_p,\phi_u\rangle,\qquad y_2=\langle Y_p,\phi_v\rangle,$$
$$\mu_1^*=\mu_1\langle\psi_u,\psi_u\rangle+\mu_2\langle\psi_u,\psi_v\rangle,$$
$$\mu_2^*=\mu_1\langle\psi_u,\psi_v\rangle+\mu_2\langle\psi_v,\psi_v\rangle,$$

$$\mu_3^* = \mu_3\langle \psi_u, \psi_u\rangle + \mu_4\langle \psi_u, \psi_v\rangle,$$
$$\mu_4^* = \mu_3\langle \psi_u, \psi_v\rangle + \mu_4\langle \psi_v, \psi_v\rangle.$$

This completes the proof.

Theorem 5. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M and k_1, k_2 be principle curvatures of M. Let θ_1, θ_2 be the angles between the unit vector X_p and ϕ_u, ϕ_v, respectively and α_1, α_2 be the angles between the unit vector Y_p and ϕ_u, ϕ_v, respectively. Then $f_*(X_p)$ and $f_*(Y_p)$ are conjugate tangent vectors if and only if

$$\mu_1^* \cos\theta_1 \cos\alpha_1 + \mu_2^* \cos\theta_1 \cos\alpha_2$$
$$+\mu_3^* \cos\theta_2 \cos\alpha_1 + \mu_4^* \cos\theta_2 \cos\alpha_2 = 0$$

Proof. Let θ_1 be the angle between X_p and ϕ_u and θ_2 be the angle between X_p and ϕ_v. Then we have

$$\cos\theta_1 = \langle X_p, \phi_u\rangle = x_1. \tag{10}$$

and

$$\cos\theta_2 = \langle X_p, \phi_v\rangle = x_2. \tag{11}$$

Similarly, let α_1 be the angle between Y_p and ϕ_u and α_2 be the angle between Y_p and ϕ_v. Then we get

$$\cos\theta_1 = \langle Y_p, \phi_u\rangle = y_1. \tag{12}$$

and

$$\cos\theta_2 = \langle Y_p, \phi_v\rangle = y_2. \tag{13}$$

Substituting (10), (11), (12) and (13) into (7) completes the proof.

5. EULER THEOREM AND DUPIN INDICATRIX FOR k-KINEMATIC SURFACES

Theorem 6. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M and k_1, k_2 be principle curvatures of M. Let $X_p \in T_p(M)$ and $k_n^g(f^*(X_p))$ be the normal curvature of M^g in the direction $f^*(X_p)$. Then

$$k_n^g(f^*(X_p)) = \frac{\mu_1^* x_1^2 + \mu_2^* x_1 x_2 + \mu_3^* x_2^2}{\lambda_1^* x_1^2 + 2\lambda_2^* x_1 x_2 + \lambda_3^* x_2^2}, \tag{14}$$

where

$$x_1 = \langle X_p, \phi_u\rangle, \qquad x_2 = \langle X_p, \phi_v\rangle,$$
$$\mu_1^* = \mu_1\langle \psi_u, \psi_u\rangle + \mu_2\langle \psi_u, \psi_v\rangle,$$
$$\mu_2^* = \mu_1\langle \psi_u, \psi_v\rangle + \mu_2\langle \psi_v, \psi_v\rangle + \mu_3\langle \psi_u, \psi_u\rangle + \mu_4\langle \psi_u, \psi_v\rangle,$$
$$\mu_3^* = \mu_3\langle \psi_u, \psi_v\rangle + \mu_4\langle \psi_v, \psi_v\rangle,$$
$$\lambda_1^* = \langle \psi_u, \psi_u\rangle, \quad \lambda_2^* = \langle \psi_u, \psi_v\rangle, \quad \lambda_3^* = \langle \psi_v, \psi_v\rangle.$$

Proof. Let $X_p \in T_p(M)$. Then, we have $X_p = x_1\phi_u + x_2\phi_v$,where $x_1 = \langle X_p, \phi_u\rangle$, $x_2 = \langle X_p, \phi_v\rangle$. It follows that

$$f_*(X_p) = x_1 f_*(\phi_u) + x_2 f_*(\phi_v)$$
$$= x_1\psi_u + x_2\psi_v$$

and

$$S^g(f_*(X_p)) = x_1 S^g(f_*(\phi_u)) + x_2 S^g(f_*(\phi_v))$$
$$= (\mu_1 x_1 + \mu_3 x_2)\psi_u + (\mu_2 x_1 + \mu_4 x_2)\psi_v.$$

By an easy calculation we get

$$\|f_*(X_p)\|^2 = x_1^2\langle \psi_u, \psi_u\rangle + 2x_1x_2\langle \psi_u, \psi_v\rangle + x_2^2\langle \psi_v, \psi_v\rangle$$
$$= \lambda_1^* x_1^2 + 2\lambda_2^* x_1 x_2 + \lambda_3^* x_2^2$$

and

$$\langle S^g(f_*(X_p)), f_*(X_p)\rangle = (\mu_1\langle \psi_u, \psi_u\rangle + \mu_2\langle \psi_u, \psi_v\rangle)x_1^2$$
$$+(\mu_3\langle \psi_u, \psi_v\rangle + \mu_4\langle \psi_v, \psi_v\rangle)x_2^2$$
$$+(\mu_1\langle \psi_u, \psi_v\rangle + \mu_2\langle \psi_v, \psi_v\rangle$$
$$+\mu_3\langle \psi_u, \psi_u\rangle + \mu_4\langle \psi_u, \psi_v\rangle)x_1x_2$$
$$= \mu_1^* x_1^2 + \mu_2^* x_1 x_2 + \mu_3^* x_2^2.$$

Therefore we obtain

$$k_n^g(f^*(X_p)) = \frac{\mu_1^* x_1^2 + \mu_2^* x_1 x_2 + \mu_3^* x_2^2}{\lambda_1^* x_1^2 + 2\lambda_2^* x_1 x_2 + \lambda_3^* x_2^2}.$$

Theorem 7. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M and k_1, k_2 be principle curvatures of M. Let $X_p \in T_p(M)$ and $k_n^g(f^*(X_p))$ be the normal curvature of M^g in the direction $f^*(X_p)$. If we denote the angle between the unit vector X_p and ϕ_u by θ_1 and the angle between the unit vector X_p and ϕ_v by θ_2 then

$$k_n^g(f^*(X_p)) = \frac{\mu_1^* \cos^2\theta_1 + \mu_2^* \cos\theta_1 \cos\theta_2 + \mu_3^* \cos^2\theta_2}{\lambda_1^* \cos^2\theta_1 + 2\lambda_2^* \cos\theta_1 \cos\theta_2 + \lambda_3^* \cos^2\theta_2},$$

Proof. Substituting (5) and (6) into (14) gives the result.

Theorem 8. Let M^g be a k-kinematic surface of a surface M and $\{\phi_u, \phi_v\}$ be orthonormal basis such that ϕ_u and ϕ_v are principle vector fields on M and k_1, k_2 be principle curvatures of M. Then

$$\mathrm{D}_{f(p)}^g = \{f^*(X_p) \in T_{f(p)}(M^g) \mid$$
$$c_1^* x_1^2 + c_2^* x_1 x_2 + c_3^* x_2^2 = \pm 1\}, \tag{15}$$

where

$f^*(X_p) = x_1\psi_u + x_2\psi_v,$

$c_1^* = \mu_1\langle\psi_u,\psi_u\rangle + \mu_2\langle\psi_u,\psi_v\rangle,$

$c_2^* = \mu_1\langle\psi_u,\psi_v\rangle + \mu_2\langle\psi_v,\psi_v\rangle + \mu_3\langle\psi_u,\psi_u\rangle + \mu_4\langle\psi_u,\psi_v\rangle,$

$c_3^* = \mu_3\langle\psi_u,\psi_v\rangle + \mu_4\langle\psi_v,\psi_v\rangle.$

Proof. Let $f^*(X_p) \in T_{f(p)}(M^g)$. Then, since $D^g_{f(p)} = \{f^*(X_p) \mid \langle S^g(f^*(X_p)), f^*(X_p)\rangle = \pm 1\}$, proof is clear.

Corollary 1. Let M^g be a k-kinematic surface of a surface M. Then the Dupin indicatrix of M^g at $f(p) \in M^g$ is

1. an ellipse if $c_2^2 - 4c_1c_3 < 0$,
2. a hyperbola if $c_2^2 - 4c_1c_3 > 0$,
3. a parabola if $c_2^2 - 4c_1c_3 = 0$.

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